



Graphical calculus of Hopf crossed modules

Kadir Emir 

Department of Mathematics and Statistics, Masaryk University, Brno, Czech Republic

Abstract

We give the graphical notion of crossed modules of Hopf algebras-will be called Hopf crossed modules for short- in a symmetric monoidal category. We use the web proof assistant Globular to visualize our (colored) string diagrams. As an application, we introduce the homotopy of Hopf crossed module maps via Globular, and give some of its functorial relations.

Mathematics Subject Classification (2010). 16T05, 16S40, 18D05, 18D10

Keywords. Globular, Hopf crossed module, symmetric monoidal category, homotopy

1. Introduction

A crossed module of groups is given by a group homomorphism $\partial: E \rightarrow G$ together with an action \triangleright of G on E satisfying the following relations for all $e, f \in E$, and $g \in G$:

$$\partial(g \triangleright e) = g \partial(e) g^{-1} \quad \text{and} \quad \partial(e) \triangleright f = e f e^{-1}.$$

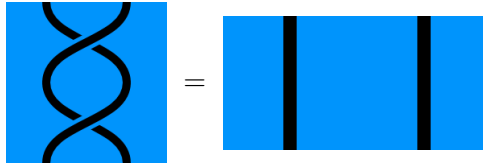
The essential example of a crossed module comes from a normal subgroup $N \trianglelefteq G$ with an inclusion map where the action is defined by conjugation. Thus crossed modules can be considered as a generalization of normal subgroups. Crossed modules are introduced by Whitehead [24] as an algebraic model for homotopy 2-types. Another result is that, the category of crossed modules is also naturally equivalent to that of cat^1 -groups proven in [14]. This property leads to a groupoid structure as an example of strict 2-groups. Crossed modules are also appear in the context of simplicial homotopy theory, since they are equivalent to the simplicial groups with Moore complex of length one. An essential result of this equivalence is that the homotopy category of n -types is equivalent to the homotopy category of simplicial groups with Moore complex of length $n - 1$, which are also the algebraic models for n -types.

A braided monoidal category [13] (or braided tensor category) is a monoidal category [21] with an isomorphism $\tau_{x,y}: A \otimes B \rightarrow B \otimes A$ called "braiding" satisfying the certain hexagon diagrams. Graphically this braiding will be denoted by:



(1.1)

where we have made the convention that the diagrams are to be read from top to bottom. Furthermore, a braided monoidal category will be called a symmetric monoidal category if the braiding (1.1) has the following property:



The well-known examples of symmetric monoidal categories are Vect, Set, and Cat.

A Hopf algebra [22] can be considered as an abstraction of the group algebra (of a group) and the universal enveloping algebra (of a Lie algebra). There exists a wide variety of variations of Hopf algebras by relaxing its properties or adding some extra structures, for instance, quasi-Hopf algebras [2, 6], quasi-triangular Hopf algebras [17], quantum groups [17, 18], the Leibniz-Hopf algebras and its dual [3, 4, 12], the Steenrod algebras [5, 19], Hopfish algebras [23], etc. Most of these notions make sense via Tannaka duality by the properties and structures on the corresponding categories. Hopf algebras are first defined over vector spaces. However, as a consequence of the relation between the category of vector spaces and monoidal categories, any Hopf algebra can also be defined over both symmetric monoidal categories and (arbitrary) braided monoidal categories; see [16] for details. The notion of Hopf crossed modules is introduced by Majid in [15] which is given by a Hopf algebra morphism $\partial: I \rightarrow H$ where I is an H -module algebra and coalgebra satisfying two Peiffer relations with an additional compatibility law that is less restricted case than being cocommutative. For any Hopf algebra H , an element $x \in H$ is said to be

- Group-like, if $\Delta(x) = x \otimes x$,
- Primitive, if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Therefore, we have the functors

$$(\)_{\text{gl}}^*: \{\text{Hopf Algebras}\} \rightarrow \{\text{Groups}\}$$

$$\text{Prim}: \{\text{Hopf Algebras}\} \rightarrow \{\text{Lie Algebras}\}$$

which preserve the crossed module structure [11] and can be extended to the corresponding functors

$$(\)_{\text{gl}}^*: \{\text{Hopf Crossed Modules}\} \rightarrow \{\text{Group Crossed Modules}\}$$


$$\text{Prim}: \{\text{Hopf Crossed Modules}\} \rightarrow \{\text{Lie Algebra Crossed Modules}\}.$$

Globular [1] is an online proof assistant for finitely-presented semistrict globular higher categories which currently operates up to the level of 4-categories. It allows one to formalize higher-categorical proofs and visualize them as string diagrams, and share them in public.

In this paper, we present the graphical calculus of Hopf crossed modules over any symmetric monoidal category, by using (colored) string diagrams via Globular. As an application, we introduce the homotopy of Hopf crossed module morphisms and prove that the functors $(\)_{\text{gl}}^*$ and Prim preserve the homotopy, from which we get the corresponding groupoid functors.

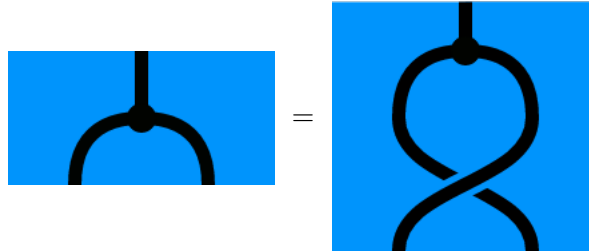
2. Hopf algebraic conventions

We recall some notions from [15, 16] in a graphical point of view. From now on, \mathcal{C} will be a fixed symmetric monoidal category and all Hopf algebras will be defined over it. We use the convention appears in [8].

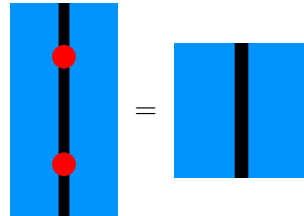
Definition 2.1. We picture any Hopf algebra by  and denote its product (\cdot) , co-product (Δ) , unit (1) , counit (ϵ) , and antipode (S) by



respectively, such that the Hopf algebra axioms [16] are satisfied. Any Hopf algebra is said to be cocommutative if and only if



Moreover, if a Hopf algebra is cocommutative (or commutative), then we have



2.1. Hopf algebra action

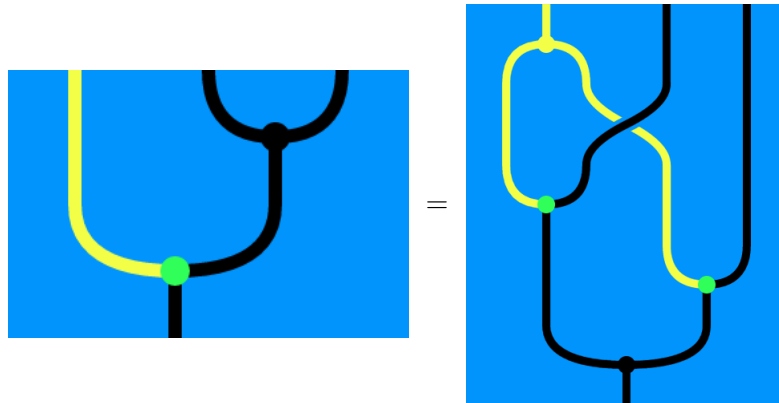
Suppose that I is a bialgebra (not necessarily a Hopf algebra) and H is a Hopf algebra. We say that I is an H -module algebra if there exists a left action $\rho: H \otimes I \rightarrow I$ of H on I



such that the following conditions hold:

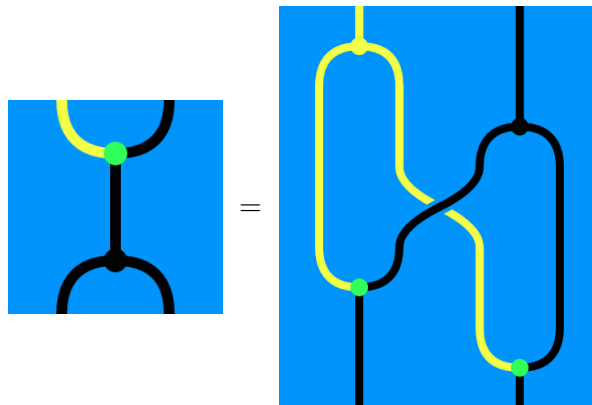
- and
-

•

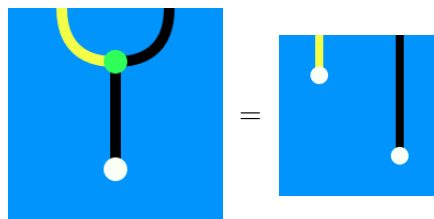


Moreover ρ makes I an H -module coalgebra if it further satisfies

•



•



In the above context, H -module algebra means a monoid in the category of H -modules. This requires the multiplication $I \otimes I \rightarrow I$ to be an H -module morphism which yields the corresponding conditions above. Similarly, an H -module coalgebra means a comonoid in the same category, equivalently the action $\rho: H \otimes I \rightarrow I$ needs to be a coalgebra morphism.

Definition 2.2 (Adjoint Action). If H is a cocommutative Hopf algebra, then H itself has a natural H -module algebra and coalgebra structure which is given by the "adjoint

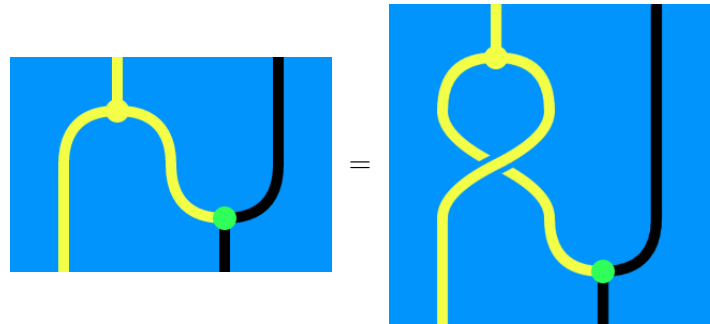
action"

(2.1)

Remark 2.3. In the cocommutative case, we have:

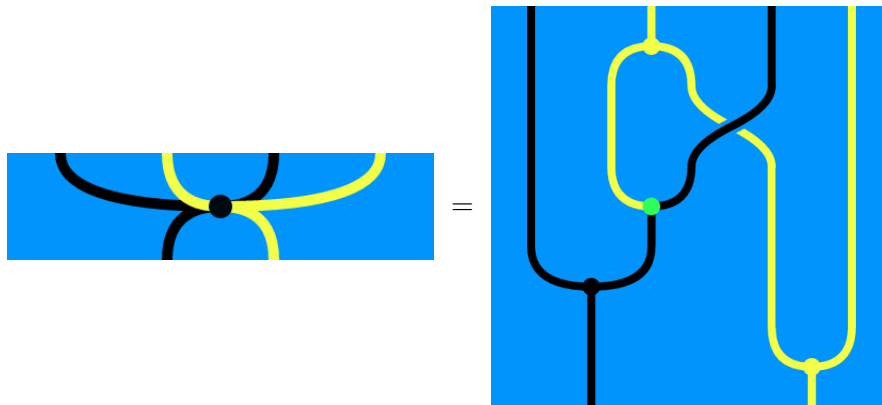
which is proven in [7]. However, it is not true in a general context because of the following equality:

Definition 2.4 (Smash Product). If I is an H -module algebra and coalgebra with action $\rho: H \otimes I \rightarrow I$ satisfying the following compatibility condition:

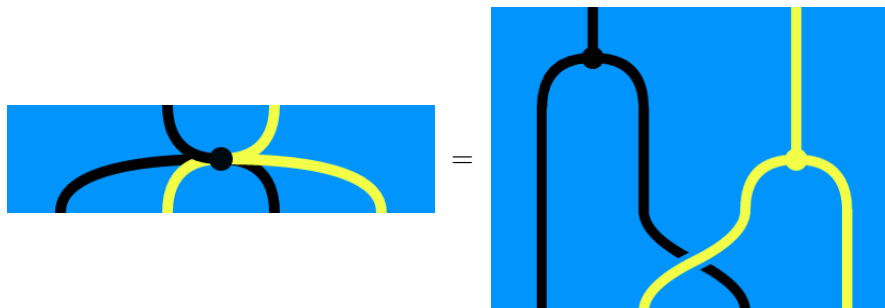

(2.2)

then we have the smash product Hopf algebra $I \otimes_{\rho} H$ with the underlying tensor product $I \otimes H$, where

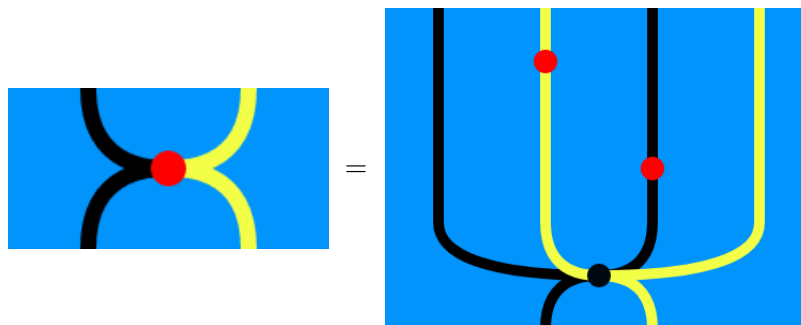
- Product:



- Coproduct:



- Antipode:

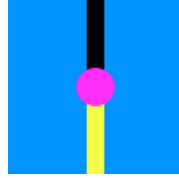


Remark that, we need the compatibility condition to make the coproduct an algebra morphism.

3. Hopf Crossed Modules

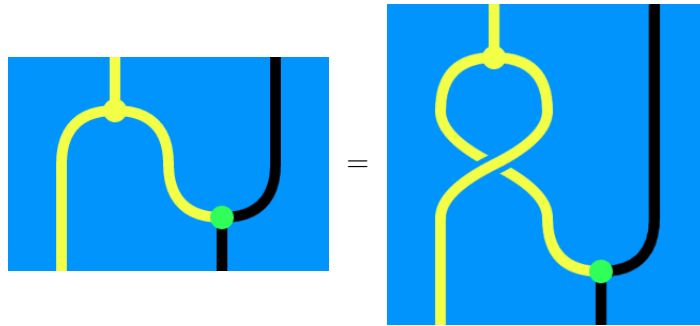
The non-diagrammatic version of the following notions appear in [11, 15].

Definition 3.1. A Hopf crossed module is a Hopf algebra morphism $\partial: I \rightarrow H$:



where I is an H -module algebra and coalgebra such that the followings hold:

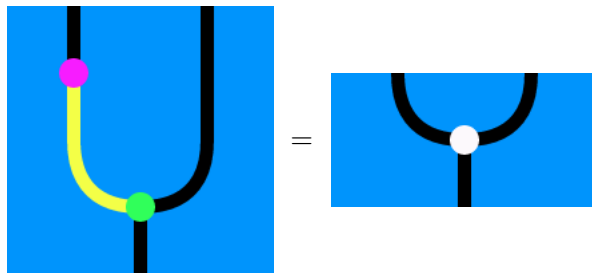
- Compatibility condition:



- Pre-crossed module condition:



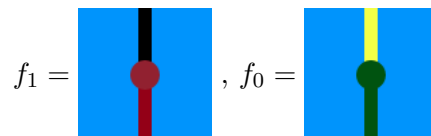
- Peiffer identity (crossed module condition):



Let $\partial: I \rightarrow H$ and $\partial': I' \rightarrow H'$ be two Hopf crossed modules



respectively. A crossed module morphism is a pair



of Hopf algebra morphisms $f_1: I \rightarrow I'$ and $f_0: H \rightarrow H'$ such that

•

(3.1)

•

(3.2)

Therefore we have the category of Hopf crossed modules.

3.1. Strict Quantum 2-Groups

A strict quantum 2-group is a pair of Hopf algebras (H_1, H_0) with the Hopf algebra morphisms $s, t: H_1 \rightarrow H_0$ and $e: H_0 \rightarrow H_1$ respectively,



which satisfies

together with an associative product which forms an embedded quantum groupoid; see [15] for details.

Remark 3.2. For a given Hopf crossed module $\partial: I \rightarrow H$, there exists a strict quantum 2-group (H_1, H) where H_1 is the smash product. However the converse is not true; namely we cannot get a Hopf crossed module structure via a strict quantum 2-group. At this point the braided crossed module notion appears, see [15, 20] for more details.

4. Globular Application: Homotopy Theory

From now on, all Hopf algebras will be colored black. Moreover, we just consider the category of cocommutative Hopf algebras over a symmetric monoidal category. This restriction will lead us to generalize the homotopy theory of crossed modules of groups and of Lie algebras in the sense of the functors:

$$(\)_{\text{gl}}^*: \{\text{Hopf Algebras}\} \rightarrow \{\text{Groups}\}, \quad \text{Prim}: \{\text{Hopf Algebras}\} \rightarrow \{\text{Lie Algebras}\}.$$


4.1. Derivation and Homotopy


Let $\mathcal{A} \doteq (\partial: I \rightarrow H)$ and $\mathcal{A}' \doteq (\partial': I' \rightarrow H')$ be two arbitrary but fixed cocommutative Hopf crossed modules. Both of them denoted are by:



Definition 4.1. Let $f_0: H \rightarrow H'$ be a cocommutative Hopf algebra morphism:



An f_0 -derivation is a coalgebra morphism $\Gamma: I \rightarrow H'$ denoted by  such that

where  denotes the action in the corresponding Hopf crossed module.

Lemma 4.2. *If Γ is an f_0 -derivation, then*

(4.2)

and, recalling the adjoint action (2.1), we have

(4.3)

Proof. Follows from (4.1) by graphical calculation. \square

Definition 4.3. Let $\mathcal{A} \doteq (\partial: I \rightarrow H)$ and $\mathcal{A}' \doteq (\partial': I' \rightarrow H')$ be two arbitrary but fixed cocommutative Hopf crossed modules, and denote both of them by:



Suppose that we have Hopf crossed module morphism $f = (f_1, f_0): \mathcal{A} \rightarrow \mathcal{A}'$ with

$$f_1 = \text{[blue rectangle with black line and purple dot]}, \quad f_0 = \text{[blue rectangle with black line and yellow dot]}$$

Define $g = (g_1, g_0)$ by the components

$$g_1 = \text{[blue rectangle with black line, purple dot on left, cyan dot on right, orange dot on right]}, \quad g_0 = \text{[blue rectangle with black line, yellow dot on left, orange dot on right, cyan dot on right]}$$

where is the f_0 -derivation $\Gamma: I \rightarrow H'$.

Theorem 4.4. g_0 and g_1 define Hopf algebra morphisms.

Proof. It is clear that g_0, g_1 are coalgebra morphisms since Γ is a coalgebra morphism. Also they are algebra morphisms [7]. Therefore they are bialgebra morphisms; and furthermore Hopf algebra morphisms from (4.2). \square

Theorem 4.5. $g = (g_1, g_0): \mathcal{A} \rightarrow \mathcal{A}'$ is a Hopf crossed module morphism.

Proof. It is already proved in [7] that the condition (3.1) holds. Also by using (4.3), the action is preserved, namely the condition (3.2) is satisfied. \square

Definition 4.6 (Homotopy). In the condition of the previous theorem, we write $f \xrightarrow{(f_0, \Gamma)} g$ or shortly $f \simeq g$, and say that (f_0, Γ) is a "homotopy (or derivation)" connecting f to g .

As a consequence of this homotopy definition, we can give the following:

Let $\mathcal{A}, \mathcal{A}'$ be Hopf crossed modules. If there exist Hopf crossed module morphisms $f: \mathcal{A} \rightarrow \mathcal{A}'$ and $g: \mathcal{A}' \rightarrow \mathcal{A}$ such that $f \circ g \simeq id_{\mathcal{A}'}$ and $g \circ f \simeq id_{\mathcal{A}}$; we say that the Hopf crossed modules \mathcal{A} and \mathcal{A}' are "homotopy equivalent", which is denoted by $\mathcal{A} \simeq \mathcal{A}'$.

4.2. A groupoid structure

Let $f = (f_1, f_0): \mathcal{A} \rightarrow \mathcal{A}'$ be a Hopf crossed module morphism. By letting Γ be the zero morphism, we get a derivation which connects f to itself. Also, the antipode of a derivation which connects f to g is again a derivation connecting g to f . Moreover, let f, g, k be Hopf crossed module morphisms between $\mathcal{A} \rightarrow \mathcal{A}'$, where Γ is a derivation connecting f to g , and Γ' is a derivation connecting g to k . Then the map $\cdot(\Gamma \otimes \Gamma')\Delta$ defines a derivation which connects f to k . These yield the following theorem.

Theorem 4.7. *Let $\mathcal{A}, \mathcal{A}'$ be two arbitrary but fixed cocommutative Hopf crossed modules. We have a groupoid HOM_{HA} whose objects are the Hopf crossed module morphisms between $\mathcal{A} \rightarrow \mathcal{A}'$, the morphisms being their homotopies.*

5. Review by Sweedler’s notation

In Sweedler’s [22] notation, we denote the coproduct of a Hopf algebra by:

$$\Delta(x) = \sum_{(x)} x' \otimes x''.$$

We can easily obtain some applications and categorical properties of the previous notions by using this notation. Hence the derivation (4.1) can also be expressed by the formula

$$\Gamma(ab) = \sum_{(b)} \left(S(f_0(b')) \triangleright_{\rho} \Gamma(a) \right) \Gamma(b''), \tag{5.1}$$

that connects Hopf crossed module morphism $f = (f_0, g_0)$ to $g = (f_0, g_0)$ where:

$$g_0(a) = \sum_{(a)} f_0(a') (\partial' \circ \Gamma)(a''), \quad g_1(x) = \sum_{(x)} f_1(x') (\Gamma \circ \partial)(x'')$$

are diagrammatically given in Definition 4.3.

Theorem 5.1. *The functors $()_{\text{gl}}^*$ and Prim preserve the homotopy and homotopy equivalence.*

Proof. The functor $()_{\text{gl}}^*$ yields $S(x) = x^{-1}$ and also $\epsilon(x) = e$. Therefore the formula (5.1) will be turned into

$$\Gamma(ab) = \left((f_0(b))^{-1} \triangleright \Gamma(a) \right) \Gamma(b),$$

which is given in [10] for the case of groups.

On the other hand, by using the properties of the functor Prim , the formula (5.1) will be turned into

$$\Gamma([a, b]) = f_0(a) \triangleright \Gamma(b) + f_0(b) \triangleright \Gamma(a) + [\Gamma(a), \Gamma(b)],$$

which is given in [9] for the case of Lie algebras. □

Recall Theorem 4.7; and the groupoid structures in [9, 10] for the case of groups and Lie algebras. As a result of the previous theorem we have the following.

Corollary 5.2. *The functors $()_{\text{gl}}^*$ and Prim can be seen as groupoid functors such as:*

$$()_{\text{gl}}^*: \text{HOM}_{\text{HA}} \rightarrow \text{HOM}_{\text{Grp}} \quad \text{and} \quad \text{Prim}: \text{HOM}_{\text{HA}} \rightarrow \text{HOM}_{\text{Lie}}.$$

Acknowledgment. The author is thankful to Jamie Vicary for the Globular lectures during AARMS 2016 Summer School in Halifax, Canada. The author supported by the projects MUNI/A/1186/2018 from Masaryk University, and also BAP/2018/2244 from Eskişehir Osmangazi University.

References

- [1] K. Bar, A. Kissinger, and J. Vicary, *Globular: an online proof assistant for higher-dimensional rewriting*, Log. Methods Comput. Sci. **14** (1), 2018.
- [2] D. Bulacu, S. Caenepeel, F. Panaite, and F. Van Oystaeyen, *Quasi-Hopf algebras. A categorical approach*, Cambridge University Press, 2019.
- [3] M. Crossley and N. Turgay, *Conjugation invariants in the Leibniz-Hopf algebra*, J. Pure Appl. Algebra, **217** (12), 2247–2254, 2013.
- [4] M. Crossley and N. Turgay, *Conjugation invariants in the mod 2 dual Leibniz-Hopf algebra*, Comm. Algebra **41** (9), 3261–3266, 2013.
- [5] M. Crossley and S. Whitehouse, *On conjugation invariants in the dual Steenrod algebra*, Proc. Amer. Math. Soc. **128** (9), 2809–2818, 2000.
- [6] V. Drinfel'd, *Quasi-Hopf algebras*, Leningr. Math. J. **1** (6), 1419–1457, 1990.
- [7] K. Emir, *Globular: Homotopy of Hopf crossed module maps*, available at: <http://globular.science/1610.001v2>.
- [8] K. Emir, *Globular: Hopf crossed modules*, available at: <http://globular.science/1611.002v1>.
- [9] K. Emir and S. Çetin, *From simplicial homotopy to crossed module homotopy in modified categories of interest*, Georgian Math. J. doi: 10.1515/gmj-2018-0069.
- [10] J. Faria Martins, *The fundamental 2-crossed complex of a reduced CW-complex*, Homology Homotopy Appl. **13** (2), 129–157, 2011.
- [11] J. Faria Martins, *Crossed modules of Hopf algebras and of associative algebras and two-dimensional holonomy*, J. Geom. Phys. **99**, 68–110, 2016.
- [12] M. Hazewinkel, *The Leibniz-Hopf algebra and Lyndon words*, CWI Report: AM-R 9612, January 1996.
- [13] A. Joyal and R. Street, *Braided monoidal categories*, Macquarie Math Reports 860081, November 1986.
- [14] J. Loday, *Spaces with finitely many non-trivial homotopy groups*, J. Pure Appl. Algebra **24** (2), 179–202, 1982.
- [15] S. Majid, *Strict quantum 2-groups*, arXiv:1208.6265.
- [16] S. Majid, *Algebras and Hopf algebras in braided categories*, in: Advances in Hopf algebras, 55–105, New York: Marcel Dekker, 1994.
- [17] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, 1995.
- [18] S. Majid, *What is .. a quantum group?*, Notices Amer. Math. Soc. **53** (1), 30–31, 2006.
- [19] J. Milnor, *The Steenrod algebra and its dual*, Ann. Math. (2) **67**, 150–171, 1958.
- [20] D. Radford, *The structure of hopf algebras with a projection*, J. Algebra **92** (2), 322–347, 1985.
- [21] P. Selinger, *A survey of graphical languages for monoidal categories*. in: New structures for physics, 289–355. Berlin: Springer, 2011.
- [22] M. Sweedler, *Hopf algebras*, Mathematics lecture note series. W. A. Benjamin, 1969.
- [23] X. Tang, A. Weinstein, and C. Zhu, *Hopfish algebras*, Pacific J. Math. **231** (1), 193–216, 2007.
- [24] J. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) **42**, 409–428, 1941.