

On ϕ -Ricci Recurrent Almost Kenmotsu Manifolds with Nullity Distributions

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(Communicated by Cihan Özgür)

ABSTRACT

The purpose of this paper is to investigate ϕ -Ricci recurrent and ϕ -Ricci symmetric almost Kenmotsu manifolds with its characteristic vector field ξ belonging to some nullity distributions. Also we obtain several corollaries. Finally, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$.

Keywords: Almost Kenmotsu manifold; ϕ -Ricci recurrence; ϕ -Ricci symmetry; generalized nullity distribution; nullity distribution.

AMS Subject Classification (2010): Primary: 53C25 ; Secondary: 53D15.

1. Introduction

The notion of k -nullity distribution ($k \in \mathbb{R}$) was introduced by Gray [12] and Tanno [16] in the study of Riemannian manifolds (M, g) , which is defined for any $p \in M$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}, \quad (1.1)$$

for any $X, Y \in T_p M$, where $T_p M$ denotes the tangent vector space of M at any point $p \in M$ and R denotes the Riemannian curvature tensor of type (1, 3).

Recently, Blair, Koufogiorgos and Papantoniou [3] introduced the (k, μ) -nullity distribution which is a generalized notion of the k -nullity distribution on a contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu) = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY]\}, \quad (1.2)$$

where $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and \mathcal{L} denotes the Lie differentiation.

Next, Dileo and Pastore [10] introduced another generalized notion of the k -nullity distribution which is named the $(k, \mu)'$ -nullity distribution on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ and is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$N_p(k, \mu)' = \{Z \in T_p M^{2n+1} : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)h'X - g(X, Z)h'Y]\}, \quad (1.3)$$

where $h' = h \circ \phi$.

In [9], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. We refer the reader to ([9],[10],[11]) for more related results on $(k, \mu)'$ -nullity distribution and (k, μ) -nullity distribution on almost Kenmotsu manifolds. In recent papers ([17],[18],[19],[20],[21]) Wang and Liu study almost Kenmotsu manifolds with nullity distributions. In [19], Wang and Liu studied ϕ -recurrent almost Kenmotsu manifolds with the characteristic vector field ξ belonging to some nullity distributions.

On the other hand, Kenmotsu [13] introduced a special class of almost contact metric manifolds named Kenmotsu manifolds nowadays. The notion of locally ϕ -symmetry was introduced by Takahashi [15] in the study of Sasakian manifolds as a weaker version of local symmetric of such manifolds. De et al [6] introduced

a generalized version of local ϕ -symmetry, called ϕ -recurrence on Sasakian manifolds. In [5], De and Sarkar studied ϕ -Ricci symmetric Sasakian manifolds. For more related results we refer the reader to De [4] and De et al ([7],[8]). Motivated by the above studies in this paper we investigate ϕ -Ricci recurrent and ϕ -Ricci symmetric almost Kenmotsu manifolds.

The paper is organized as follows:

In section 2, we give some basic knowledge on almost Kenmotsu manifolds. A few well-known results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions are provided in section 3. In the next section, we study ϕ -Ricci recurrent almost Kenmotsu manifolds with ξ belonging to some nullity distributions. Also we obtain several corollaries. In the final section, we give an example of a 5-dimensional almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$.

2. Almost Kenmotsu manifolds

Let M be a $(2n + 1)$ -dimensional differentiable manifold admits a (ϕ, ξ, η) -structure or an almost contact structure, where ϕ is an $(1, 1)$ tensor field, ξ a characteristic vector field and η an 1-form such that ([1, 2])

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity endomorphism. It is customary to include also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (2.1).

If a manifold M with a (ϕ, ξ, η) -structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y of $T_p M^{2n+1}$, then M is said to have an almost contact metric structure (ϕ, ξ, η, g) . The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y of $T_p M^{2n+1}$. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1, 2)$ -type torsion tensor N_ϕ , defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y . It is well known [13] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f , defined by $f = ce^t$ for some positive constant c . Let \mathcal{D} be the distribution orthogonal to ξ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold \mathcal{D} is an integrable distribution as η is closed. Let the two tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on an almost Kenmotsu manifold M^{2n+1} . The tensor fields l and h are symmetric and satisfy the following relations [14]

$$h\xi = 0, \quad l\xi = 0, \quad tr(h) = 0, \quad tr(h\phi) = 0, \quad h\phi + \phi h = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{2.3}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.5}$$

for any vector fields X, Y . On the other hand, according to Takahashi [15] and De and Sarkar [5] we have the following definitions.

Definition 2.1. An almost Kenmotsu manifold is said to be ϕ -symmetric, if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for any vector fields $W, X, Y, Z \in T_p M$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ , then the manifold is called locally ϕ -symmetric manifold.

Definition 2.2. An almost Kenmotsu manifold is said to be ϕ -Ricci recurrent if it satisfies

$$\phi^2((\nabla_W Q)Y) = A(W)QY, \tag{2.6}$$

for any vector fields $W, Y \in T_p M$, where A is the 1-form on M^{2n+1} and Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. In addition, if the vector fields W, Y are orthogonal to ξ , then the manifold is called locally ϕ -Ricci recurrent manifold.

Definition 2.3. An almost Kenmotsu manifold is said to be ϕ -Ricci symmetric if it satisfies

$$\phi^2((\nabla_W Q)Y) = 0, \tag{2.7}$$

for any vector fields $W, Y \in T_p M$, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. In addition, if the vector fields W, Y are orthogonal to ξ , then the manifold is called locally ϕ -Ricci symmetric manifold.

3. Properties of the nullity conditions

In this section we provide some related results on almost Kenmotsu manifolds with ξ belonging to some nullity distributions. The $(1, 1)$ -type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2). \tag{3.1}$$

Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value λ . It follows from (3.1) that $\lambda^2 = -(k + 1)$, a constant. Therefore $k \leq -1$ and $\lambda = \pm\sqrt{-k - 1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces associated with h' corresponding to the non-zero eigen value λ and $-\lambda$ respectively. We have following lemmas.

Lemma 3.1. (Proposition 3.1 and Proposition 5.1 of [14]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold satisfying either the generalized (k, μ) -nullity condition or the generalized $(k, \mu)'$ -nullity condition (the term generalized means k, μ both are smooth functions), with $h \neq 0$. Then, one has

$$h'^2 = (k + 1)\phi^2 (\Leftrightarrow h^2 = (k + 1)\phi^2), \tag{3.2}$$

$$S(X, \xi) = 2nk\eta(X), \tag{3.3}$$

for any vector field X on M^{2n+1} . Furthermore, in the case of generalized (k, μ) -nullity condition, one has

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX] \tag{3.4}$$

and in the case of generalized $(k, \mu)'$ -nullity condition, one has

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X], \tag{3.5}$$

for any $X, Y \in T_p M$. In addition if $n > 1$ then one has

$$(\nabla_X h')Y = -g(h'X + h'^2 X, Y)\xi - \eta(Y)(h'X + h'^2 X) - (\mu + 2)\eta(X)h'Y, \tag{3.6}$$

for any $X, Y \in T_p M$.

Lemma 3.2. (Proposition 4.1 and Proposition 4.3 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and $\text{Spec}(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k - 1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following:

- (a) $K(X, \xi) = k - 2\lambda$ if $X \in [\lambda]'$ and $K(X, \xi) = k + 2\lambda$ if $X \in [-\lambda]'$,
- (b) $K(X, Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$; $K(X, Y) = k + 2\lambda$ if $X, Y \in [-\lambda]'$ and $K(X, Y) = -(k + 2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,
- (c) M^{2n+1} has constant negative scalar curvature $r = 2n(k - 2n)$.

Lemma 3.3. (Lemma 3 of [19]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the generalized $(k, \mu)'$ -nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by

$$Q = -2nid + 2n(k + 1)\eta \otimes \xi + [\mu - 2(n - 1)]h'. \tag{3.7}$$

Moreover, if both k and μ are constant, then we have

$$Q = -2nid + 2n(k + 1)\eta \otimes \xi - 2nh'. \tag{3.8}$$

In both cases, the scalar curvature of M^{2n+1} is $2n(k - 2n)$.

Lemma 3.4. (Proposition 4.2 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k, -2)'$ -nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\ R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\ R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k + 2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k + 2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\ R(X_\lambda, Y_\lambda)Z_\lambda &= (k - 2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k + 2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}]. \end{aligned}$$

Lemma 3.5. (Lemma 4.1 of [10]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belonging to the $(k, -2)'$ -nullity distribution. Then for any $X, Y \in T_pM$,

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{3.9}$$

Lemma 3.6. (Theorem 4.1 of [10]) Let M be an almost Kenmotsu manifold of dimension $2n + 1$. Suppose that the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. Then $k = -1$, $h = 0$ and M is locally a warped product of an open interval and an almost Kähler manifold.

4. ϕ -Ricci recurrent almost Kenmotsu manifolds

This section is devoted to study ϕ -Ricci recurrent almost Kenmotsu manifolds with some nullity distributions. At first we prove the following theorem.

Theorem 4.1. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ ($n > 1$) be a ϕ -Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Then the manifold M^{2n+1} is ϕ -Ricci symmetric and hence locally isometric to the Riemannian product of an $(n + 1)$ -dimensional manifold of constant sectional curvature -4 and a flat n -dimensional manifold.

Proof. We suppose that the manifold M^{2n+1} is a ϕ -Ricci recurrent almost Kenmotsu manifold. Taking the covariant differentiation along arbitrary vector field $Y \in T_pM$ of (3.8) we have

$$(\nabla_Y Q)X = 2n(k + 1)[(\nabla_Y \eta)X\xi + \eta(X)\nabla_Y \xi] - 2n(\nabla_Y h')X, \tag{4.1}$$

for any vector fields $X, Y \in T_pM$. Using (2.3) we obtain from the above equation

$$\begin{aligned} (\nabla_Y Q)X &= 2n(k + 1)[g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + Y\eta(X) \\ &\quad + g(X, h'Y)\xi + h'Y\eta(X)] - 2n(\nabla_Y h')X, \end{aligned} \tag{4.2}$$

for any vector fields $X, Y \in T_pM$. Applying ϕ^2 on both sides of (4.2) we obtain

$$\begin{aligned} \phi^2((\nabla_Y Q)X) &= 2n(k + 1)[- \eta(X)Y + \eta(X)\eta(Y)\xi - \eta(X)h'Y] \\ &\quad - 2n\phi^2((\nabla_Y h')X), \end{aligned} \tag{4.3}$$

for any vector fields $X, Y \in T_pM$. Making use of (3.9) we get from (4.3)

$$\begin{aligned} \phi^2((\nabla_Y Q)X) &= 2n(k + 1)[- \eta(X)Y + \eta(X)\eta(Y)\xi - \eta(X)h'Y] \\ &\quad + 2n\eta(X)[-h'Y + (k + 1)(Y - \eta(Y)\xi)], \end{aligned} \tag{4.4}$$

from which we have

$$\phi^2((\nabla_Y Q)X) = -2n(k + 2)\eta(X)h'Y, \tag{4.5}$$

for any vector fields $X, Y \in T_pM$. By virtue of equations (4.5) and (2.6) we obtain

$$-2n(k + 2)\eta(X)h'Y = A(Y)QX, \tag{4.6}$$

for any vector fields $X, Y \in T_pM$. Substituting $X = \xi$ in (4.6) yields

$$-(k + 2)h'Y = kA(Y)\xi, \tag{4.7}$$

for any vector field $Y \in T_pM$. Taking inner product of (4.7) with ξ we have

$$kA(Y) = 0, \tag{4.8}$$

for any vector field $Y \in T_pM$. Dileo and Pastore [10] proved that in an almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$ -nullity distribution with $h' \neq 0$ then $k < -1$. Hence (4.8) implies $A = 0$, that is, the manifold M^{2n+1} is ϕ -Ricci symmetric. Using the relation $A = 0$, it follows from (4.7) that $k = -2$. Noticing the fact $\lambda^2 = -(k + 1)$ and $k = -2$ we have $\lambda = \pm 1$. Without losing generality we may choose $\lambda = 1$. Then we can write from Lemma 3.4

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_\lambda &= -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= 0, \end{aligned}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.2 that $K(X, \xi) = -4$ for any $X \in [\lambda]'$ and $K(X, \xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.2 we see that $K(X, Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X, Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X, Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [10] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1 - \lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1} . Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1} . Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof. \square

Since ϕ -recurrent implies ϕ -Ricci recurrent, we have the following:

Corollary 4.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ -recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Then the manifold is locally isometric to the Riemannian product of an $(n + 1)$ -dimensional manifold of constant sectional curvature -4 and a flat n -dimensional manifold.*

The above corollary have been proved by Wang and Liu [19].

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be a ϕ -Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Then the following statements are equivalent:*

- (i) M^{2n+1} is ϕ -Ricci symmetric;
- (ii) k is a constant;
- (iii) ξ belongs to the $(k, \mu)'$ -nullity distribution.

Proof. Let us suppose M^{2n+1} be a ϕ -Ricci recurrent almost Kenmotsu manifold with ξ belonging to the generalized $(k, \mu)'$ -nullity distribution and $h' \neq 0$. Taking the covariant differentiation along arbitrary vector field $W \in T_pM$ of (3.7) we get

$$\begin{aligned} (\nabla_W Q)Y &= 2nW(k)\eta(Y)\xi + 2n(k + 1)[(\nabla_W \eta)Y\xi + \eta(Y)\nabla_W \xi] \\ &\quad + W(\mu)h'Y + [\mu - 2(n - 1)](\nabla_W h')Y, \end{aligned} \tag{4.9}$$

for any vector fields $W, Y \in T_pM$. Applying ϕ^2 on both sides of (4.9), we have

$$\begin{aligned} \phi^2((\nabla_W Q)Y) &= -W(\mu)h'Y + [\mu - 2(n - 1)]\phi^2((\nabla_W h')Y) \\ &\quad + 2n(k + 1)\eta(Y)\{-W + \eta(W)\xi - h'W\}, \end{aligned} \tag{4.10}$$

for any vector fields $W, Y \in T_pM$. In view of (4.10) and (2.6) we have

$$\begin{aligned} -W(\mu)h'Y + [\mu - 2(n - 1)]\phi^2((\nabla_W h')Y) \\ + 2n(k + 1)\eta(Y)\{-W + \eta(W)\xi - h'W\} = A(W)QY, \end{aligned} \tag{4.11}$$

for any vector fields $W, Y \in T_pM$. Making use of (3.6) and (4.11) we obtain

$$\begin{aligned} -W(\mu)h'Y + [\mu - 2(n - 1)][-\eta(Y)\{-h'W + (k + 1)(W - \eta(W)\xi)\}] \\ + (\mu + 2)\eta(W)h'Y - 2n(k + 1)[\eta(Y)W - \eta(Y)\eta(W)\xi + \eta(Y)h'W] \\ = A(W)QY, \end{aligned} \tag{4.12}$$

for any vector fields $W, Y \in T_pM$. Taking inner product with ξ of (4.12) yields

$$kA(W) = 0, \tag{4.13}$$

for any vector field $W \in T_pM$. From (3.2) we see that the smooth function k satisfies $k < -1$. Hence it follows from (4.13) that $A = 0$. Thus M^{2n+1} is ϕ -Ricci symmetric. Therefore (i) \Rightarrow (ii) is proved.

Taking into account $A = 0$ we have from (4.12)

$$\begin{aligned} & -W(\mu)h'Y + [\mu - 2(n - 1)][-\eta(Y)\{-h'W + (k + 1)(W - \eta(W)\xi)\}] \\ & + (\mu + 2)\eta(W)h'Y - 2n(k + 1)[\eta(Y)W - \eta(Y)\eta(W)\xi + \eta(Y)h'W] \\ & = 0, \end{aligned} \tag{4.14}$$

for any vector fields $W, Y \in T_pM$. Substituting $Y = \xi$ in (4.14) we obtain

$$[\mu - 2(n - 1)][h'W - (k + 1)(W - \eta(W)\xi)] - 2n(k + 1)[W - \eta(W)\xi + h'W] = 0, \tag{4.15}$$

for any vector field $W \in T_pM$. Now taking inner product of (4.15) with the vector field X we have

$$\begin{aligned} & [\mu - 2(n - 1)][g(h'W, X) - (k + 1)(g(W, X) - \eta(W)\eta(X))] \\ & - 2n(k + 1)[g(W, X) - \eta(W)\eta(X) + g(h'W, X)] = 0, \end{aligned} \tag{4.16}$$

for any vector fields $W, X \in T_pM$. Consider a local orthonormal basis $\{e_i : i = 1, 2, \dots, 2n + 1\}$ of tangent space at each point of the manifold M^{2n+1} . Setting $X = W = e_i$ in (4.16) and taking summation over $i : 1 \leq i \leq 2n + 1$, we get

$$(k + 1)(\mu + 2) = 0. \tag{4.17}$$

Again from (3.2) we see that the smooth function k satisfies $k < -1$. Therefore it follows from (4.17) that $\mu = -2$. Pastore and Saltarelli [14] proved that in an almost Kenmotsu manifold with generalized (k, μ) '-nullity distribution and $h' \neq 0$, then

$$\xi(k) = -2(k + 1)(\mu + 2) \tag{4.18}$$

holds. In view of $\mu = -2$ we have from (4.18), $k = \text{constant}$. Thus (ii) \Rightarrow (iii) is proved. Conversely, (iii) \Rightarrow (i) is proved in Theorem 4.1. This completes the proof. \square

Since ϕ -recurrent implies ϕ -Ricci recurrent, we have the following:

Corollary 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ ($n > 1$) be a ϕ -recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (k, μ) '-nullity distribution and $h' \neq 0$. Then the following statements are equivalent:*

- (i) M^{2n+1} is ϕ -symmetric;
- (ii) k is a constant;
- (iii) ξ belongs to the (k, μ) '-nullity distribution.

The above corollary have been proved by Wang and Liu [19].

Now we prove some special theorems on almost Kenmotsu manifolds with ξ belonging to the (k, μ) -nullity and generalized (k, μ) -nullity distributions.

Theorem 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ -Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. Then the manifold M^{2n+1} is an Einstein one.*

Proof. Let us suppose that the manifold M^{2n+1} is an ϕ -Ricci recurrent almost Kenmotsu manifold with ξ belonging to the (k, μ) -nullity distribution. In view of Lemma 3.6 and the equation (2.5) we get

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.19}$$

for any $X, Y \in T_pM$. Contracting Y in (4.19) we have

$$S(X, \xi) = -2n\eta(X), \tag{4.20}$$

for any $X, Y \in T_pM$. Thus, we have

$$Q\xi = -2n\xi. \tag{4.21}$$

Taking the covariant differentiation along arbitrary vector field $Z \in T_pM$ of (4.21) we get

$$(\nabla_Z Q)\xi = -2n(\nabla_Z \xi). \tag{4.22}$$

Using Lemma 3.6 the Equation (2.3) becomes

$$\nabla_Z \xi = Z - \eta(Z)\xi, \tag{4.23}$$

for any $Z \in T_pM$. From (4.22) and (4.23) we have

$$(\nabla_Z Q)\xi = -2n[Z - \eta(Z)\xi], \tag{4.24}$$

where Z be any vector field on M . Applying ϕ^2 on both sides of (4.24) we see that

$$\phi^2((\nabla_Z Q)\xi) = -2n[-Z + \eta(Z)\xi], \tag{4.25}$$

for any $Z \in T_pM$. In view of (4.25) and (2.6) we obtain

$$-2n[-Z + \eta(Z)\xi] = A(Z)Q\xi, \tag{4.26}$$

where $Z \in T_pM$. Taking the inner product of (4.26) with ξ yields $A(Z) = 0$, for any $Z \in T_pM$, thus we have $A = 0$ and consequently M^{2n+1} is ϕ -Ricci symmetric. Taking into account $A = 0$, we can write from (4.25) and (4.26) that $\phi^2((\nabla_Z Q)\xi) = 0$, for any $Z \in T_pM$, from which it follows that

$$-(\nabla_Z Q)\xi + \eta((\nabla_Z Q)\xi)\xi = 0. \tag{4.27}$$

Taking the inner product of (4.27) with the vector field U and making use of $g((\nabla_Z Q)\xi, U) = (\nabla_Z S)(\xi, U)$ we have

$$(\nabla_Z S)(\xi, U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0, \tag{4.28}$$

for any $U, Z \in T_pM$. Again using (4.24) we see from (4.28) that

$$(\nabla_Z S)(\xi, U) = 0, \tag{4.29}$$

which implies

$$\nabla_Z S(\xi, U) - S(\nabla_Z \xi, U) - S(\xi, \nabla_Z U) = 0, \tag{4.30}$$

for any $U, Z \in T_pM$. Making use of (4.20) and (4.23) into (4.30) we can write $S(U, Z) = -2ng(U, Z)$, for any $U, Z \in T_pM$. Therefore the manifold M^{2n+1} is an Einstein manifold. This completes the proof. \square

In [19], Wang and Liu proved that a ϕ -recurrent almost Kenmotsu manifold M^{2n+1} with the characteristic vector field ξ belonging to the (k, μ) -nullity distribution is of constant sectional curvature -1 , which implies that the manifold is an Einstein one of the form $S(U, Z) = -2ng(U, Z)$, for any $U, Z \in T_pM$. Also, ϕ -recurrent implies ϕ -Ricci recurrent. Thus, we have the following:

Corollary 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ -recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the (k, μ) -nullity distribution. Then the manifold M^{2n+1} is an Einstein one.*

Theorem 4.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a ϕ -Ricci recurrent almost Kenmotsu manifold with the characteristic vector field ξ belonging to the generalized (k, μ) -nullity distribution and $h \neq 0$. Then the manifold M^{2n+1} is an Einstein one.*

Proof. Suppose that an almost Kenmotsu manifold M^{2n+1} is ϕ -Ricci recurrent and ξ belongs to the generalized (k, μ) -nullity distribution with $h \neq 0$. Making use of (3.3) we get

$$Q\xi = 2nk\xi. \tag{4.31}$$

Taking the covariant differentiation along arbitrary vector field $Z \in T_pM$ of (4.31) we get

$$(\nabla_Z Q)\xi = 2nk(\nabla_Z \xi) + 2nZ(k)\xi. \tag{4.32}$$

Using (2.3) in (4.32) yields

$$(\nabla_Z Q)\xi = 2nk(Z - \eta(Z)\xi - \phi hZ) + 2nZ(k)\xi, \tag{4.33}$$

for any vector field $Z \in T_pM$. Applying ϕ^2 on both sides of (4.33) we have

$$\phi^2((\nabla_Z Q)\xi) = 2nk(-Z + \eta(Z)\xi + \phi hZ), \tag{4.34}$$

for any vector field $Z \in T_pM$. On the other hand, the assumption that M^{2n+1} is ϕ -Ricci recurrent implies $\phi^2((\nabla_Z Q)Y) = A(Z)QY$, for any $Y, Z \in T_pM$. Substituting $Y = \xi$ into this relation we obtain

$$\phi^2((\nabla_Z Q)\xi) = A(Z)Q\xi. \tag{4.35}$$

In view of (4.34) and (4.35) we get

$$2nk(-Z + \eta(Z)\xi + \phi hZ) = A(Z)Q\xi, \tag{4.36}$$

for any vector field $Z \in T_pM$. Taking inner product of (4.36) with ξ implies $kA(Z) = 0$ for any vector field $Z \in T_pM$. Also, from (3.2) we see that the smooth function k satisfies the condition $k < -1$. Therefore, $A = 0$ and consequently M^{2n+1} is ϕ -Ricci symmetric. Taking into account $A = 0$, we can write from (4.35) that $\phi^2((\nabla_Z Q)\xi) = 0$, for any $Z \in T_pM$, from which it follows that

$$-(\nabla_Z Q)\xi + \eta((\nabla_Z Q)\xi)\xi = 0. \tag{4.37}$$

Taking the inner product of (4.37) with the vector field U and using $g((\nabla_Z Q)\xi, U) = (\nabla_Z S)(\xi, U)$ we have

$$(\nabla_Z S)(\xi, U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0, \tag{4.38}$$

for any vector fields $U, Z \in T_pM$. This implies

$$\nabla_Z S(\xi, U) - S(\nabla_Z \xi, U) - S(\xi, \nabla_Z U) - \eta((\nabla_Z Q)\xi)\eta(U) = 0. \tag{4.39}$$

Applying (3.3), (2.3) and (4.33) into (4.39) we obtain

$$\begin{aligned} 2nZ(k)\eta(U) + 2nk\nabla_Z\eta(U) - S(Z, U) + 2nk\eta(Z)\eta(U) \\ + S(\phi hZ, U) - 2nk\eta(\nabla_Z U) - 2nZ(k)\eta(U) = 0, \end{aligned} \tag{4.40}$$

from which it follows that

$$S(U, Z) - S(\phi hZ, U) - 2nkg(U, Z) + 2nkg(\phi hZ, U) = 0, \tag{4.41}$$

for any vector fields $U, Z \in T_pM$. Now replacing Z by ϕhZ in (4.41) and noticing the fact $h^2 = (k + 1)\phi^2$ we have

$$(k + 1)S(U, Z) + S(\phi hZ, U) - 2nk(k + 1)g(U, Z) - 2nkg(\phi hZ, U) = 0, \tag{4.42}$$

for any vector fields $U, Z \in T_pM$. Adding (4.41) and (4.42) yields

$$(k + 2)[S(U, Z) - 2nkg(U, Z)] = 0, \tag{4.43}$$

for any vector fields $U, Z \in T_pM$. Clearly, it follows from (4.43) that either $k = -2$ or, $S(U, Z) = 2nkg(U, Z)$, for any vector fields $U, Z \in T_pM$. Now we prove that the former case can not occur. Indeed, if we assume that the former case is true, that is, $k = -2$, a constant, then $\xi(k) = 0$. Here we recall a result due to Pastore and Saltarelli [14]. They proved that in an almost Kenmotsu manifold with generalized (k, μ) -nullity distribution and $h \neq 0$, the relation $\xi(k) = -4(k + 1)$ holds. Therefore substituting $k = -2$ in this relation we have $\xi(k) = 4$. Thus, we have $\xi(k) = 0$ and $\xi(k) = 4$, which is absurd. Hence, we get the desired result. \square

5. Example of a 5-dimensional almost Kenmotsu manifold

In this section, we construct an example of an almost Kenmotsu manifold such that ξ belongs to the (k, μ) -nullity distribution. We consider 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in \mathbb{R}^5 . Let ξ, e_2, e_3, e_4, e_5 are five vector fields in \mathbb{R}^5 which satisfies [10]

$$\begin{aligned} [\xi, e_2] &= -2e_2, \quad [\xi, e_3] = -2e_3, \quad [\xi, e_4] = 0, \quad [\xi, e_5] = 0, \\ [e_i, e_j] &= 0, \quad \text{where } i, j = 2, 3, 4, 5. \end{aligned}$$

Let g be the Riemannian metric defined by

$$g(\xi, \xi) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$$

$$\text{and } g(\xi, e_i) = g(e_i, e_j) = 0 \text{ for } i \neq j; i, j = 2, 3, 4, 5.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \xi),$$

for any $Z \in \chi(M)$. Let ϕ be the (1, 1)-tensor field defined by

$$\phi(\xi) = 0, \phi(e_2) = e_4, \phi(e_3) = e_5, \phi(e_4) = -e_2, \phi(e_5) = -e_3.$$

Using the linearity of ϕ and g we have

$$\eta(\xi) = 1, \phi^2 Z = -Z + \eta(Z)\xi$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Moreover,

$$h'\xi = 0, h'e_2 = e_2, h'e_3 = e_3, h'e_4 = -e_4, h'e_5 = -e_5.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula we get the following:

$$\begin{aligned} \nabla_\xi \xi &= 0, \nabla_\xi e_2 = 0, \nabla_\xi e_3 = 0, \nabla_\xi e_4 = 0, \nabla_\xi e_5 = \xi, \\ \nabla_{e_2} \xi &= 2e_2, \nabla_{e_2} e_2 = -2\xi, \nabla_{e_2} e_3 = 0, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = 0, \\ \nabla_{e_3} \xi &= 2e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -2\xi, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_5 = 0, \\ \nabla_{e_4} \xi &= 0, \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = 0, \nabla_{e_4} e_4 = 0, \nabla_{e_4} e_5 = 0, \\ \nabla_{e_5} \xi &= 0, \nabla_{e_5} e_2 = 0, \nabla_{e_5} e_3 = 0, \nabla_{e_5} e_4 = 0, \nabla_{e_5} e_5 = 0. \end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h'X,$$

for any $X \in \chi(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.

By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$\begin{aligned} R(\xi, e_2)\xi &= 4e_2, R(\xi, e_2)e_2 = -4\xi, R(\xi, e_3)\xi = 4e_3, R(\xi, e_3)e_3 = -4\xi, \\ R(\xi, e_4)\xi &= R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0, \\ R(e_2, e_3)e_2 &= 4e_3, R(e_2, e_3)e_3 = -4e_2, R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0, \\ R(e_2, e_5)e_2 &= R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0, \\ R(e_3, e_5)e_3 &= R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0. \end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the (k, μ) '-nullity distribution, with $k = -2$ and $\mu = -2$.

References

- [1] Blair, D.E., Contact manifold in Riemannian geometry. Lecture Notes on Mathematics. Springer, Berlin, 509, 1976.
- [2] Blair, D.E., Riemannian geometry on contact and symplectic manifolds. *Progr. Math.* 203, Birkhäuser, 2010.
- [3] Blair, D.E., Koufogiorgos, T. and Papantoniou, B.J., Contact metric manifolds satisfying a nullity condition. *Israel J. Math.* 91 (1995), 189-214.
- [4] De, U.C., On ϕ -symmetric Kenmotsu manifolds. *Int. Electron. J. Geom.* 1 (2008), 33-38.
- [5] De, U.C., Sarkar, A., On ϕ -Ricci symmetric Sasakian manifolds. *Proc. of the Jangjeon Math. Soc.* 11 (2008), 47-52.

- [6] De, U.C., Shaikh, A.A. and Biswas, S., On ϕ -recurrent Sasakian manifolds. *Novi Sad J. Math.* 33 (2003), 43-48.
- [7] De, U.C., Yildiz, A. and Yaliniz, A.F., On ϕ -recurrent Kenmotsu manifolds. *Turkish J. Math.* 33 (2009), 17-25.
- [8] De, U.C., Yildiz, A. and Yaliniz, A.F., Locally ϕ -symmetric normal almost contact metric manifolds of dimension 3. *Applied Math. Letters.* 22 (2009), 723-727.
- [9] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds and local symmetry. *Bull. Belg. Math. Soc. Simon Stevin.* 14 (2007), 343-354.
- [10] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds and nullity distributions. *J. Geom.* 93 (2009), 46-61.
- [11] Dileo, G. and Pastore, A.M., Almost Kenmotsu manifolds with a condition of η -parallelism. *Differential Geom. Appl.* 27 (2009), 671-679.
- [12] Gray, A., Spaces of constancy of curvature operators. *Proc. Amer. Math. Soc.* 17 (1966), 897-902.
- [13] Kenmotsu, K., A class of almost contact Riemannian manifolds. *Tohoku Math. J.* 24 (1972), 93-103.
- [14] Pastore, A.M. and Saltarelli, V., Generalized nullity distributions on almost Kenmotsu manifolds. *Int. Electron. J. Geom.* 4 (2011), 168-183.
- [15] Takahashi, T., Sasakian ϕ -symmetric spaces. *Tohoku Math. J.* 29 (1977), 91-113.
- [16] Tanno, S., Some differential equations on Riemannian manifolds. *J. Math. Soc. Japan.* 30 (1978), 509-531.
- [17] Wang, Y. and Liu, X., Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions. *Filomat.* 28 (2014), 839-847.
- [18] Wang, Y. and Liu, X., Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions. *Ann. Polon. Math.* 112 (2014), 37-46.
- [19] Wang, Y. and Liu, X., On ϕ -recurrent almost Kenmotsu manifolds. *Kuwait J. Sci.* 42 (2015), 65-77.
- [20] Wang, Y. and Liu, X., On a type of almost Kenmotsu manifolds with harmonic curvature tensors. *Bull. Belg. Math. Soc. Simon Stevin.* 22 (2015), 15-24.
- [21] Wang, Y. and Liu, X., On almost Kenmotsu manifolds satisfying some nullity distributions. *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.* 86 (2016), 347-353.

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