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SOME REMARKS ON THE ORDER SUPERGRAPH OF THE POWER GRAPH OF A FINITE GROUP

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Abstract. Let G be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$. In an earlier paper, the main properties of this graph was obtained. The aim of this paper is to investigate the Hamiltonianity, Eulerianness and 2-connectedness of this graph.

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1. Introduction

Let G be a finite group and $x \in G$. The order of x is denoted by $o(x)$ and the least common multiple of all element orders in G is the exponent of G which is denoted by $Exp(G)$. If there is an element $a \in G$ such that $o(a) = Exp(G)$, then G is called full exponent. The set of all element orders of G is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. Set $\Xi_i(G)$ to be the set of all elements of G of order i and $\Omega_i(G) = |\Xi_i(G)|$. Also $nse(G) = \{\Omega_i(G) | i \in \pi_e(G)\}$. An $EPPO$-group is a group that all elements have prime power order and an EPO-group is a group with elements of prime order.

Throughout this paper graph means simple graph. Suppose Γ is such a graph. The number of vertices adjacent to x is the degree of x and is denoted by $deg_\Gamma(x)$. If the graph Γ can not be disconnected by removing less than k vertices, then Γ is called k-connected. It is clear that every Hamiltonian graph is 2-connected. A set of all vertices in Γ such that no two of which are adjacent is an independent set for Γ. The independent number of Γ, $\alpha(\Gamma)$, is the cardinality of an independent set with maximum size. A set S of vertices of a graph Γ is a vertex cover for Γ, if every edge of Γ has at least one vertex in S as an endpoint. The vertex cover

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number, $\beta(\Gamma)$, is the size of a minimum vertex cover of graph. In the graph Γ with n vertices always we have $\alpha(\Gamma) + \beta(\Gamma) = n$.

The directed power graph of a group G is a graph with vertex set G and there is a directed edge connecting x to y if and only if y is a power of x. This directed graph was introduced in the seminal paper of Kelarev and Quinn in 1999 [13]. In the mentioned paper, the authors considered the directed power graph of groups and gave a complete description of the structure of this graph for a finite abelian group. The same authors [15], extended their results to all semigroups. We refer to [14,16], for some properties of the directed power graph of semigroups.

Suppose A is a simple graph and $\mathcal{G} = \{\Gamma_a\}_{a\in A}$ is a set of graphs labeled by vertices of A. Following Sabidussi [20, p. 396], the A-join of $\mathcal{G}$ is the graph Δ with the following vertex and edge sets:

$$\begin{aligned} V(\Delta) &= \{(x,y) \mid x \in V(A) \ \&\ y \in V(\Gamma_x)\}, \\ E(\Delta) &= \{(x,y)(x',y') \mid xx' \in E(A) \ or\ else\ x = x' \ \&\ yy' \in E(\Gamma_x)\}. \end{aligned}$$

If A is an p-vertex labeled graph then the A-join of $\Delta_1, \Delta_2, \ldots, \Delta_p$ is denoted by $A[\Delta_1, \Delta_2, \ldots, \Delta_p]$.

The undirected power graph of a finite group G, $\mathcal{P}(G)$, was introduced by Chakrabarty et al. [4]. This graph has G as its vertex set and two vertices x and y are adjacent if and only if one is a power of the other. The main properties of this graph were investigated by Cameron [2] and Cameron and Ghosh [3]. Define the graph $\mathcal{S}(G)$ with vertex set G such that two vertices x and y are adjacent if and only if $o(x)|o(y)$ or $o(y)|o(x)$. This graph is called the main supergraph of $\mathcal{P}(G)$. Some basic properties of this graph are studied in [11]. In [9], the automorphism group of this graph computed in general and in [10] its eigenvalues and Laplacian eigenvalues were computed. Set $\pi_e(G) = \{a_1, \ldots, a_k\}$ and define the graph Δ_G with vertex set $\pi_e(G)$ and edge set $E(\Delta_G) = \{xy \mid x, y \in \pi_e(G), x|y \ or\ y|x\}$. In [8,9], the authors proved that $\mathcal{S}(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}]$, where K_n denotes the complete graph on n vertices.

The proper power graph $\mathcal{P}^*(G)$ and its proper main supergraph $\mathcal{S}^*(G)$ are defined as graphs constructed from $\mathcal{P}(G)$ and $\mathcal{S}(G)$ by removing identity element of G, respectively.

Suppose G is a finite group, $X \subseteq G$ and $C \subseteq G - \{1\}$. Following Williams [25], the prime graph $\Lambda(G)$ is a simple graph that vertices are primes dividing the order of the group. Two vertices p and q are adjacent if and only if G contains an element of order pq. The commuting graph $C(G, X)$ is a simple graph with vertex set X, and two vertices $x, y \in X$ are adjacent, whenever $xy = yx$. In this paper, we will assume that $X = G - \{1\}$ and the corresponding commuting graph is denoted by $\Delta(G)$. The directed Cayley graph $\overrightarrow{X(G,C)}$ is a graph with vertex set G and edge set $\{(g,h)|g^{-1}h \in C \cup C^{-1}\}$. It is well-known that Cayley graphs are regular and vertex-transitive.

Suppose Γ_1 and Γ_2 are two graphs. The Cartesian product Γ_1 and Γ_2, $\Gamma_1 \Box \Gamma_2$, is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that two vertices (a,b) and (x,y) are adjacent in $\Gamma_1 \Box \Gamma_2$ if $a = x$ and $by \in E(H)$ or $b = y$ and $ax \in E(G)$. The tensor product $\Gamma_1 \times \Gamma_2$ of graphs Γ_1 and Γ_2 is a graph with the same vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two vertices (a,b) and (x,y) are adjacent in $\Gamma_1 \times \Gamma_2$ if and only if $by \in E(H)$ and $ax \in E(G)$.

Let Γ be a graph and $M \subseteq V(\Gamma)$. M is called a module if for any $x \notin M$, $M \subseteq N(x)$ or $M \cap N(x) = \emptyset$. The trivial modules are empty set, singletons and the whole set V. A graph in which all modules are trivial is said to be primitive. A strong module is a module M such that for any other module M', either $M \cap M' = \emptyset$ or $M \subseteq M'$ or $M' \subseteq M$. We now assume that M and M' two disjoint modules. If any vertex of M is adjacent to all vertices of M', then we say M and M' are adjacent, and if there is no an edge such that one of its end points is belong to M and another in M' then we say M and M' are non-adjacent.

For a module M, if $M \subset S$ and there is no module M' such that $M \subset M' \subset S$, then the module M is maximal with respect to a set S of vertices. We shall assume $S = V$, if the set S is not specified. Let for $1 \leq i \leq k$, M_i be a module of graph Γ and $P = \{M_1, \ldots, M_k\}$ be a partition of the vertex set of a graph, then P is a modular partition of Γ. A non-trivial modular partition $P = \{M_1, \ldots, M_k\}$ which only contains maximal strong modules is a maximal modular partition. Notice that each graph has a unique maximal modular partition. Quotient graph whose vertices are modules belonging to the modular partition P of graph Γ is denoted by Γ/P. In this graph, two vertices of Γ/P are adjacent if and only if the corresponding modules are adjacent in Γ [7].

Theorem 1.1. (*Modular Decomposition Theorem*)[5,6] *For any graph Γ, one of the following three conditions is satisfied:*

- Γ *is not connected.*
- $\overline{\Gamma}$ *is not connected.*
- Γ *and* $\overline{\Gamma}$ *are connected and the quotient graph* Γ/P*, with* P *the maximal modular partition of* Γ*, is a primitive graph.*

Throughout this paper we refer to [19] for group theory concepts and for graph theoretical concepts and notations, we refer to [24]. For the sake of completeness, in what follows we mention the presentation of the dihedral group D_{2n}, the semi-dihedral group SD_{8n}, the dicyclic group T_{4n} and the group V_{8n}.

$$\begin{aligned} D_{2n} &= \langle a,b \mid a^n = b^2 = e,\ bab = a^{-1}\rangle, \\ SD_{8n} &= \langle a,b \mid a^{4n} = b^2 = e,\ bab = a^{2n-1}\rangle, \\ T_{4n} &= \langle a,b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle, \\ V_{8n} &= \langle a,b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle. \end{aligned}$$

It is easy to see the dicyclic group T_{4n} has order $4n$ and the groups SD_{8n} and V_{8n} have order $8n$.

2. Main results

A vertex in a graph Γ is said to be even, if its degree is an even integer. There is a condition for Eulerian of a graph Γ which states that Γ is Eulerian if and only if all of its degrees are even.

Theorem 2.1. *Let G be a finite group. The graph $\mathcal{S}(G)$ is Eulerian if and only if G is an odd order group.*

Proof. Suppose G is a group of order n. Then the degree of identity has to be $n-1$ and so n is odd. Conversely, suppose n is odd and $\pi_e(G) = \{a_1, \ldots, a_k\}$. Choose the non-identity vertex x in $\mathcal{S}(G)$ and assume that $o(x) = a_i$. Then

$$deg_{\mathcal{S}(G)}(x) = \Omega_{a_i}(G) + \sum_{a_i|a_j \neq e\ or\ (a_j|a_i\ \&\ a_i \neq a_j)} \Omega_{a_j}(G).$$

If k_i, $1 \leq i \leq k$, denotes the number of cyclic subgroups of order a_i then $\Omega_{a_i}(G) = k\phi(a_i)$, that ϕ is the Euler's totient function. Since G has odd order, it does not have involutions and $\phi(m)$, $m \geq 3$, is even. Thus for each a_i, $a_i \in \pi_e(G)$, $\Omega_{a_i}(G)$ is even. Therefore, degree of every vertex in $\mathcal{S}(G)$ is even and $\mathcal{S}(G)$ is Eulerian. $\square$

In the next theorem, the relationship between connectedness of $\mathcal{S}^*(G)$ and $\Lambda(G)$ is studied.

Theorem 2.2. ([11]) *If the prime graph of a group G is disconnected then $\mathcal{S}^*(G)$ is disconnected. In particular, $\mathcal{S}(G)$ is not Hamiltonian.*

Theorem 2.3. *Let G be a finite group. If Δ_G is Hamiltonian then $\mathcal{S}(G)$ will be Hamiltonian.*

Proof. Suppose Δ_G is Hamiltonian and $T : e \sim a_1 \sim \ldots \sim a_k \sim e$ is a Hamiltonian cycle in Δ_G. Set $\Xi(G) = \{x_{i1}, x_{i2}, \ldots, x_{i\Omega_{a_i}(G)}\}$. We construct a Hamiltonian cycle T' in $\mathcal{S}(G)$ as follows:

$$\begin{aligned} T' : \quad & e \sim x_{11} \sim \ldots \sim x_{1\Omega_{a_1}(G)} \sim x_{21} \sim \ldots \sim x_{2\Omega_{a_2}(G)} \sim \ldots \sim \\ & x_{k1} \sim \ldots \sim x_{k\Omega_{a_k}(G)} \sim e, \end{aligned}$$

and so $\mathcal{S}(G)$ is Hamiltonian, as desired. □

Corollary 2.4. *The main supergraph of the power graph of the following simple groups are not Hamiltonian:*

(1) ${}^2F_4(q)$, *where* $q = 2^{2m+1}$ *and* $m \geq 1$*;*
(2) ${}^2G_2(q)$, *where* $q = 3^{2m+1}$ *and* $m \geq 0$*;*
(3) $A_1(q)$, $A_2(q)$, $B_2(q)$, $C_2(q)$ *and* $S_4(q)$, *where* q *is an odd prime power;*
(4) $F_4(2^m)$, $m \geq 1$ *and* $U_3(q)$, *where* q *is a prime power.*

Proof. Apply Theorems 2.34 and 2.35 from [11]. □

It is easy to see that the main supergraph of the power graph of the cyclic group of order p, p is prime, is Hamiltonian. This simple result and Corollary 2.4 suggest the following conjecture:

Conjecture 2.5. *The main supergraph of the power graph of a non-abelian finite simple group is not Hamiltonian.*

Theorem 2.6. *If* G *is full exponent then* $\mathcal{S}(G)$ *is* 2*-connected.*

Proof. Suppose x is an element of order $Exp(G)$. Then e and x are adjacent to all elements of the group. This proves that $\mathcal{S}(G)$ is 2-connected. □

Theorem 2.7. *If* G *is an abelian group, then* $\mathcal{S}(G)$ *is* 2*-connected.*

Proof. To prove the theorem, it is enough to show that $\mathcal{S}^*(G)$ is connected. Choose non-identity elements $x, y \in \mathcal{S}^*(G)$. Since G is abelian, $xy = yx$. If x and y are adjacent in $\mathcal{S}(G)$, then are adjacent in $\mathcal{S}^*(G)$ too. This implies that $o(x) \nmid o(y)$ and $o(y) \nmid o(x)$. Our main proof will consider the following two cases:

(1) $o(x)$ and $o(y)$ are coprime. Since $o(xy) = o(x)o(y)$, $o(x) \mid o(xy)$ and $o(y) \mid o(xy)$. Thus $x \sim xy \sim y$ is a path in $\mathcal{S}^*(G)$ and so x, y are vertices of a connected component of $\mathcal{S}^*(G)$.
(2) $n = (o(x), o(y)) \neq 1$. Without loss of generality, we can assume that $o(x) > o(y)$. Since $o(x) \equiv n \ (mod\ o(y))$, $y^n = (xy)^{o(x)}$. On the other hand,

$x^{o(y)} = (xy)^{o(y)}$ and so $y \sim y^n \sim (xy)^{o(x)} \sim xy \sim (xy)^{o(y)} \sim x^{o(y)} \sim x$ is a path in $\mathcal{S}^*(G)$. Hence x and y are in a connected component of $\mathcal{S}^*(G)$.

This proves that $\mathcal{S}^*(G)$ is connected. □

Lemma 2.8. *Let G and H be groups such that $(|G|, |H|) = 1$. Then $\mathcal{S}(G \times H) = \mathcal{S}(G) \times \mathcal{S}(H)$.*

Proof. Suppose (x, y) and (a, b) are adjacent vertices in $\mathcal{S}(G \times H)$. Then $o((x, y)) \mid o((a, b))$ or $o((a, b)) \mid o((x, y))$. Since G and H have coprime order, $o(x)o(y) \mid o(a)o(b)$ or $o(a)o(b) \mid o(x)o(y)$. On the other hand, $(o(a), o(y)) = (o(b), o(x)) = 1$. Hence $o(a)o(b) \mid o(x)o(y)$ implies that $o(a) \mid o(x)$ and $o(b) \mid o(y)$. Similarly, $o(x)o(y) \mid o(a)o(b)$ implies that $o(x) \mid o(a)$ and $o(y) \mid o(b)$. Therefor, a, x are adjacent in $\mathcal{S}(G)$, and b, y are adjacent in $\mathcal{S}(H)$ which proves that (a, b) and (x, y) are adjacent in $\mathcal{S}(G) \times \mathcal{S}(H)$. A similar argument as above shows that if (a, b) and (x, y) are adjacent in $\mathcal{S}(G) \times \mathcal{S}(H)$, then $ax \in E(\mathcal{S}(G))$ and $by \in E(\mathcal{S}(H))$. □

The proof of the previous lemma shows that in general $\mathcal{S}(G) \times \mathcal{S}(H)$ is a subgraph of $\mathcal{S}(G \times H)$. By [12, Theorem 5.29], if G and H are non-empty graphs, then $G \times H$ is connected if and only if both of G and H are connected and at least one of them are non-bipartite. Moreover, if G and H are connected and bipartite, then $G \times H$ has exactly two connected components. In the following theorem, we apply this result to prove that the main supergraph of the power graph of a nilpotent group is 2-connected.

Theorem 2.9. *If G is nilpotent, then $\mathcal{S}(G)$ is 2-connected.*

Proof. Since G is nilpotent, $G \cong P_1 \times \ldots \times P_r$, where P_i's are all Sylow P_i-subgroups of G. By Lemma 2.8, $\mathcal{S}(G) \cong \mathcal{S}(P_1 \times \ldots \times P_r) = \mathcal{S}(P_1) \times \ldots \times \mathcal{S}(P_r)$ and so $\mathcal{S}^*(G) \cong \mathcal{S}^*(P_1 \times \ldots \times P_r) = \mathcal{S}^*(P_1) \times \ldots \times \mathcal{S}^*(P_r)$. Since $\mathcal{S}^*(P_i)$, $1 \leq i \leq r$, are complete, they are non-bipartite and connected. This shows that $\mathcal{S}^*(G)$ is connected, as desired. □

Theorem 2.10. *Let G be a finite group. If $xy \in E(\Delta(G))$ then x and y are in the same component of $\mathcal{S}^*(G)$.*

Proof. By definition, $V(\Delta(G)) = V(\mathcal{S}^*(G))$. Suppose, x, y are adjacent vertices of $\Delta(G)$. So $xy = yx$. If $o(x) \mid o(y)$ or $o(y) \mid o(x)$ then x and y are adjacent in $\mathcal{S}^*(G)$. We now assume that $o(x) \nmid o(y)$ and $o(y) \nmid o(x)$. We consider two cases that $(o(x), o(y)) = 1$ or $(o(x), o(y)) \neq 1$.

(1) $(o(x), o(y)) = 1$. In this case, $o(xy) = o(x)o(y)$. This gives a path $x \sim xy \sim y$ in $\mathcal{S}^*(G)$, as desired.

(2) $(o(x), o(y)) \neq 1$. Choose the prime number p such that $p \mid o(x)$ and $p \mid o(y)$. If $t \in G$ has order p then $x \sim t \sim y$ is a path in $\mathcal{S}^*(G)$.

This completes the proof. □

Corollary 2.11. *If $\Delta(G)$ is complete then $\mathcal{S}(G)$ is 2-connected.*

It is clear that if G and H are groups with the same order such that for each divisor d of $|G|$, $\Omega_d(G) = \Omega_d(H)$ then $\mathcal{S}(G) \cong \mathcal{S}(H)$. The converse of this result is not generally correct. To prove, we consider $G = Z_4 \times Z_4$ and $H = Z_2 \times Z_4 \times Z_2$. Since G and H are 2-groups, $\mathcal{S}(G) \cong \mathcal{S}(H)$. But $\Omega_4(G) = 8 < 12 = \Omega_4(H)$ and $\Omega_2(G) = 7 > 3 = \Omega_2(H)$. On the other hand, it is possible to find finite groups G and H such that $\mathcal{S}(G) \cong \mathcal{S}(H)$, but $\pi_e(G) \neq \pi_e(H)$. An example is the pair $(G, H) = (D_8, Z_8)$. Finally, it is possible to construct the pair (G, H) of finite groups such that $\pi_e(G) = \pi_e(H)$, but $\mathcal{S}(G) \ncong \mathcal{S}(H)$. To see this, it is enough to assume that $G = D_{20}$ and $H = Z_2 \times Z_{10}$. In what follows, we prove that in the group under same specific conditions the equality of spectrum and order implies that the main supergraph are isomorphic.

Theorem 2.12. (See [1,23]). *Suppose G_1 is a finite group and G_2 is one of the following finite groups:*

(1) *A finite simple group,*
(2) *A symmetric group S_n, $n \geq 3$,*
(3) *Automorphism group of a sporadic simple group,*

then $G_1 \cong G_2$ if and only if $|G_1| = |G_2|$ and $\pi_e(G_1) = \pi_e(G_2)$.

Corollary 2.13. *If G_1 is a finite group and G_2 is one of the following finite groups:*

(1) *A finite simple group,*
(2) *A symmetric group S_n, $n \geq 3$,*
(3) *Automorphism group of a sporadic simple group.*

If $|G_1| = |G_2|$ and $\pi_e(G_1) = \pi_e(G_2)$ then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$.

In the following result, the finite groups G in which the main supergraph $\mathcal{S}(G)$ is vertex transitive are classified.

Theorem 2.14. *Let G be a finite group, then $\mathcal{S}(G)$ is a vertex transitive if and only if G is a p-group. There is no group G such that $\overrightarrow{\mathcal{S}(G)}$ is vertex transitive.*

Proof. If G is a p-group then $\mathcal{S}(G)$ is complete and so it is a Cayley graph. Conversely, we assume that $\mathcal{S}(G)$ is vertex-transitive, where G has order n. Since $deg_{\mathcal{S}(G)}(e) = n - 1$, $\mathcal{S}(G)$ has to be complete and so G is a p-group.

We now assume that G is a finite group such that $\overrightarrow{\mathcal{S}(G)}$ is vertex transitive. Then each vertex of $\overrightarrow{\mathcal{S}(G)}$ will have the in-degree $n-1$ and out-degree zero which is impossible. □

The present authors [11], proved that for each finite group G we have $|\pi(G)| \leq \alpha(\mathcal{S}(G)) \leq |\pi_e(G)| - 1$ with right-hand equality if and only if G is an EPO-group. Applying this result, we have:

Theorem 2.15. *If G is a finite group of order n then $n+1-|\pi_e(G)| \leq \beta(\mathcal{S}(G)) \leq n - |\pi(G)|$. The left-hand equality is attained if and only if G is an EPO-group.*

Theorem 2.16. *Let G be a finite group. $\overline{\mathcal{S}^*(G)}$ is complete if and only if $G \cong Z_2$.*

Proof. Suppose $\overline{\mathcal{S}^*(G)}$ is a complete graph. Then G is an EPO-group and there is a unique elements of each order. So, $\mathcal{S}(G)$ is a star graph and by [11, Corollary 2.18], $G \cong Z_2$. The converse is obvious. □

Theorem 2.17. *Let G be a finite group of order> 2. Then G has full exponent if and only if $\overline{\mathcal{S}^*(G)}$ is disconnected.*

Proof. If G is full exponent group of order n, $n > 2$ [11, Theorem 2.15], there are at least two elements of degree $n-1$ in $\mathcal{S}(G)$. This proves that $\overline{\mathcal{S}^*(G)}$ is disconnected. To prove the converse, we show that if G is not a full exponent group of order n, $n > 2$, then $\overline{\mathcal{S}^*(G)}$ is connected. Suppose $|G| = {p_1}^{n_1} \ldots {p_k}^{n_k}$. If $k = 1$, then $\mathcal{S}(G)$ is complete and $\overline{\mathcal{S}^*(G)}$ is an empty graph, as desired. Suppose $k \geq 2$. Define

$$V_i = \{g \in G | 1 \neq o(g) \mid {p_i}^{n_i}\}.$$

Then for each i, the graph $\overline{\mathcal{S}^*(G)}$ has an induced subgraph isomorphic to $\overline{K_{|V_i|}}$ in such a way that every element $x \in V_i$ is adjacent to every element $y \in V_j$, $i \neq j$. Thus the induced subgraph $[\bigcup_{i=1}^{k} V_i]$ is connected. Suppose $x, y \notin \bigcup_{i=1}^{k} V_i$, $o(x) = {q_1}^{\alpha_1} \ldots {q_r}^{\alpha_r}$ and $o(y) = {q_1}^{\beta_1} \ldots {q_s}^{\beta_s}$, $r \leq s$. If $(o(x), o(y)) = 1$, then $xy \in E(\overline{\mathcal{S}^*(G)})$ as desired. We now assume that $d = (o(x), o(y)) \neq 1$. If there exists a prime number p such that $p \nmid d$ then we choose an element z of order p in G. So $x \sim z \sim y$ is a path connecting x and y in $\mathcal{S}(G)$. Hence, it is enough to assume that, for any i, $1 \leq i \leq k$, $p_i \mid d$. Suppose $o(x) = {p_1}^{\gamma_1} \ldots {p_k}^{\gamma_k}$ and $o(y) = {p_1}^{\delta_1} \ldots {p_k}^{\delta_k}$. If $o(x) \nmid o(y)$ and $o(y) \nmid o(x)$ then $xy \in E(\overline{\mathcal{S}^*(G)})$. Suppose $o(x) \mid o(y)$ and choose i such that $\gamma_i \neq \alpha_i$. Then $x \sim x_i \sim y$ is a path in $\overline{\mathcal{S}^*(G)}$. This completes the proof. □

Theorem 2.18. *If G is a full exponent group, then the number of connected components of $\overline{\mathcal{S}^*(G)}$ is $c(\overline{\mathcal{S}^*(G)}) = \varphi(G)+1$, where $\varphi(G) = |\{a \in G | o(a) = Exp(G)\}|$.*

Proof. Suppose G is a full exponent group of order ${p_1}^{n_1} \ldots {p_k}^{n_k}$, where, p_i, $1 \leq i \leq k$ are distinct primes and $k > 1$. Similar to the Theorem 2.17, we define

$$V_i = \{g \in G | 1 \neq o(g) \mid {p_i}^{n_i}\}.$$

By Theorem 2.17, the induced subgraph on $\bigcup_{i=1}^k V_i$ is connected. Suppose $x, y \notin \bigcup_{i=1}^k V_i$. If $o(x), o(y) \notin \{|G|, Exp(G)\}$ then by similar argument as in Theorem 2.17, there exits an element $u \in \bigcup_{i=1}^k V_i$, such that $x \sim u \sim y$ is path in $\overline{\mathcal{S}^*(G)}$. We now assume that $o(x) \in \{|G|, Exp(G)\}$. Then $\{x\}$ is a component of $\overline{\mathcal{S}^*(G)}$ and so the number of connected components is $\varphi(G) + 1$. □

Theorem 2.19. *Let G be a finite group. Then,*

1. *if $Exp(G) = m$, then $c(\overline{\mathcal{S}^*(G)}) = k\phi(m) + 1$, where k is the number of cyclic subgroups of order m in G;*
2. *if G is nilpotent, then $c(\overline{\mathcal{S}^*(G)}) = \prod_{i=1}^k \varphi(G_i) + 1$, where G_i's are Sylow subgroups of G;*
3. *$c(\overline{\mathcal{S}^*(G)}) = \phi(|G|) + 1$ if and only if the number of cyclic subgroups of order $Exp(G)$ in G is $\frac{|G|}{Exp(G)}$.*

Proof. Apply Theorems 2.2, 2.6, 2.8 and 3.2 from [22]. □

Corollary 2.20. *The following hold:*

1. *if $2^k \neq n \geq 3$ is an even positive integer then $c(\overline{\mathcal{S}^*(D_{2n})}) = \phi(n) + 1$, and if n is odd then $\overline{\mathcal{S}^*(D_{2n})}$ is connected;*
2. *if $n \geq 3$, $\overline{\mathcal{S}^*(S_n)}$ is connected and if $n \geq 4$, then $\overline{\mathcal{S}^*(A_n)}$ is connected;*
3. *if $n = 2^k$, then $c(\overline{\mathcal{S}^*(SD_{8n})}) = 8n - 1$ and if $n \neq 2^k$, then $c(\overline{\mathcal{S}^*(SD_{8n})}) = \phi(4n) + 1$;*
4. *if n is odd, then $\overline{\mathcal{S}^*(T_{4n})}$ is connected and if $n = 2^k$, then $c(\overline{\mathcal{S}^*(T_{4n})}) = 4n - 1$. If $n \neq 2^k$ and n is an even number, then $c(\overline{\mathcal{S}^*(T_{4n})}) = \phi(2n) + 1$;*
5. *if n is odd, then $\overline{\mathcal{S}^*(V_{8n})}$ is connected. If $n = 2^k$, then $c(\overline{\mathcal{S}^*(V_{8n})}) = 8n - 1$ and if $n \neq 2^k$ and n is an even number, then $c(\overline{\mathcal{S}^*(V_{8n})}) = \phi(2n) + 1$.*

By the graph structure of $\mathcal{S}(G) = \Delta_G[K_{\Omega_{a_1}(G)}, \ldots, K_{\Omega_{a_k}(G)}]$ and definition of module, one can see that every $K_{\Omega_{a_i}(G)}$, $1 \leq i \leq k$, in $\mathcal{S}(G)$ is a maximal strong module. Also $P = \{V(K_{\Omega_{a_1}(G)}), \ldots, V(K_{\Omega_{a_k}(G)})\}$ is a modular partition of $\mathcal{S}(G)$ and quotient graph $\mathcal{S}(G)/P$ is isomorphic to Δ_G.

Theorem 2.21. *Let G_1 and G_2 be two finite groups. We also assume that these groups are not full exponent, they are not p-groups, for some prime number p, and the graphs Δ_{G_1} and Δ_{G_2} are primitive. If $\mathcal{S}^*(G_1) \cong \mathcal{S}^*(G_2)$, then $|G_1| = |G_2|$ and $nse(G_1) = nse(G_2)$.*

Proof. By Theorem 1.1, since Δ_{G_1} and Δ_{G_2} are primitive, $\mathcal{S}^*(G_1)$, $\mathcal{S}^*(G_2)$, $\overline{\mathcal{S}^*(G_1)}$ and $\overline{\mathcal{S}^*(G_2)}$ are connected. In addition, each graph has a unique maximal modular partition and $\mathcal{S}^*(G_1) \cong \mathcal{S}^*(G_2)$ implies that $\Delta_{G_1} \cong \Delta_{G_2}$ and so $|G_1| = |G_2|$. This shows that $nse(G_1) = nse(G_2)$, as desired. □

Theorem 2.22. [17,18,21] *Suppose G, H are finite groups and one of the following are satisfied:*

- *H is a sporadic simple group;*
- *H is a Mathieu group;*
- *H is the symmetric group S_r, where r is prime number.*

If $|G| = |H|$ and $nse(G) = nse(H)$, then $G \cong H$.

Theorem 2.23. *Suppose G_1 and G_2 satisfy the conditions of Theorem 2.21. We also assume that one of the following conditions are satisfied:*

- *G_1 is a sporadic simple group;*
- *G_1 is a Mathieu group;*
- *G_1 is the symmetric group S_r, where r is prime number.*

If $\mathcal{S}^*(G_1) \cong \mathcal{S}^*(G_2)$, then $G_1 \cong G_2$.

Proof. Apply Theorems 2.21 and 2.22. □

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