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(α, m_1, m_2)-CONVEXITY AND SOME INEQUALITIES OF HERMITE-HADAMARD TYPE

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Abstract. In this paper, we introduce a new class of extended (α, m_1, m_2)-convex functions. Some algebraic properties of these class functions have been investigated. Some new Hermite-Hadamard type inequalities are derived. Results represent significant refinement and improvement of the previous results. Also, the author establish a new integral identity and, by this identity, Hölder's and power mean inequality, discover some new Hermite-Hadamard type inequalities for functions whose first derivatives are (α, m_1, m_2)-convex. Our results are new and coincide with the previous results in special cases.

1. INTRODUCTION

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

This definition is well known in the literature. One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The following double inequality is well known as the Hadamard inequality in the literature.

Definition 2. *$f : [a,b] \to \mathbb{R}$ be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

Some refinements of the Hermite-Hadamard inequality on convex functions have been investigated by [3, 9, 14] and the Authors obtained a new refinement of the Hermite-Hadamard inequality for convex functions.

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Definition 3. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex function, where $m \in [0, 1]$; if we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that the function f is m-concave function if $(-f)$ is m -convex.

Obviously, for $m = 1$ the above definition recaptures the concept of standard convex functions on $[a, b]$; and for $m = 0$ the concept star-shaped functions.

The interested reader can find more about partial ordering of convexity in [13]. For many papers connected with m-convex and (α, m)-convex functions see ([1, 8, 11, 15]) and the references therein. There are similar inequalities for s-convex and h-convex functions in [7] and [14], respectively.

Definition 4. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is sad to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \le t^{\alpha} f(x) + m(1 - t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \le 0$.

It can be easily seen that for $(\alpha, m) = (1, m)$, (α, m)-convexity reduces to m-convexity; $(\alpha, m) = (a, 1)$, (α, m)-convexity reduces to α-convexity and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the concept of usual convexity defined on $[0, b]$, $b > 0$.

Definition 5. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (m_1, m_2)-convex, if*

$$f(m_1 tx + m_2(1-t)y) \le m_1 t f(x) + m_2(1-t)f(y)$$

for all $x, y \in I$, $t \in [0, 1]$ and $(m_1, m_2) \in (0, 1]^2$.

Definition 6. *Let $f : [0, b] \to \mathbb{R}$. If $f(tx) \le tf(x)$ is valid for all $x \in [0, b]$, then we say that $f(x)$ is a starshaped function on $[0, b]$.*

Definition 7. *Let $f : [0, b] \to \mathbb{R}$ and $m_1 \in (0, 1]$. If $f(m_1 tx) \le m_1 t f(x)$ is valid for all $x \in [0, b]$ and $t \in [0, 1]$, then we say that the function $f(x)$ is a m_1-starshaped function on $[0, b]$. Specially, for $m_1 = 1$, we have $f(tx) \le tf(x)$.*

In [4], Kadakal proved the following theorem for (m_1, m_2)-convex functions.

Theorem 8. *Let the function $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, be a (m_1, m_2)-convex functions with $m_1, m_2 \in (0, 1]^2$. If $0 \le a < b < b^*$ and $f \in L[a, b]$, then the following inequalities holds:*

$$\frac{1}{b-a} \int_a^b f(x)dx \le \min \left\{ \frac{m_1 f\left(\frac{a}{m_1}\right) + m_2 f\left(\frac{b}{m_2}\right)}{2}, \frac{m_1 f\left(\frac{b}{m_1}\right) + m_2 f\left(\frac{a}{m_2}\right)}{2} \right\}.$$

2. (α, m_1, m_2)-CONVEX FUNCTIONS AND SOME PROPERTIES

In this section, we will begin by setting some algebraic properties for (α, m_1, m_2)-convex functions.

Definition 9. *$f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m_1, m_2)-convex function, if*

$$f\left(m_1 tx + m_2(1-t)y\right) \leq m_1 t^\alpha f(x) + m_2(1-t^\alpha) f(y)$$

for all $x, y \in I$, $t \in [0,1]$ and $(\alpha, m_1, m_2) \in (0,1]^3$.

We will denote by $K^\alpha_{m_1,m_2}(b)$ the class of all (α, m_1, m_2)-convex functions on interval I for which $f(0) \leq 0$. Also, we note that, for any $t \in [0,1]$ and $(\alpha, m_1, m_2) \in (0,1]^3$, we have

$$\begin{aligned} f(0) &= f\left(m_1 t^\alpha .0 + m_2(1-t^\alpha).0\right) \\ &\leq m_1 t^\alpha f(0) + m_2(1-t^\alpha) f(0) \\ 0 &\leq \left[m_1 t^\alpha + m_2(1-t^\alpha) - 1\right] f(0). \end{aligned}$$

Since $m_1 t^\alpha + m_2(1-t^\alpha) - 1 \leq 0$, we get $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m_1, m_2) \in \{(0,0,0), (\alpha,1,0), (1,1,0), (1,1,m), (1,1,1), (\alpha,1,1), (\alpha,1,m)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex, α-convex and (α, m)-convex functions respectively.

Definition 10. *Let $f : [0, b] \to \mathbb{R}$ and $(\alpha, m_1) \in (0,1]^2$. If*

$$f(m_1 tx) \leq m_1 t^\alpha f(x)$$

is valid for all $x \in [0, b]$ and $t \in [0,1]$, then we say that $f(x)$ is (α, m_1)-starshaped function on interval $[0, b]$. Specially, for $m_1 = 1$ and $\alpha = 1$, we have $f(tx) \leq tf(x)$.

Remark 11. *In Definition 9, if we choose $m_2 = 0$, we get the concept of (α, m_1)-starshaped functions on interval $[0, b]$.*

Proposition 12. *If the function f is in the class $K^\alpha_{m_1,m_2}(b)$, then it is (α, m_1)-starshaped.*

Proof. For any $x \in [0, b]$, $t \in [0,1]$ and $(\alpha, m_1, m_2) \in (0,1]^3$, we have

$$\begin{aligned} f\left(m_1 tx\right) &= f\left(m_1 tx + m_2(1-t).0\right) \\ &\leq m_1 t^\alpha f(x) + m_2(1-t^\alpha) f(0) \\ &\leq m_1 t^\alpha f(x). \end{aligned}$$

Specially, for $m_1 = 1$, we have $f(tx) \leq t^\alpha f(x)$. □

Theorem 13. *Let $f, g : [0, b] \to \mathbb{R}$. If f and g are (α, m_1, m_2)-convex, then*

(i) $f + g$ is (α, m_1, m_2)-convex,
(ii) For $c \in \mathbb{R}$ $(c \geq 0)$ cf is (α, m_1, m_2)-convex.

Proof. *(i)* For $x, y \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} & (f+g)(m_1 tx + m_2(1-t)y) \\ = & f(m_1 tx + m_2(1-t)y) + g(m_1 tx + m_2(1-t)y) \\ \leq & m_1 f(x) + m_2(1-t^\alpha) f(y) + m_1 t^\alpha g(x) + m_2(1-t^\alpha) g(y) \\ \leq & m_1 t^\alpha (f+g)(x) + m_2(1-t^\alpha)(f+g)(y). \end{aligned}$$

(ii) For $c \in \mathbb{R}$ $(c \geq 0)$, we obtain

$$\begin{aligned} (cf)(m_1 tx + m_2(1-t)y) & \leq c\left[m_1 t^\alpha f(x) + m_2(1-t^\alpha) f(y)\right] \\ & \leq m_1 t^\alpha (cf)(x) + m_2(1-t^\alpha)(cf)(y). \end{aligned}$$

This completes the proof of theorem. □

Theorem 14. *Let f be a (m_1, m_2)-convex function. If the function g is a (α, m_1, m_2)-convex and increasing, then the function gof is a (α, m_1, m_2)-convex.*

Proof. For $x, y \in I$ and $t \in [0,1]$, we get

$$\begin{aligned} (gof)(m_1 tx + m_2(1-t)y) & = g(f(m_1 tx + m_2(1-t)y)) \\ & \leq g(m_1 t f(x) + m_2(1-t) f(y)) \\ & \leq m_1 t^\alpha g(f(x)) + m_2(1-t^\alpha) g(f(y)) \\ & = m_1 t^\alpha (gof)(x) + m_2(1-t^\alpha)(gof)(y). \end{aligned}$$

This completes the proof of theorem. □

Theorem 15. *Let $f, g : I \to \mathbb{R}$ are both nonnegative and monotone increasing. If f and g are (α, m_1, m_2)-convex functions, then fg is (α, m_1, m_2)-convex function.*

Proof. If $x \leq y$ (the case $y \leq x$ is similar) then $[f(x) - f(y)][g(y) - g(x)] \leq 0$ which implies

$$f(x)g(y) + f(y)g(x) \leq f(x)g(x) + f(y)g(y). \tag{1}$$

On the other hand for $x, y \in I$ and $t \in [0,1]$,

$$\begin{aligned} & (fg)(m_1 tx + m_2(1-t)y) \\ = & f(m_1 tx + m_2(1-t)y)\, g(m_1 tx + m_2(1-t)y) \\ \leq & \left[m_1 t^\alpha f(x) + m_2(1-t^\alpha) f(y)\right]\left[m_1 t^\alpha g(x) + m_2(1-t^\alpha) g(y)\right] \\ = & m_1 m_1 t^{2\alpha} f(x)g(x) + m_1 m_2 t^\alpha (1-t^\alpha) f(x)g(y) + m_2 m_1 t^\alpha (1-t^\alpha) f(y)g(x) \\ & + m_2 m_2 (1-t^\alpha)^2 f(y)g(y) \\ = & m_1^2 t^{2\alpha} f(x)g(x) + m_1 m_2 t^\alpha (1-t^\alpha)\left[f(x)g(y) + f(y)g(x)\right] + m_2^2 (1-t^\alpha)^2 f(y)g(y). \end{aligned}$$

Using now (2.1), we obtain,

$$\begin{aligned} & (fg)(m_1 tx + m_2(1-t)y) \\ \leq & m_1^2 t^{2\alpha} f(x)g(x) + m_1 m_2 t^\alpha (1-t^\alpha)\left[f(x)g(x) + f(y)g(y)\right] \\ & + m_2^2 (1-t^\alpha)^2 f(y)g(y) \end{aligned}$$

$$= \quad m_1t^\alpha\left[m_1t^\alpha+m_2(1-t^\alpha)\right]f(x)g(x)+m_2(1-t^\alpha)\left[m_1t^\alpha+m_2(1-t^\alpha)\right]f(y)g(y).$$

Since $m_1t^\alpha+m_2(1-t^\alpha)\leq m\leq 1$, where $m=\max\{m_1,m_2\}$. Therefore, we get

$$\begin{aligned}(fg)(m_1tx+m_2(1-t)y) &\leq m_1t^\alpha f(x)g(x)+m_2(1-t^\alpha)f(y)g(y)\\ &= m_1t^\alpha(fg)(x)+m_2(1-t^\alpha)(fg)(y).\end{aligned}$$

This completes the proof of theorem. □

Theorem 16. *Let $f:[0,b^*]\to\mathbb{R}$ a finite function on $\frac{a}{m_1},\frac{b}{m_2}\in[0,b^*]$, (α,m_1,m_2)-convex with $\alpha,m_1,m_2\in(0,1]$. Then f is on bounded any closed interval $[a,b]$.*

Proof. Let

$$M=\max\left\{m_1f\left(\frac{a}{m_1}\right),m_2f\left(\frac{b}{m_2}\right),m_2f\left(\frac{a}{m_2}\right),m_1f\left(\frac{b}{m_1}\right)\right\},$$

so for any $z=ta+(1-t)b$ in interval $[a,b]$, we get

$$\begin{aligned}f(z) &= f(ta+(1-t)b)\\ &= f\left(m_1t\frac{a}{m_1}+m_2(1-t)\frac{b}{m_2}\right)\\ &\leq m_1t^\alpha f\left(\frac{a}{m_1}\right)+m_2(1-t^\alpha)f\left(\frac{b}{m_2}\right)\\ &\leq M.\end{aligned}$$

Thus, the function f is upper bounded in interval $[a,b]$.

Now we notice that any $z\in[a,b]$ can be written as $\frac{a+b}{2}+t$ for $|t|\leq\frac{b-a}{2}$, hence

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right)\\ &\leq m_1\frac{1}{2^\alpha}f\left(\frac{\frac{a+b}{2}+t}{m_1}\right)+m_2\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{\frac{a+b}{2}-t}{m_2}\right).\end{aligned}$$

In other word, we get

$$\begin{aligned}f\left(\frac{\frac{a+b}{2}+t}{m_1}\right) &\geq \frac{2^\alpha}{m_1}f\left(\frac{a+b}{2}\right)-\frac{2^\alpha}{m_1}m_2\left(1-\frac{1}{2^\alpha}\right)f\left(\frac{\frac{a+b}{2}-t}{m_2}\right)\\ &\geq \frac{2^\alpha}{m_1}f\left(\frac{a+b}{2}\right)-\frac{2^\alpha}{m_1}\left(1-\frac{1}{2^\alpha}\right)M\\ &= \frac{2^\alpha}{m_1}f\left(\frac{a+b}{2}\right)-\frac{2^\alpha-1}{m_1}M\end{aligned}$$

and similarly,

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\left(\frac{a+b}{2}+t\right)+\frac{1}{2}\left(\frac{a+b}{2}-t\right)\right)$$

$$\leq \quad m_2 \frac{1}{2^\alpha} f\left(\frac{\frac{a+b}{2}+t}{m_2}\right) + m_1\left(1-\frac{1}{2^\alpha}\right) f\left(\frac{\frac{a+b}{2}-t}{m_1}\right),$$

hence, we obtain

$$\begin{aligned} f\left(\frac{\frac{a+b}{2}+t}{m_2}\right) &\geq \frac{2^\alpha}{m_2} f\left(\frac{a+b}{2}\right) - \frac{2^\alpha}{m_2} m_1 \left(1-\frac{1}{2^\alpha}\right) f\left(\frac{\frac{a+b}{2}-t}{m_1}\right) \\ &\geq \frac{2^\alpha}{m_1} f\left(\frac{a+b}{2}\right) - \frac{2^\alpha}{m_1}\left(1-\frac{1}{2^\alpha}\right) M \\ &= \frac{2^\alpha}{m_1} f\left(\frac{a+b}{2}\right) - \frac{2^\alpha - 1}{m_1} M, \end{aligned}$$

and since $\frac{a+b}{2}+t$ is arbitrary in $[a,b]$, the function f is also bounded below in $[a,b]$. This completes the proof of theorem. □

Theorem 17. *Let $\alpha, m_1, m_2 \in [0,1]$, $b > 0$ and $f_\beta : [0,b] \to \mathbb{R}$ be an arbitrary family of (α, m_1, m_2)-convex functions and let $f(x) = \sup_\beta f_\beta(x)$. If*

$$J = \left\{ u \in [0,b] : \frac{u}{m_1}, \frac{u}{m_2} \in [0,b] \;\; \text{and } f(u), f\left(\frac{u}{m_1}\right), f\left(\frac{u}{m_2}\right) < \infty \right\}$$

is nonempty, then J is an interval and f is (α, m_1, m_2)-convex on J.

Proof. Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned} f\left(tx+(1-t)y\right) &= \sup_\beta f_\beta(x)\left(m_1 t \frac{x}{m_1} + m_2(1-t)\frac{y}{m_2}\right) \\ &\leq \sup_\beta \left[m_1 t^\alpha f_\beta\left(\frac{x}{m_1}\right) + m_2(1-t^\alpha) f_\beta\left(\frac{y}{m_2}\right)\right] \\ &\leq m_1 t^\alpha \sup_\beta f_\beta\left(\frac{x}{m_1}\right) + m_2(1-t^\alpha) \sup_\beta f_\beta\left(\frac{y}{m_2}\right) \\ &= m_1 t^\alpha f\left(\frac{x}{m_1}\right) + m_2(1-t^\alpha) f\left(\frac{y}{m_2}\right) < \infty. \end{aligned}$$

This shows simultaneously that J is an interval since it contains every point between any two of its points.

Now, we show that the function f is (m_1, m_2)-convex on J: If $t \in [0,1]$ and $x, y \in J$, then

$$\begin{aligned} f\left(m_1 tx + m_2(1-t)y\right) &= \sup_\beta f_\beta\left(m_1 tx + m_2(1-t)y\right) \\ &\leq \sup_\beta \left[m_1 t^\alpha f_\beta(x) + m_2(1-t^\alpha) f_\beta(y)\right] \\ &\leq m_1 t^\alpha \sup_\beta f_\beta(x) + m_2(1-t^\alpha) \sup_\beta f_\beta(y) \\ &= m_1 t^\alpha f(x) + m_2(1-t^\alpha) f(y) \end{aligned}$$

and that the function f is (α, m_1, m_2)-convex on J.

This completes the proof of theorem. □

3. HERMITE-HADAMARD INEQUALITY FOR (α, m_1, m_2)-CONVEX FUNCTIONS

The goal of this paper is to develop concepts of the (α, m_1, m_2)-convex functions and to establish some inequalities of Hermite-Hadamard type for these classes of functions.

Theorem 18. *Let $f : I \to \mathbb{R}$ be a (α, m_1, m_2)-convex function with $(\alpha, m_1, m_2) \in (0, 1]^3$. If $0 \le a < b < \infty$ and $f \in L[a, b]$, then the following inequalities hold:*

i. $f\left(\frac{a+b}{2}\right) \le \frac{1}{2^\alpha}\frac{m_1^2}{b-a}\int_a^b f(m_1 x)\,dx + \left(1 - \frac{1}{2^\alpha}\right)\frac{m_2^2}{b-a}\int_a^b f(m_2 y)\,dy$

ii. $\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{\frac{m_1 f\left(\frac{a}{m_1}\right) + \alpha m_2 f\left(\frac{b}{m_2}\right)}{\alpha+1}, \frac{m_1 f\left(\frac{b}{m_1}\right) + \alpha m_2 f\left(\frac{a}{m_2}\right)}{\alpha+1}\right\}$

Proof. i. By the (α, m_1, m_2)-convexity of the function f, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left[m_1 t\frac{a}{m_1} + m_2(1-t)\frac{b}{m_2}\right] + \left[m_1(1-t)\frac{a}{m_1} + m_2 t\frac{b}{m_2}\right]}{2}\right) \\ &= f\left(\frac{1}{2}m_1\left[t\frac{a}{m_1} + \frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right] + \frac{1}{2}m_2\left[t\frac{b}{m_2} + \frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right]\right) \\ &\le \frac{1}{2^\alpha}m_1 f\left(t\frac{a}{m_1} + \frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right) \\ &\quad + \left(1 - \frac{1}{2^\alpha}\right)m_2 f\left(t\frac{b}{m_2} + \frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right). \end{aligned}$$

Now, if we take integral the last inequality on $t \in [0, 1]$ and choose $m_1 x = ta + (1-t)b$ and $m_2 y = tb + (1-t)a$, we deduce

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2^\alpha}\frac{m_1^2}{b-a}\int_a^b f(m_1 x)\,dx + \left(1 - \frac{1}{2^\alpha}\right)\frac{m_2^2}{b-a}\int_a^b f(m_2 y)\,dy.$$

ii. By using the (α, m_1, m_2)-convexity of the function f, if the variable is changed as $u = ta + (1-t)b$

$$\begin{aligned} \int_0^1 f(ta + (1-t)b)\,dt &= \frac{1}{b-a}\int_0^1 f(u)\,du \\ &\le \int_0^1 \left[t^\alpha m_1 f\left(\frac{a}{m_1}\right) + (1 - t^\alpha)\,m_2 f\left(\frac{b}{m_2}\right)\right]dt \\ &= \frac{m_1 f\left(\frac{a}{m_1}\right) + \alpha m_2 f\left(\frac{b}{m_2}\right)}{\alpha+1} \end{aligned}$$

and similarly for $z = tb + (1-t)a$, then

$$\begin{aligned}\int_0^1 f(tb+(1-t)a)\,dt &= \frac{1}{b-a}\int_0^1 f(z)\,dz \\ &\le \int_0^1 \left[t^\alpha m_1 f\left(\frac{b}{m_1}\right) + (1-t^\alpha) m_2 f\left(\frac{a}{m_2}\right)\right] dt \\ &= \frac{m_1 f\left(\frac{b}{m_1}\right) + \alpha m_2 f\left(\frac{a}{m_2}\right)}{\alpha+1}.\end{aligned}$$

So, we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{\frac{m_1 f\left(\frac{a}{m_1}\right) + \alpha m_2 f\left(\frac{b}{m_2}\right)}{\alpha+1}, \frac{m_1 f\left(\frac{b}{m_1}\right) + \alpha m_2 f\left(\frac{a}{m_2}\right)}{\alpha+1}\right\}.$$

This completes the proof of theorem. □

Remark 19. *Under the conditions of Theorem 18, if* $m_1 = 1,\ m_2 = m$*, then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{mf\left(\frac{b}{m}\right) + f(a)}{2}\right\}.$$

This inequality is the Hermite-Hadamard inequality for the m*-convex functions* [15].

Remark 20. *Under the conditions of Theorem 18,*

i) If $\alpha = m_1 = m_2 = 1$*, then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

This inequality is the Hermite-Hadamard inequality for the convex functions [5].

(ii) If $\alpha = m_1 = 1,\ m_2 = m$*, then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a) + \alpha mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + \alpha mf\left(\frac{a}{m}\right)}{2}\right\}.$$

This inequality is the Hermite-Hadamard inequality for the m*-convex functions* [2].

(iii) If $\alpha = s,\ m_1 = m_2 = 1$*, then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+sf(b)}{s+1}, \frac{f(b)+sf(a)}{s+1}\right\}.$$

This inequality is the Hermite-Hadamard inequality for the s*-convex functions in the first sense* [10].

(iv) If $m_1 = 1,\ m_2 = m$*, then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a) + \alpha mf\left(\frac{b}{m}\right)}{\alpha+1}, \frac{f(b) + \alpha mf\left(\frac{a}{m}\right)}{\alpha+1}\right\}.$$

This is the Hermite-Hadamard inequality for the (α, m)*-convex functions* [12].

Theorem 21. *Let the function* $f : [0, b^*] \to \mathbb{R},\ b^* > 0$*,be a* (α, m_1, m_2)*-convex functions with* $\alpha, m_1, m_2 \in (0,1]^3$*. If* $m = \min\{m_1, m_2\}$*,* $0 \leq a < b < \frac{b}{m} < b^*$ *and* $f \in L\left[a, \frac{b}{m}\right]$*, then the following inequalities holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^\alpha}\frac{m_1}{b-a}\int_a^b f\left(\frac{x}{m_1}\right)dx + \left(1-\frac{1}{2^\alpha}\right)\frac{m_2}{b-a}\int_a^b f\left(\frac{y}{m_2}\right)dy$$

Proof. By the (α, m_1, m_2)-convexity of the function f, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{\left[m_1 t\frac{a}{m_1} + m_2(1-t)\frac{b}{m_2}\right] + \left[m_1(1-t)\frac{a}{m_1} + m_2 t\frac{b}{m_2}\right]}{2}\right) \\ &= f\left(\frac{1}{2}m_1\left[t\frac{a}{m_1} + \frac{m_2}{m_1}(1-t)\frac{b}{m_2}\right] + \frac{1}{2}m_2\left[t\frac{b}{m_2} + \frac{m_1}{m_2}(1-t)\frac{a}{m_1}\right]\right) \\ &\leq \frac{1}{2^\alpha}m_1 f\left(\frac{ta+(1-t)b}{m_1}\right) + \left(1-\frac{1}{2^\alpha}\right)m_2 f\left(\frac{tb+(1-t)a}{m_2}\right). \end{aligned}$$

Now, if we take integral the last inequality on $t \in [0,1]$ and choose $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we deduce

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^\alpha}\frac{m_1}{b-a}\int_a^b f\left(\frac{x}{m_1}\right)dx + \left(1-\frac{1}{2^\alpha}\right)\frac{m_2}{b-a}\int_a^b f\left(\frac{y}{m_2}\right)dy$$

This completes the proof of theorem. □

Remark 22. *Under the conditions of Theorem 21, if* $\alpha = m_1 = m_2 = 1$*, then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx$$

This inequality is the left hand side of Hermite-Hadamard inequality for the convex functions [13].

4. SOME NEW INEQUALITIES FOR (α, m_1, m_2)-CONVEXITY

In [6], Kirmaci used the following lemma to prove Theorems.

Lemma 23. *Let* $f : I^* \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I^**,* $a, b \in I^*$ *(*I^* *is the interior of* I*) with* $a < b$*. If* $f' \in L[a,b]$*, then we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ = &\ (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)\,dt\right] \end{aligned}$$

The main purpose of this section is to establish new estimations and refinements of the Hermite-Hadamard inequality for functions whose first derivatives in absolute value are (α, m_1, m_2)-convex. For this, we will use the following lemma.

Lemma 24. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1 a, m_2 b \in I^\circ$ with $m_1 a < m_2 b$. If $f' \in L[m_1 a, m_2 b]$, then the following equality*

$$\begin{aligned} & \frac{1}{m_2 b - m_1 a} \int_{m_1 a}^{m_2 b} f(x)dx - f\left(\frac{m_1 a + m_2 b}{2}\right) \\ = & \ (m_2 b - m_1 a)\left[\int_0^{\frac{1}{2}} t f'\left(m_1 t a + m_2(1-t)b\right) dt \right. \\ & \left. + \int_{\frac{1}{2}}^1 (t-1) f'\left(m_1 t a + m_2(1-t)b\right) dt\right] \end{aligned}$$

holds for $t \in [0,1]$ and $m_1, m_2 \in (0,1]^2$.

Proof. By integration by parts and then by changing of variable $x = m_1 t a + m_2(1-t)b$, we get

$$\begin{aligned} & \int_0^{\frac{1}{2}} t f'\left(m_1 t a + m_2(1-t)b\right) dt + \int_{\frac{1}{2}}^1 (t-1) f'\left(m_1 t a + m_2(1-t)b\right) dt \\ = & -\frac{f\left(m_1 t a + m_2(1-t)b\right)}{m_2 b - m_1 a} t\Big|_0^{\frac{1}{2}} + \frac{1}{m_2 b - m_1 a}\int_0^{\frac{1}{2}} f\left(m_1 t a + m_2(1-t)b\right) dt \\ & - \frac{f\left(m_1 t a + m_2(1-t)b\right)}{m_2 b - m_1 a}(t-1)\Big|_{\frac{1}{2}}^1 + \frac{1}{m_2 b - m_1 a}\int_{\frac{1}{2}}^1 f\left(m_1 t a + m_2(1-t)b\right) dt \\ = & \frac{1}{(m_2 b - m_1 a)^2}\int_{m_1 a}^{m_2 b} f(x)\,dx - \frac{1}{m_2 b - m_1 a} f\left(\frac{m_1 a + m_2 b}{2}\right). \end{aligned}$$

Thus, the proof of lemma is completed. ◻

Theorem 25. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1 a, m_2 b \in I^\circ$ with $m_1 a < m_2 b$ and $f' \in L[m_1 a, m_2 b]$. If $|f'|$ is (α, m_1, m_2) -convex on the interval $[m_1 a, m_2 b]$, then the following equality*

$$\begin{aligned} & \left|\frac{1}{m_2 b - m_1 a}\int_{m_1 a}^{m_2 b} f(x)dx - f\left(\frac{m_1 a + m_2 b}{2}\right)\right| \\ \leq & \ (m_2 b - m_1 a)\left\{\left[\frac{1}{(\alpha+1)(\alpha+2)}\left(1 - \frac{1}{2^{\alpha+1}}\right)\right] m_1 |f'(a)| \right. \\ & \left. + \left[\frac{1}{4} + \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2^{\alpha+1}} - 1\right)\right] m_2 |f'(b)|\right\} \end{aligned}$$

holds for $t \in [0,1]$ and $\alpha, m_1, m_2 \in (0,1]^3$.

Proof. Using Lemma 24 and the inequality

$$|f'(m_1ta+m_2(1-t)b)| \leq m_1t^{\alpha}|f'(a)| + m_2(1-t^{\alpha})|f'(b)|$$

we get

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq &\left| (m_2b-m_1a)\right. \\
&\left.\times\left[\int_0^{\frac{1}{2}} tf'(m_1ta+m_2(1-t)b)\,dt + \int_{\frac{1}{2}}^1 (t-1)f'(m_1ta+m_2(1-t)b)\,dt\right]\right| \\
\leq &(m_2b-m_1a)\left[\int_0^{\frac{1}{2}} |t|\left(m_1t^{\alpha}|f'(a)| + m_2(1-t^{\alpha})|f'(b)|\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^1 |t-1|\left(m_1t^{\alpha}|f'(a)| + m_2(1-t^{\alpha})|f'(b)|\right)dt\right] \\
\leq &(m_2b-m_1a)\left[\int_0^{\frac{1}{2}} \left(m_1t^{\alpha+1}|f'(a)| + m_2\left(t-t^{\alpha+1}\right)|f'(b)|\right)dt\right. \\
&\left.+\int_{\frac{1}{2}}^1 \left(m_1\left(t^{\alpha}-t^{\alpha+1}\right)f'(a) + m_2(1-t)(1-t^{\alpha})|f'(b)|\right)dt\right]
\end{aligned}$$

$$\begin{aligned}
= &(m_2b-m_1a)\left[\frac{m_1|f'(a)|}{(\alpha+2)2^{\alpha+2}} + \left(\frac{1}{8}-\frac{1}{(\alpha+2)2^{\alpha+2}}\right)m_2|f'(b)|\right. \\
&+\left(\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\alpha+3}{(\alpha+1)(\alpha+2)2^{\alpha+2}}\right)m_1|f'(a)| \\
&\left.+\left(\frac{1}{8}-\frac{1}{(\alpha+1)(\alpha+2)}+\frac{\alpha+3}{(\alpha+1)(\alpha+2)2^{\alpha+2}}\right)m_2|f'(b)|\right] \\
= &(m_2b-m_1a)\left\{\left[\frac{1}{(\alpha+1)(\alpha+2)}\left(1-\frac{1}{2^{\alpha+1}}\right)\right]m_1|f'(a)|\right. \\
&\left.+\left[\frac{1}{4}+\frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{1}{2^{\alpha+1}}-1\right)\right]m_2|f'(b)|\right\}
\end{aligned}$$

where

$$\begin{aligned}
\int_0^{\frac{1}{2}} t^{\alpha+1}dt &= \frac{1}{(\alpha+2)2^{\alpha+2}}, \\
\int_0^{\frac{1}{2}} \left(t-t^{\alpha+1}\right)dt &= \frac{1}{8}-\frac{1}{(\alpha+2)2^{\alpha+2}},
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-t^{\alpha+1}\right)dt &= \frac{1}{(\alpha+1)(\alpha+2)}-\frac{\alpha+3}{(\alpha+1)(\alpha+2)2^{\alpha+2}},\\
\int_{\frac{1}{2}}^{1}(1-t)(1-t^{\alpha})\,dt &= \frac{1}{8}-\frac{1}{(\alpha+1)(\alpha+2)}+\frac{\alpha+3}{(\alpha+1)(\alpha+2)2^{\alpha+2}}.
\end{aligned}$$

This completes the proof of theorem. □

Remark 26. *Under the conditions of Theorem 25, if we take $\alpha = m_1 = m_2 = 1$, then our result coincides with* [6].

Theorem 27. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $m_1a, m_2b \in I^\circ$ with $m_1a < m_2b$ and $f' \in L[m_1a, m_2b]$, and let $q > 1$. If $|f'|$ is (α, m_1, m_2)-convex on the interval $[m_1a, m_2b]$, then the following equality*

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx-f\left(\frac{m_1a+m_2b}{2}\right)\right|\\
\leq\ &(m_2b-m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left\{\left[\frac{m_1|f'(a)|^q-m_2|f'(b)|^q}{(\alpha+1)2^{\alpha+1}}+\frac{m_2|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\left(2^{\alpha+1}-1\right)\frac{\left(m_1|f'(a)|^q-m_2|f'(b)|^q\right)}{(\alpha+1)2^{\alpha+1}}+\frac{m_2|f'(b)|^q}{2}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

holds for $t \in [0,1]$ and $\alpha, m_1, m_2 \in (0,1]^3$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 24, Hölder's integral inequality and the inequality

$$|f'(m_1ta+m_2(1-t)b)|^q \leq m_1t^{\alpha}|f'(a)|^q+m_2(1-t^{\alpha})|f'(b)|^q,$$

we obtain

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b}f(x)dx-f\left(\frac{m_1a+m_2b}{2}\right)\right|\\
\leq\ &(m_2b-m_1a)\left[\int_0^{\frac{1}{2}}|t|\,|f'(m_1ta+m_2(1-t)b)|\,dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1}|t-1|\,|f'(m_1ta+m_2(1-t)b)|\,dt\right]\\
\leq\ &(m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|f'(m_1ta+m_2(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^{\frac{1}{2}}|t-1|^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}|m_1t^{\alpha}f'(a)+m_2(1-t^{\alpha})f'(b)|^q\,dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq (m_2b-m_1a) \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(m_1t^{\alpha}\left|f'(a)\right|^q+m_2(1-t^{\alpha})\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^1 |t-1|^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(m_1t^{\alpha}\left|f'(a)\right|^q+m_2(1-t^{\alpha})\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right] \\
&= (m_2b-m_1a)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\left\{\left[\frac{m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q}{(\alpha+1)2^{\alpha+1}}+\frac{m_2\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\left(2^{\alpha+1}-1\right)\frac{\left(m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q\right)}{(\alpha+1)\,2^{\alpha+1}}+\frac{m_2\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^{\frac{1}{2}} t^p dt &= \int_{\frac{1}{2}}^1 |t-1|^p\,dt=\frac{1}{(p+1)2^{p+1}} \\
\int_0^{\frac{1}{2}} t^{\alpha} dt &= \frac{1}{(\alpha+1)2^{\alpha+1}}, \qquad \int_0^{\frac{1}{2}}(1-t^{\alpha})\,dt=\frac{1}{2}-\frac{1}{(\alpha+1)2^{\alpha+1}}, \\
\int_{\frac{1}{2}}^1 t^{\alpha} dt &= \frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right),\ \int_{\frac{1}{2}}^1(1-t^{\alpha})dt=\frac{1}{2}-\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right).
\end{aligned}
$$

□

Remark 28. *Under the conditions of Theorem 25, if we take* $\alpha=m_1=m_2=1$ *, then our result coincides with* [6].

Theorem 29. *Let* $f: I^{\circ}\subseteq\mathbb{R}\rightarrow\mathbb{R}$ *be a differentiable mapping on* I°, $m_1a, m_2b\in I^{\circ}$ *with* $m_1a<m_2b$ *and* $f'\in L[m_1a,m_2b]$, *and let* $q\geq 1$. *If* $|f'|$ *is* (α,m_1,m_2)*-convex on the interval* $[m_1a,m_2b]$, *then the following equality*

$$
\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx-f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
&\leq (m_2b-m_1a)\,2^{\frac{3}{q}-3}\left[\left(\frac{m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q}{(\alpha+2)2^{\alpha+2}}+\frac{m_2\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q}{(\alpha+1)(\alpha+2)}\left(1-\frac{\alpha+3}{2^{\alpha+2}}\right)+\frac{m_2\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

holds for $t\in[0,1]$ *and* $\alpha,m_1,m_2\in(0,1]^3$, *where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 24, well-known power mean inequality and the inequality

$$\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q \leq m_1t^\alpha \left|f'(a)\right|^q + m_2(1-t^\alpha)\left|f'(b)\right|^q,$$

we get

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq\ & (m_2b-m_1a)\left[\int_0^{\frac{1}{2}} |t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt\right. \\
& \left.+\int_{\frac{1}{2}}^1 |t-1|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|dt\right] \\
\leq\ & (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}}|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t|\left|f'\left(m_1ta+m_2(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t-1|\left|m_1t^\alpha f'(a)+m_2(1-t^\alpha)f'(b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
\leq\ & (m_2b-m_1a)\left[\left(\int_0^{\frac{1}{2}}tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t\left[m_1t^\alpha\left|f'(a)\right|^q+m_2(1-t^\alpha)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t-1|\left[m_1t^\alpha\left|f'(a)\right|^q+m_2(1-t^\alpha)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right] \\
=\ & (m_2b-m_1a)\left[\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(m_1\left|f'(a)\right|^q\int_0^{\frac{1}{2}}t^{\alpha+1}dt+m_2\left|f'(b)\right|^q\int_0^{\frac{1}{2}}(t-t^{\alpha+1})dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(m_1\left|f'(a)\right|^q\int_{\frac{1}{2}}^1(1-t)t^\alpha dt+m_2\left|f'(b)\right|^q\int_{\frac{1}{2}}^1(1-t)(1-t^\alpha)dt\right)^{\frac{1}{q}}\right] \\
=\ & (m_2b-m_1a)\,2^{\frac{3}{q}-3}\left[\left(\frac{m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q}{(\alpha+2)2^{\alpha+2}}+\frac{m_2\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\frac{m_1\left|f'(a)\right|^q-m_2\left|f'(b)\right|^q}{(\alpha+1)(\alpha+2)}\left(1-\frac{\alpha+3}{2^{\alpha+2}}\right)+\frac{m_2\left|f'(b)\right|^q}{8}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where integrals can be calculated as above. □

Corollary 30. *Under the conditions of Theorem 29, if we take $q=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{m_2b-m_1a}\int_{m_1a}^{m_2b} f(x)dx - f\left(\frac{m_1a+m_2b}{2}\right)\right| \\
\leq\ & (m_2b-m_1a)\left[\frac{m_1\left|f'(a)\right|}{(\alpha+1)(\alpha+2)}\left(1-\frac{1}{2^{\alpha+1}}\right)\right. \\
&\left.+m_2\left|f'(b)\right|\left(\frac{1}{4}-\frac{1}{(\alpha+1)(\alpha+2)}\left(1-\frac{1}{2^{\alpha+1}}\right)\right)\right]
\end{aligned}$$

References

[1] Bakula, M. K., Özdemir, M. E. and Pečarić, J. Hadamard tpye inequalities for m-convex and (α, m) -convex functions, *J. Inequal. Pure and Appl. Math.*, 9(4)(2008), 1-25.

[2] Dragomir, S.S. On Some New Inequalities of Hermite-Hadamard Type for m-Convex Functions, *Tamkang Journal of Mathematics*, Volume 33, Number 1, Spring 2002, 45–55.

[3] Dragomir, S.S. and Pearce, CEM. Selected Topics on Hermite-Hadamard Inequalities and Its Applications, RGMIA Monograph (2002).

[4] Kadakal, H., (m_1, m_2)-convexity and some new Hermite-Hadamard type inequalities, *Journal of Applied and Engineering Mathematics*, (accepted for publications), 2019.

[5] Kavurmaci, H., Avci M. and Özdemir, M.E. New inequalities of Hermite-Hadamard type for convex functions with applications, *Journal of Inequalities and Applications*, 2011, 2011:86.

[6] Kirmaci, U. S., Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. and Comput.*, 147 (2004) 137–146.

[7] Kırmacı, U. S., Bakula, M. K., Özdemir, M. E. and Pe čarić, J. Hadamard-tpye inequalities for s-convex functions, *Appl. Math. and Comp.*, 193(2007), 26–35.

[8] Lara, T., Rosales, E. and Sánchez, J.L. New Properties of m-Convex Functions, *International Journal of Mathematical Analysis*, Vol. 9, 2015, No. 15, 735–742.

[9] Mihaly, B. Hermite-Hadamard-type inequalities for generalized convex functions. *J. Inequal, Pure Appl. Math.*, 9(3), Article ID 63 (2008) (PhD thesis).

[10] Özdemir, M. E., Latif M. A. and Akdemir, A. O. On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates, *Journal of Inequalities and Applications*, 2012, 2012:21.

[11] Özdemir, M. E., Set E. and Sarıkaya, M. Z. Some new Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions, *Hacettepe J. of. Math. and Statistics*, 40(2011), 219–229.

[12] Park, J., New Ostrowski-Like type inequalities for differentiable (s, m)-convex mappings, *International Journal of Pure and Applied Mathematics*, Volume 78 No. 8 2012, 1077–1089.

[13] Pečarić, J. E., Proschan, F and Tong, Y. L. Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, INC, 1992.

[14] Sarikaya, M.Z., Sağlam A. and Yıldırım, H. On some Hadamard-type inequalities for h-convex functions, *J. Math. Ineq.*, 2(3)(2008), 335–341.

[15] Toader, G. The hierarchy of convexity and some classic inequalities, *J. Math. Ineq.*,3(3)(2009), 305–313.

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