

Differential Relations for the Solutions to the NLS Equation and Their Different Representations

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Abstract

Solutions to the focusing nonlinear Schrödinger equation (NLS) of order N depending on $2N - 2$ real parameters in terms of wronskians and Fredholm determinants are given. These solutions give families of quasi-rational solutions to the NLS equation denoted by v_N and have been explicitly constructed until order $N = 13$. These solutions appear as deformations of the Peregrine breather P_N as they can be obtained when all parameters are equal to 0. These quasi rational solutions can be expressed as a quotient of two polynomials of degree $N(N + 1)$ in the variables x and t and the maximum of the modulus of the Peregrine breather of order N is equal to $2N + 1$. Here we give some relations between solutions to this equation. In particular, we present a connection between the modulus of these solutions and the denominator part of their rational expressions. Some relations between numerator and denominator of the Peregrine breather are presented.

Keywords: Fredholm determinants, NLS equation, Peregrine breathers, Rogue waves, Wronskians
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1. Introduction

We consider the one dimensional focusing nonlinear Schrödinger equation (NLS) which can be written in the form

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (1.1)$$

The first results concerning the NLS equation date from the works of Zakharov and Shabat in 1972 who solved it using the inverse scattering method [1, 2]. Its and Kotlyarov first constructed periodic and almost periodic algebro-geometric solutions to the focusing NLS equation in 1976 [3, 4]. Ma found in 1979 the first breather type solution of the NLS equation [5]. In 1983, the first quasi rational solutions of NLS equation were constructed by Peregrine [6]. In 1986, Eleonski, Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and got the first higher order analogue of the Peregrine breather [7, 8, 9]. Other analogues of the Peregrine breathers of order 3 and 4 were constructed using Darboux transformations, in a series of articles by Akhmediev et al. [10, 11, 12, 13].

Recently, many works about NLS equation have been published using different methods. We can quote the works of Matveev et al. [14, 15] in 2010 for the representation of the solutions in terms of wronskians; those of Gaillard [16, 17, 18] for the solutions given in terms of wronskians and Fredholm determinants, and their quasi-rational solutions limit of order N depending on $2N - 2$ parameters. Akhmediev gave quasi rational solutions using Darboux transformation in several papers [19, 20, 21]. Guo, Ling and Liu in 2012 gave an other representation of the solutions as a ratio of two determinants [22] using generalized Darboux transformation. A new approach has been done by Ohta and Yang in [23] using Hirota bilinear method. Smirnov [24]

gave solutions with an algebro-geometric approach. Other types of solutions were given by Zhao et al. in [25].

We give some relations between the modulus of these solutions and the denominator part of their rational expression. Some relations between numerator and denominator of the rational solutions are given.

2. Different representations of solutions to the NLS equation

2.1 Solutions of the NLS equation in terms of of Fredholm determinant

We have to define the following notations.

The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of the parameters $\lambda_\nu, 1 \leq \nu \leq 2N$; they are defined by the formulas :

$$\begin{aligned} \kappa_\nu &= 2\sqrt{1-\lambda_\nu^2}, & \delta_\nu &= \kappa_\nu \lambda_\nu, & \gamma_\nu &= \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}},; \\ x_{r,\nu} &= (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, & r &= 1, 3. \end{aligned} \quad (2.1)$$

The parameters $-1 < \lambda_\nu < 1, \nu = 1, \dots, 2N$, are real numbers such that

$$\begin{aligned} -1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1 \\ \lambda_{N+j} &= -\lambda_j, & j &= 1, \dots, N. \end{aligned} \quad (2.2)$$

The condition (2.2) implies that

$$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}, \quad j = 1, \dots, N.$$

Complex numbers $e_\nu, 1 \leq \nu \leq 2N$ are defined in the following way :

$$\begin{aligned} e_j &= i \sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} - \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ e_{j+N} &= i \sum_{l=1}^{N-1} a_l (j\varepsilon)^{2l+1} + \sum_{l=1}^{N-1} b_l (j\varepsilon)^{2l+1}, \\ 1 &\leq j \leq N-1. \end{aligned} \quad (2.3)$$

$\varepsilon, a_\nu, b_\nu, \nu = 1 \dots 2N$ are arbitrary real numbers.

Let I be the unit matrix, and

$$\varepsilon_j = j \quad 1 \leq j \leq N, \quad \varepsilon_j = N + j, \quad N + 1 \leq j \leq 2N.$$

Let's consider the matrix $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$ defined by :

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \quad (2.4)$$

With these notations, the solution to the NLS equation takes the form [16, 17, 18] :

Theorem 2.1. *The function v defined by*

$$v(x, t) = \frac{\det(I + D_3(x, t))}{\det(I + D_1(x, t))} e^{2it - i\varphi}.$$

is a solution to the focusing NLS equation depending on $2N - 1$ real parameters $a_j, b_j, \varepsilon, 1 \leq j \leq N - 1$ with the matrix $D_r = (d_{jk}^{(r)})_{1 \leq j, k \leq 2N}$ defined by

$$d_{\nu\mu}^{(r)} = (-1)^{\varepsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu).$$

where $\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$ being defined in (2.1), (2.2) and (2.3).

2.2 Wronskian representation

For this, we need to define the following notations :

$$\phi_{r,v} = \sin \Theta_{r,v}, \quad 1 \leq v \leq N, \quad \phi_{r,v} = \cos \Theta_{r,v}, \quad N+1 \leq v \leq 2N, \quad r = 1, 3,$$

with the arguments

$$\Theta_{r,v} = \kappa_v x / 2 + i \delta_v t - i x_{r,v} / 2 + \gamma_v y - i e_v / 2, \quad 1 \leq v \leq 2N.$$

The functions $\phi_{r,v}$ are defined by

$$\phi_{r,v} = \sin \Theta_{r,v}, \quad 1 \leq v \leq N, \quad \phi_{r,v} = \cos \Theta_{r,v}, \quad N+1 \leq v \leq 2N, \quad r = 1, 3, \quad (2.5)$$

We denote $W_r(y)$ the wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,v})_{v,\mu \in [1, \dots, 2N]}].$$

We consider the matrix $D_r = (d_{v\mu})_{v,\mu \in [1, \dots, 2N]}$ defined in (2.4). Then we have the following statement [17] :

Theorem 2.2.

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0),$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{v=1}^{2N} \Theta_{r,v})}{\prod_{v=2}^{2N} \prod_{\mu=1}^{v-1} (\gamma_v - \gamma_\mu)}.$$

With these notations, we have the following result [17] :

Theorem 2.3. The function v defined by

$$v(x, t) = \frac{W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0)} e^{2it - i\varphi}.$$

is a solution to the focusing NLS equation depending on $2N - 1$ real parameters $a_j, b_j, \varepsilon, 1 \leq j \leq N - 1$ with ϕ_v^r defined in (2.5)

$$\begin{aligned} \phi_{r,v} &= \sin(\kappa_v x / 2 + i \delta_v t - i x_{r,v} / 2 + \gamma_v y - i e_v / 2), \quad 1 \leq v \leq N, \\ \phi_{r,v} &= \cos(\kappa_v x / 2 + i \delta_v t - i x_{r,v} / 2 + \gamma_v y - i e_v / 2), \quad N+1 \leq v \leq 2N, \quad r = 1, 3, \end{aligned}$$

$\kappa_v, \delta_v, x_{r,v}, \gamma_v, e_v$ being defined in (2.1), (2.2) and (2.3).

We can give another representation of the solutions to the NLS equation depending only on terms $\gamma_v, 1 \leq v \leq 2N$. From the relations (2.1), we can express the terms κ_v, δ_v and $x_{r,v}$ in function of γ_v , for $1 \leq v \leq 2N$ and we obtain :

$$\begin{aligned} \kappa_j &= \frac{4\gamma_j}{(1 + \gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1 - \gamma_j^2)}{(1 + \gamma_j^2)^2}, \quad x_{r,j} = (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ \kappa_j &= \frac{4\gamma_j}{(1 + \gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1 - \gamma_j^2)}{(1 + \gamma_j^2)^2}, \quad x_{r,j} = (r-1) \ln \frac{\gamma_j + i}{\gamma_j - i}, \quad N+1 \leq j \leq 2N. \end{aligned}$$

We have the following new representation [17, 26] :

Theorem 2.4. The function v defined by

$$v(x, t) = \frac{\det[(\partial_y^{\mu-1} \tilde{\phi}_{3,v}(0))_{v,\mu \in [1, \dots, 2N]}]}{\det[(\partial_y^{\mu-1} \tilde{\phi}_{1,v}(0))_{v,\mu \in [1, \dots, 2N]}]} e^{2it - i\varphi} \quad (2.6)$$

is a solution to the NLS equation (1.1) depending on $2N - 1$ real parameters $a_j, b_j, \varepsilon, 1 \leq j \leq N - 1$. The functions $\tilde{\phi}_{r,v}$ are defined by

$$\begin{aligned}\tilde{\phi}_{r,j}(y) &= \sin \left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i \frac{(r-1)}{2} \ln \frac{\gamma_j - i}{\gamma_j + i} + \gamma_j y - ie_j \right), \\ \tilde{\phi}_{r,N+j}(y) &= \cos \left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i \frac{(r-1)}{2} \ln \frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j}y - ie_{N+j} \right),\end{aligned}\tag{2.7}$$

where $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, 1 \leq j \leq N$.

λ_j is an arbitrary real parameter such that $0 < \lambda_j < 1, \lambda_{N+j} = -\lambda_j, 1 \leq j \leq N$.

The terms e_v are defined by (2.3),

where a_j and b_j are arbitrary real numbers, $1 \leq j \leq N - 1$.

Remark 2.1. In the formula (2.6), the determinants $\det[(\partial_y^{\mu-1} f_v(0))_{v,\mu \in [1,\dots,2N]}]$ are the wronskians of the functions f_1, \dots, f_{2N} evaluated in $y = 0$. In particular $\partial_y^0 f_v$ means f_v .

2.3 Families of quasi-rational solutions of NLS equation in terms of a quotient of two determinants

The following notations are used :

$$X_v = \kappa_v x/2 + i\delta_v t - ix_{3,v}/2 - ie_v/2,$$

$$Y_v = \kappa_v x/2 + i\delta_v t - ix_{1,v}/2 - ie_v/2,$$

for $1 \leq v \leq 2N$, with $\kappa_v, \delta_v, x_{r,v}$ defined in (2.1).

Parameters e_v are defined by (2.3).

Below the following functions are used :

$$\begin{aligned}\varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k,\end{aligned}\tag{2.8}$$

for $1 \leq k \leq N$, and

$$\begin{aligned}\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k},\end{aligned}\tag{2.9}$$

for $1 \leq k \leq N$.

We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N, 1 \leq k \leq 2N$ in the same way, the term X_k is only replaced by Y_k .

$$\begin{aligned}\psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k,\end{aligned}\tag{2.10}$$

for $1 \leq k \leq N$, and

$$\begin{aligned}\psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k},\end{aligned}\tag{2.11}$$

for $1 \leq k \leq N$.

Then we get the following result [26] :

Theorem 2.5. The function v defined by

$$v(x, t) = \frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} e^{2it - i\varphi}$$

is a quasi-rational solution of the NLS equation (1.1) depending on $2N - 2$ real parameters $a_j, b_j, 1 \leq j \leq N - 1$, where

$$\begin{aligned} n_{j1} &= \varphi_{j,1}(x,t,0), 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ n_{jN+1} &= \varphi_{j,N+1}(x,t,0), 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{j1} &= \psi_{j,1}(x,t,0), 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ d_{jN+1} &= \psi_{j,N+1}(x,t,0), 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \varepsilon^{2k-2}}(x,t,0), \\ & & & 2 \leq k \leq N, 1 \leq j \leq 2N \end{aligned}$$

The functions φ and ψ are defined in (2.8), (2.9), (2.10), (2.11).

3. Structure of the multi-parametric solutions to the NLS equation of order N depending on $2N - 2$ parameters

3.1 The quotient of two polynomials of degree $(N(N+1))$ in x and t by an exponential depending on t

Here we present a result which states the structure of the quasi-rational solutions of the NLS equation. It was only conjectured in preceding works [16, 18]. Moreover we obtain here families of deformations of the N th Peregrine breather depending on $2N - 2$ parameters.

In this section we use the notations defined in the previous sections. The functions φ and ψ are defined in (2.8), (2.9), (2.10), (2.11).

The structure of the quasi rational solutions to the NLS equation is given by [27] :

Theorem 3.1. *The function v defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} e^{2it - i\varphi} \quad (3.1)$$

is a quasi-rational solution of the NLS equation (1.1) quotient of two polynomials $R(x,t)$ and $S(x,t)$ depending on $2N - 2$ real parameters a_j and $b_j, 1 \leq j \leq N - 1$.

$R(x,t)$ and $S(x,t)$ are polynomials of degrees $N(N+1)$ in x and t .

Remark 3.1. *The polynomials $R(x,t)$ and $S(x,t)$ have the same coefficients of degrees $N(N+1)$ in $2x$ and $4t$ equal to 1. The polynomial $B(x,t)$ does not have any real root.*

3.2 The structure of the Peregrine breather of order n

There is any freedom to choose γ_j in such a way that the conditions on λ_j are checked. We know from previous works [16, 18] that the (analogue) Peregrine breathers are obtained when all the parameters a_j and b_j are equal to 0. In order to get the more simple expressions in the determinants, we choose particular solutions in the previous families.

Here we choose $\gamma_j = j\varepsilon$ as simple as possible in order to have the conditions on λ_j checked, and we have [26, 27] :

Theorem 3.2. *The function v_0 defined by*

$$v_{n,0}(x,t) = \left(\frac{\det((n_{jk})_{j,k \in [1,2N]})}{\det((d_{jk})_{j,k \in [1,2N]})} e^{2it - i\varphi} \right)_{(a_j=b_j=0, 1 \leq j \leq N-1)} \quad (3.2)$$

is the Peregrine breather of order N solution of the NLS equation (1.1) whose highest amplitude in modulus is equal to $2N + 1$.

Remark 3.2. *The previous result is given in the frame where the limit of the modulus of the solution when x or t tend to infinity is equal to 1. We know that if $v(x,t)$ is a solution to the NLS equation then $u(x,t) = av(ax, a^2t)$ is also a solution to the NLS equation, for any arbitrary real a .*

Remark 3.3. *In (3.2), the matrices $(n_{jk})_{j,k \in [1,2N]}$ and $(d_{jk})_{j,k \in [1,2N]}$ are defined in (3.1).*

We have seen in previous section that solutions of NLS equation given by (2.7) can be written in function uniquely of terms γ . We recall that the terms γ_j are given by $\gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $1 \leq j \leq N$; λ_j is an arbitrary real parameter such that $0 < \lambda_j < 1$, $\lambda_{N+j} = -\lambda_j$, $1 \leq j \leq N$. We can rewrite the result given in (2.7) in a simplest formulation as follows [26, 27] :

Theorem 3.3. *The function v defined by*

$$v(x,t) = \frac{\det((f_{jk}^{(3)})_{j,k \in [1,2N]})}{\det((f_{jk}^{(1)})_{j,k \in [1,2N]})} e^{2it - i\varphi}$$

is a quasi-rational solution of the NLS equation (1.1) depending on $2N - 2$ real parameters a_j, b_j , $1 \leq j \leq N - 1$ where

$$\begin{aligned} f_{jk}^{(r)} &= \frac{\partial^{2(k-1)}}{\partial \varepsilon^{2(k-1)}} \left(\gamma^{A^{j-1}} \sin \left[\frac{2\gamma}{1+\gamma^2} x + 4i \frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2} t - i \frac{r-1}{2} \ln \frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1} (a_l + ib_l) \varepsilon^{2l+1} + (j-1) \frac{\pi}{2} \right] \right)_{(\varepsilon=0)}, \\ f_{jN+k}^{(r)} &= \frac{\partial^{2(k-1)}}{\partial \varepsilon^{2(k-1)}} \left(\gamma^{2N-4j-1} \cos \left[\frac{2\gamma}{1+\gamma^2} x - 4i \frac{\gamma(1-\gamma^2)}{(1+\gamma^2)^2} t + i \frac{r-1}{2} \ln \frac{\gamma-i}{\gamma+i} + \sum_{l=1}^{N-1} (a_l - ib_l) \varepsilon^{2l+1} + (j-1) \frac{\pi}{2} \right] \right)_{(\varepsilon=0)}, \\ 1 \leq k \leq N, \quad 1 \leq j \leq 2N, \quad r \in \{1,3\}, \quad \varepsilon \in]0,1[, \quad \gamma = \varepsilon(1-\varepsilon^2)^{1/2}. \end{aligned}$$

Remark 3.4. *In the previous theorem, the expression $\frac{\partial^0}{\partial \varepsilon^0} f(x)$ means $f(x)$.*

The solution to the NLS equation can be written in the form

$$v_N(x,t) = \frac{R_N(x,t)}{S_N(x,t)} e^{2it} = \left(1 + \frac{A_N(x,t)}{B_N(x,t)} \right) e^{2it} \tag{3.3}$$

and the Peregrine breather in the form

$$v_{N,0}(x,t) = \frac{T_N(x,t)}{U_N(x,t)} e^{2it} = \left(1 + \frac{P_N(x,t)}{Q_N(x,t)} \right) e^{2it} \tag{3.4}$$

where the index 0 means that all the parameters are equal to 0.

4. Differential relation for the NLS equation

In previous works [26, 27], we have proven that the solutions v_N to the NLS equation can be written in the form

$$v_N(x,t) = \left(1 + \frac{A_N(x,t)}{B_N(x,t)} \right) e^{2it}.$$

We have a very simple relation between the square of the modulus of v_N and the denominator part B_N . This relation appears in a paper of Ling and Zhao [25] where the solutions to the NLS equation are given in the frame of the generalized Darboux transformation. Here this result and its proof are given in a general frame by the following theorem :

Theorem 4.1. *The solutions $v_N(x,t) = \left(1 + \frac{A_N(x,t)}{B_N(x,t)} \right) e^{2it}$ to the NLS equation verify the following relation*

$$|v_N(x,t)|^2 = 1 + (\ln B_N(x,t))_{xx}, \tag{4.1}$$

where the subscript $_{xx}$ means the double derivation with respect to x .

Proof. For simplicity with omit the references to N and (x,t) to the solution v and the polynomials A and B . If we substitute v by $\left(1 + \frac{A}{B} e^{2it} \right)$ in the expression $X = iv_t + v_{xx} + 2|v|^2 v$, we get

$$\begin{aligned} & \frac{2}{B^3} (|A+B|^2(A+B) + AB_x^2 - AB_{xx}B) \\ & + \frac{1}{B^2} (i(BA_t - AB_t) - 2B^2 - 2AB - 2A_x B_x + AB_{xx} + BA_{xx}) = 0 \end{aligned}$$

This can give the two following relations

$$|A+B|^2(A+B) + (A+B)B_x^2 - (A+B)B_{xx}B - (A+B)B^2 = 0$$

and

$$i(A_t B - AB_t) + (A_{xx}B - 2A_x B_x + AB_{xx}) + (B_{xx}B - B_x^2) = 0$$

The first relation can be rewritten as

$$|A+B|^2 + B_x^2 - B_{xx}B - B^2 = 0$$

Then the square of the modulus of v_N can be written as

$$|v_N|^2 = \frac{|A+B|^2}{B^2} = 1 + (\ln B)_{xx},$$

which proves relation (4.1). □

5. Relations between rational part of the solutions to the NLS equation

With the preceding notations, we get the following statement

Theorem 5.1. *The polynomials of the solutions v_N to the NLS equation defined by (3.3) $v_N(x,t) = \frac{R_N(x,t)}{S_N(x,t)} e^{2it}$ verify the following relations*

$$\begin{aligned} & (i(R_N)_t + (R_N)_{xx} - 2R_N)S_N^2 - ((S_N)_{xx} + i(S_N)_t)R_N S_N \\ & - 2(R_N)_x(S_N)_x S_N + 2((S_N)_x)^2 + R_N \overline{R_N} R_N = 0. \end{aligned}$$

Proof. It is sufficient to replace in the equation (1.1) $v_N(x,t)$ by $\frac{R_N(x,t)}{S_N(x,t)} e^{2it}$. □

Proposition 5.1. *The coordinates of extrema (x_0, t_0) of solutions v_N to the NLS equation defined by (3.3) $v_N(x,t) = \frac{R_N(x,t)}{S_N(x,t)} e^{2it}$ verify the following relations*

$$\begin{aligned} & (R_N)_x(x_0, t_0) \overline{R_N}(x_0, t_0) S_N(x_0, t_0) + (\overline{R_N})_x(x_0, t_0) R_N(x_0, t_0) S_N(x_0, t_0) \\ & - 2(S_N)_x(x_0, t_0) R_N(x_0, t_0) \overline{R_N}(x_0, t_0) = 0, \end{aligned}$$

$$\begin{aligned} & (R_N)_t(x_0, t_0) \overline{R_N}(x_0, t_0) S_N(x_0, t_0) + (\overline{R_N})_t(x_0, t_0) R_N(x_0, t_0) S_N(x_0, t_0) \\ & - 2(S_N)_t(x_0, t_0) R_N(x_0, t_0) \overline{R_N}(x_0, t_0) = 0. \end{aligned}$$

$$(R_N)_x(x_0, t_0) S_N(x_0, t_0) - (S_N)_x(x_0, t_0) R_N(x_0, t_0) = 0.$$

$$(R_N)_t(x_0, t_0) S_N(x_0, t_0) - (S_N)_t(x_0, t_0) R_N(x_0, t_0) + 2i S_N(x_0, t_0) R_N(x_0, t_0) = 0.$$

where \bar{a} means the complex conjugate of a .

Proof. It is sufficient to compute the partial derivatives of (1.1) $v_N(x,t)$ defined by $\frac{R_N(x,t)}{S_N(x,t)} e^{2it}$. □

Remark 5.1. *As a consequence of the result on the highest modulus of the P_N breather defined by (3.4) $v_{N,0}(x,t) = \frac{T_N(x,t)}{U_N(x,t)} e^{2it}$, we get*

$$T_N(0,0) = (2N+1)U_N(0,0).$$

6. Conclusion

Different representations of the solutions to the NLS equation have been summarized in this paper, as well as the structure of the quasi rational solutions. Some differential relations have been given in this text for the NLS equation.

From different studies realized by the author, [26]-[32], it seems that the maximums of the modulus of the solutions to the NLS equation are in connection with the zeros of the Yablonski-Vorob'ev polynomials [33, 34].

It would be relevant to study this conjecture.

It would be also relevant to search other types of equations verified by the polynomials (P_N, Q_N) , (R_N, S_N) , (A_N, B_N) or (T_N, U_N) .

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