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Lambert Series in the Summation of Reciprocals in Gaussian Fibonacci Sequences

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Abstract. In this paper, we consider infinite sums derived from the reciprocals of the Gaussian Fibonacci numbers. New expressions of these sums are obtained in terms of Lambert series.

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1. Introduction

Classical Fibonacci numbers have been often use in different sciences such as biology, physics and economy. In the literature, there have been many studies on Fibonacci numbers and generalizations, see [1–4, 6, 7, 9, 12]. In [8], Horadam has defined complex Fibonacci numbers and the generalization of the classical Fibonacci numbers to complex numbers, which has given rise to new problems. Also, very interesting relationships and similarities between Fibonacci numbers and Gaussian Fibonacci numbers are given in [11].

For any integer $n \geq 2$, the Gaussian Fibonacci numbers GF_n are defined by

$$GF_0 = i,\ GF_1 = 1;\ GF_n = GF_{n-1} + GF_{n-2}\,. \tag{1.1}$$

The n^{th} Gaussian Fibonacci number is given by the equality

$$GF_n = F_n + iF_{n-1},$$

where i is the imaginary unit which satisfies $i^2 = -1$.

There exist a Binet formula for the Gaussian Fibonacci numbers,

$$GF_n = \gamma\alpha^n + \lambda\beta^n \tag{1.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}, \gamma = \frac{2\sqrt{5}+(5+\sqrt{5})i}{10}$ and $\lambda = \frac{-2\sqrt{5}+(5-\sqrt{5})i}{10}$.

Here, α and β satisfy the following equations

$$\alpha + \beta = 1, \alpha - \beta = \sqrt{5}, \alpha\beta = -1. \tag{1.3}$$

Recent studies show that there has been an increasing interest on reciprocal sums of the Fibonacci numbers. In [5], Elsner et al. investigated the algebraic relations for reciprocal sums of the Fibonacci numbers. Also, Ohtsuka and

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Nakamura obtained the partial infinite sums of the reciprocals of Fibonacci numbers [13]. Indeed, the floor function has been used in all of the above studies. However, Horadam obtained infinite sums of the reciprocals of Fibonacci numbers and some generalizations by the help of Lambert series in [10].

Now, we recall the definition of Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}.$$

More particularly, we speak of the Lambert series and generalized Lambert series respectively,

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \quad \text{and} \quad |x| < 1,$$

$$L(a,x) = \sum_{n=1}^{\infty} \frac{ax^n}{1-ax^n} \quad \text{and} \quad |x| < 1, |ax| < 1.$$

The purpose of this paper is to express the infinite sums

$$\sum_{n=1}^{\infty} \frac{1}{GF_n}, \quad \sum_{n=1}^{\infty} \frac{1}{GF_{2n}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{GF_{2n-1}}$$

in terms of Lambert series.

2. Infinite Sums of the Reciprocals of Fibonacci Numbers

In this section, we first obtain the reciprocal of Gaussian Fibonacci numbers with the help of the Binet formula. Then, we express the infinite sums of these reciprocals in terms of the Lambert series.

Theorem 2.1. *i. If n is even, then*

$$\sum_{n=1}^{\infty} \frac{1}{GF_n} = \frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}} \left\{ L\left(i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}, \beta\right) - L\left(-\frac{\lambda}{\gamma}, \beta^2\right) \right\},$$

ii. If n is odd, then

$$\sum_{n=1}^{\infty} \frac{1}{GF_n} = \frac{-\sqrt{\gamma}}{\gamma\sqrt{\lambda}} \left\{ L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}}, \beta\right) - L\left(\frac{\lambda}{\gamma}, \beta^2\right) \right\}.$$

Proof. Firstly, we let n be even. We consider

$$\begin{aligned} \frac{1}{GF_n} &= \frac{1}{\gamma \cdot \alpha^n + \lambda \cdot \beta^n} \\ &= \frac{1}{\gamma\left(\alpha^n + \frac{\lambda}{\gamma} \cdot \beta^n\right)} \\ &= \frac{1}{\gamma} \cdot \frac{\beta^n}{(\alpha\beta)^n + \frac{\lambda}{\gamma} \cdot \beta^{2n}}. \end{aligned}$$

Since n is even, then $(\alpha\beta)^n = (-1)^n = 1$ and therefore,

$$\begin{aligned}\frac{1}{GF_n} &= \frac{1}{\gamma}\cdot\frac{\beta^n}{1+\frac{\lambda}{\gamma}\cdot\beta^{2n}}\\ &= \frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\frac{i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}{1-i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}-\frac{\frac{-\lambda}{\gamma}\cdot\beta^{2n}}{1+\frac{\lambda}{\gamma}\cdot\beta^{2n}}\right).\end{aligned}$$

Thus, we have

$$\sum_{n=1}^{\infty}\frac{1}{GF_n}=\frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\sum_{n=1}^{\infty}\left(\frac{i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}{1-i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}\right)-\sum_{n=1}^{\infty}\left(\frac{\frac{-\lambda}{\gamma}\cdot\beta^{2n}}{1+\frac{\lambda}{\gamma}\cdot\beta^{2n}}\right)\right).$$

By the help of Lambert series, we get

$$\sum_{n=1}^{\infty}\frac{1}{GF_n}=\frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(i\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta\right)-L\left(-\frac{\lambda}{\gamma},\beta^2\right)\right\},$$

where $\left|i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta\right|<1$, and $\left|-\frac{\lambda}{\gamma}\cdot\beta^2\right|<1$.
Secondly, we assume that n is odd. Then, $(\alpha\beta)^n=(-1)^n=-1$. Now, we get

$$\begin{aligned}\frac{1}{GF_n} &= -\frac{1}{\gamma}\cdot\frac{\beta^n}{1-\frac{\lambda}{\gamma}\cdot\beta^{2n}}\\ &= \frac{-\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\frac{\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}{1-\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}-\frac{\frac{\lambda}{\gamma}\cdot\beta^{2n}}{1-\frac{\lambda}{\gamma}\cdot\beta^{2n}}\right).\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{GF_n}=\frac{-\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\sum_{n=1}^{\infty}\left(\frac{\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}{1-\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^n}\right)-\sum_{n=1}^{\infty}\left(\frac{\frac{\lambda}{\gamma}\cdot\beta^{2n}}{1-\frac{\lambda}{\gamma}\cdot\beta^{2n}}\right)\right)$$

and we have

$$\sum_{n=1}^{\infty}\frac{1}{GF_n}=\frac{-\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta\right)-L\left(\frac{\lambda}{\gamma},\beta^2\right)\right\},$$

where $\left|\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta\right|<1$, and $\left|\frac{\lambda}{\gamma}\cdot\beta^2\right|<1$. □

Theorem 2.2. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{1}{GF_{2n}}=\frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta^2\right)-L\left(-\frac{\lambda}{\gamma},\beta^4\right)\right\}.$$

Proof. In the Eqn. (1.2), we consider $2n$ instead of n, then

$$\begin{aligned}\frac{1}{GF_{2n}} &= \frac{1}{\gamma\cdot\alpha^{2n}+\lambda\cdot\beta^{2n}}\\ &= \frac{1}{\gamma\left(\alpha^{2n}+\frac{\lambda}{\gamma}\cdot\beta^{2n}\right)}\\ &= \frac{1}{\gamma}\cdot\frac{\beta^{2n}}{(\alpha\beta)^{2n}+\frac{\lambda}{\gamma}\cdot\beta^{4n}}.\end{aligned}$$

Since $(\alpha\beta)^{2n}=1$, we have

$$\begin{aligned}\frac{1}{GF_{2n}} &= \frac{1}{\gamma}\cdot\frac{\beta^{2n}}{1+\frac{\lambda}{\gamma}\cdot\beta^{4n}}\\ &= \frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\frac{i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^{2n}}{1-i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^{2n}}-\frac{\frac{-\lambda}{\gamma}\cdot\beta^{4n}}{1+\frac{\lambda}{\gamma}\cdot\beta^{4n}}\right).\end{aligned}$$

Therefore, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{GF_{2n}}=\frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left(\sum_{n=1}^{\infty}\left(\frac{i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^{2n}}{1-i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^{2n}}\right)-\sum_{n=1}^{\infty}\left(\frac{\frac{-\lambda}{\gamma}\cdot\beta^{4n}}{1+\frac{\lambda}{\gamma}\cdot\beta^{4n}}\right)\right),$$

which can be written in terms of Lambert series as:

$$\sum_{n=1}^{\infty}\frac{1}{GF_{2n}}=\frac{i\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta^2\right)-L\left(-\frac{\lambda}{\gamma},\beta^4\right)\right\},$$

where $\left|i\frac{\sqrt{\lambda}}{\sqrt{\gamma}}\cdot\beta^2\right|<1$, and $\left|-\frac{\lambda}{\gamma}\cdot\beta^4\right|<1$.

□

Lemma 2.3. *We assume α,β,λ and γ are defined as in (1.2). Then,*

$$\sum_{n=1}^{\infty}\frac{\beta^{2n-1}}{1-\frac{\lambda}{\gamma}\cdot\beta^{4n-2}}=\sum_{n=1}^{\infty}\frac{\beta^{n}}{1-\frac{\lambda}{\gamma}\cdot\beta^{2n}}-\sum_{n=1}^{\infty}\frac{\beta^{2n}}{1-\frac{\lambda}{\gamma}\cdot\beta^{4n}}.$$

Theorem 2.4. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{1}{GF_{2n-1}}=-\frac{\sqrt{\gamma}}{\gamma\sqrt{\lambda}}\left\{L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta\right)+L\left(\frac{\lambda}{\gamma},\beta^4\right)-L\left(\frac{\lambda}{\gamma},\beta^2\right)-L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}},\beta^2\right)\right\}.$$

Proof. In (1.2), we consider $2n-1$ instead of n, then

$$\begin{aligned}\frac{1}{GF_{2n-1}} &= \frac{1}{\gamma\cdot\alpha^{2n-1}+\lambda\cdot\beta^{2n-1}}\\ &= \frac{1}{\gamma\left(\alpha^{2n}+\frac{\lambda}{\gamma}\cdot\beta^{2n}\right)}\\ &= \frac{1}{\gamma}\cdot\frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1}+\frac{\lambda}{\gamma}\cdot\beta^{4n-2}}.\end{aligned}$$

Since $(\alpha\beta)^{2n-1} = -1$, we have

$$\frac{1}{GF_{2n-1}} = -\frac{1}{\gamma} \cdot \frac{\beta^{2n-1}}{1-\frac{\lambda}{\gamma}\cdot\beta^{4n-2}}.$$

Therefore, we get

$$\sum_{n=1}^{\infty} \frac{1}{GF_{2n-1}} = -\frac{1}{\gamma} \cdot \sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{1 - \frac{\lambda}{\gamma} \cdot \beta^{4n-2}}$$

and from the previous Lemma, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{GF_{2n-1}} &= -\frac{1}{\gamma} \cdot \left(\sum_{n=1}^{\infty} \frac{\beta^n}{1 - \frac{\lambda}{\gamma} \cdot \beta^{2n}} - \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - \frac{\lambda}{\gamma} \cdot \beta^{4n}} \right) \\ &= -\frac{\sqrt{\gamma}}{\gamma\sqrt{\lambda}} \left(\sum_{n=1}^{\infty} \frac{\frac{\sqrt{\lambda}}{\sqrt{\gamma}} \cdot \beta^n}{1 - \frac{\sqrt{\lambda}}{\sqrt{\gamma}} \cdot \beta^n} - \sum_{n=1}^{\infty} \frac{\frac{\lambda}{\gamma} \cdot \beta^{2n}}{1 - \frac{\lambda}{\gamma} \cdot \beta^{2n}} - \sum_{n=1}^{\infty} \frac{\frac{\sqrt{\lambda}}{\sqrt{\gamma}} \cdot \beta^{2n}}{1 - \frac{\sqrt{\lambda}}{\sqrt{\gamma}} \cdot \beta^{2n}} + \sum_{n=1}^{\infty} \frac{\frac{\lambda}{\gamma} \cdot \beta^{4n}}{1 - \frac{\lambda}{\gamma} \cdot \beta^{4n}} \right).\end{aligned}$$

By the help of Lambert series, we have

$$\sum_{n=1}^{\infty} \frac{1}{GF_{2n-1}} = -\frac{\sqrt{\gamma}}{\gamma\sqrt{\lambda}} \left\{ L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}}, \beta\right) + L\left(\frac{\lambda}{\gamma}, \beta^4\right) - L\left(\frac{\lambda}{\gamma}, \beta^2\right) - L\left(\frac{\sqrt{\lambda}}{\sqrt{\gamma}}, \beta^2\right) \right\}.$$

□

3. Conclusion

In this work, we studied the reciprocal of Gaussian Fibonacci numbers with the help of the Binet formula. Then, infinite sums of these reciprocals have been expressed in terms of the Lambert series. Our results can be applied to Gaussian Lucas numbers and Gaussian Pell numbers by using a similar method.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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