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On The Reformulated Zagreb Coindex

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Abstract − The reformulated Zagreb coindex of G is specified the degrees d_i and d_j. This paper includes some inequalities for the reformulated Zagreb index and reformulated Zagreb coindex. Also, some bounds are reported consepting the eigenvalues and complement of eigenvalues.

Keywords − *Reformulated Zagreb index, coindex*

1. Introduction

Let G be a simple connected graph on the vertex set $V(G)$ and the edge set $E(G)$. Also, let the degree of v_i denoted by d_i. The Reformulated Zagreb matrix of G is described with $[RZ(G)] = [rz]_{ij}$ where $[rz]_{ij} = (d_i + d_j - 2)^2$ if the vertices i is adjacent to j and $[rz]_{ij} = 0$ if otherwise.

The Reformulated Zagreb index $RZ(G)$ of G [7] is a general sum-connectivity index where

$$RZ(G) = \sum_{i,j \in E(G)} (d_G(i) + d_G(j) - 2)^2. \qquad (1)$$

The Zagreb coindex of G is described in [3],

$$\bar{Z}_1(G) = \sum_{v_i, v_j \notin E(G)} (d_G(i) + d_G(j)). \qquad (2)$$

In this study, different bounds are set using the degrees, the edges and the vertices. Also, some relations deal with the complement of eigenvalues of $[RZ]_{ij}$ are obtained. In Section 2, the reformulated Zagreb coindex is defined and different inequalities for this index are found.

2. Preliminaries

In this section, some back-ground material that is needed for later sections will be given.

Lemma 2.1. [4] Let $\lambda_1(M)$ be the spectral radius and $M = (m_{ij})$ be an nxn irreducible nonnegative matrix. Let $R_i(M) = \sum_{j=1}^{m} m_{ij}$. [8] Then,

$$(minR_i(M) : 1 \leq i \leq n) \leq \lambda_1(M) \leq (maxR_i(M) : 1 \leq i \leq n) \qquad (3)$$

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Lemma 2.2. [2] Let V be the vertex set, $v_i \in V$, m_i be the average degree of the vertices adjacent to v_i. Then [2]

$$\lambda_1(G) \leq max(\sqrt{m_i m_j} : 1 \leq i, j \leq n, v_i, v_j \in E) \tag{4}$$

Lemma 2.3. [6] If G is a regular graph then,

$$Z_1(G) \geq \frac{4m^2}{n}.$$

Lemma 2.4. [5] If G is a regular graph then,

$$\bar{Z}_1(G) \leq \frac{-4m^2}{n} + 2m(n-1).$$

See [1] and [8] for details.

3. MAIN SERULTS

3.1. On eigenvalues

Some inequalities deal with the first eigenvalue of $[RZ(G)]$ are given in this subsection. In addition, a bound for the complement of this eigenvalue is outlined.

Theorem 3.1. If G is a simple, connected graph then

$$\lambda_1^{RZ}(G) \leq \sqrt{(F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

where $\lambda_1^{RZ}(G)$ is the first eigenvalue of $[RZ(G)]$, $F_1 = (nd_i^2 + 4m^2)$ and $F_2 = ((nd_j^2 + 4m^2)$.

Proof. Let $D(G)^{-1}RZ(G)D(G) = F^{+}(G)$ and $X = (x_1, x_2, ..., x_n)^T$ be an eigenvector of $RZ^{+}(G)$. Also, $x_i = 1$ and $0 < x_k \leq 1$ for every k. Let $x_j = max_k(x_k : v_i v_k \in E)$ where i is adjacent to k. Let $RZ^{+}(G)X = \lambda_1^{RZ}(G)X$. If $i - th$ equation from above equation is get, then

$$\begin{aligned} \lambda_1^{RZ}(G)x_i &= \sum_k (d_i + d_k - 2)^2 x_k \\ &\leq (nd_i^2 + 4d_i(m-n) + 4n - 8m + 4m^2)x_k. \end{aligned}$$

Using Lemma 2.1, it is known that

$$\lambda_1^{RZ}(G)x_i \leq (nd_i^2 + 4d_i(m-n) + 4n - 8m + 4m^2)x_k.$$

The $j - th$ equation of the same equation,

$$\lambda_1^{RZ}(G)x_j \leq (nd_j^2 + 4d_j(m-n) + 4n - 8m + 4m^2)x_k.$$

From Lemma 2.2, the inequality holds that

$$\lambda_1^{RZ}(G) \leq \sqrt{(F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

□

Corollary 3.2. Let G be a graph on n vertices and m edges. Then,

$$\bar{\lambda_1}^{RZ}(G) \leq \sqrt{K - (F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m)}.$$

where $K = (F_1 + 4(m-n)d_i + 4n - 8m)(F_2 + 4(m-n)d_j + 4n - 8m) + (\bar{F_1} + 2(n^2 - 3n - 2m)(n - 1 - d_i) + 8(n-m) - 4n^2)(\bar{F_2} + 2(n^2 - 3n - 2m)(n - 1 - d_j) + 8(n-m) - 4n^2)$.

Proof. Cauchy-Schwarz inequality and Theorem 3.1 gives that

$$\begin{aligned}(\lambda_1^{RZ}(G)+\bar{\lambda_1}^{RZ}(G))^2 &\leq (\lambda_1^F(G))^2+(\bar{\lambda_1}^F(G))^2\\ &\leq (F_1+4(m-n)d_i+4n-8m)(F_2+4(m-n)d_j+4n-8m)\\ &+(\bar{F_1}+2(n^2-3n-2m)(n-1-d_i)+8(n-m)-4n^2)\\ &(\bar{F_2}+2(n^2-3n-2m)(n-1-d_j)+8(n-m)-4n^2).\end{aligned}$$

Since $\bar{m}+m=\dfrac{n^2-n}{2}$ then $\bar{F_1}=n(n-1-d_i)^2+2(n(n-1)-2m)$ and $\bar{F_2}=n(n-1-d_j)^2+2(n(n-1)-2m)$. It is implies that

$$\bar{\lambda_1}^{RZ}(G)\leq\sqrt{K-(F_1+4(m-n)d_i+4n-8m)(F_2+4(m-n)d_j+4n-8m)}.$$

□

3.2. Reformulated Zagreb Coindex

In this subsection, the reformulated Zagreb coindex is concerned with. Thus, some bounds concepting this indices are obtained.

Definition 3.3. The Reformulated Zagreb coindex $\bar{RZ}(G)$ defined as

$$\bar{RZ}(G)=\sum_{v_i,v_j\notin E(G)}(d_G(i)+d_G(j)-2)^2. \tag{5}$$

Theorem 3.4. Let G be a graph on n vertices and m edges. Then,

$$RZ(\bar{G})\geq 2(n^2-n-2m)(n-2)^2-4(n-2)Z_1(\bar{G})+(Z_1(\bar{G}))^2.$$

Proof. It is known that, $RZ(\bar{G})=\sum_{v_i,v_j\in E(\bar{G})}(d_{\bar{G}}(i)+d_{\bar{G}}(j)-2)^2$. Since $d_{\bar{G}}(i)=(n-1-d_i)$ and $d_{\bar{G}}(j)=(n-1-d_j)$ then,

$$\begin{aligned}RZ(\bar{G}) &= \sum_{v_i,v_j\in E(\bar{G})}((n-1-d_i)+(n-1-d_j)-2)^2\\ &= \sum_{v_i,v_j\in E(\bar{G})}4(n^2-2n+1)-(4n-4)\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j+2)+\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j+2)^2\end{aligned}$$

Since G has $\binom{n}{2}-m=\dfrac{n^2-n-2m}{2}$ edges, then

$$\begin{aligned}RZ(\bar{G}) &= 4(n^2-2n+1)(\frac{n^2-n-2m}{2})-4(n-1)(Z_1(\bar{G})+2(\frac{n^2-n-2m}{2}))\\ &+\sum_{v_i,v_j\in E(\bar{G})}(d_i+d_j)^2+4Z_1(\bar{G})+4(\frac{n^2-n-2m}{2})\\ &\geq 2(n^2-n-2m)(n-2)^2+4(n-2)Z_1(\bar{G})+(Z_1(\bar{G}))^2.\end{aligned}$$

□

Corollary 3.5. If G is a regular graph on n vertices and m edges. Then,

$$\begin{aligned}RZ(\bar{G}) &\geq 2(n^2-n-2m)(n-2)^2-4(n-2)(\frac{-4m^2}{n}+2m(n-1))\\ &+(\frac{-4m^2}{n}+2m(n-1))^2.\end{aligned}$$

PROOF. Since $Z_1(\bar{G}) = \bar{Z}_1(G)$ then

$$RZ(\bar{G}) \geq 2(n^2-n-2m)(n-2)^2 - 4(n-2)\bar{Z}_1(G) + (\bar{Z}_1(G))^2$$

Using Lemma 2.4, it is concluded that

$$\begin{aligned} RZ(\bar{G}) \;&\geq\; 2(n^2-n-2m)(n-2)^2 - 4(n-2)(\frac{-4m^2}{n} + 2m(n-1)) \\ &+(\frac{-4m^2}{n} + 2m(n-1))^2. \end{aligned}$$

□

Theorem 3.6. Let G be a graph on n vertices and m edges. Then,

$$RZ(G) + \bar{RZ}(G) = (n-2)Z_1(G) + 2m(2m-4n+5) - 2n.$$

PROOF.

$$\begin{aligned} RZ(G) + \bar{RZ}(G) \;=\; & \sum_{v_i,v_j \in E(G)} (d_i+d_j-2)^2 + \sum_{v_i,v_j \notin E(G)} (d_i+d_j-2)^2 \\ =& \frac{1}{2}(\sum_{v_i \in V(G)} \sum_{v_j \in V(G)} (d_i+d_j-2)^2 - \sum_{v_j \in V(G)} (d_j+d_j-2)^2) \\ =& \frac{1}{2}(\sum_{v_i \in V(G)} \sum_{j \in V(G)} (d_i^2+d_j^2+2d_id_j-4d_i-4d_j+4) - 4\sum_{v_j \in V(G)} (d_j^2-2d_j+1) \\ =& \frac{1}{2}(n\sum_{v_i \in V(G)} d_i^2 + \sum_{v_j \in V(G)} d_j^2 + 2\sum_{v_i \in V(G)} d_i \sum_{v_j \in V(G)} d_j - 4n\sum_{v_i \in V(G)} d_i \\ & -4n\sum_{v_j \in V(G)} d_j + \sum_{v_i \in V(G)} \sum_{v_j \in V(G)} 4 - 4\sum_{v_j \in V(G)} d_j^2 + 8\sum_{v_j \in V(G)} d_j - \sum_{v_j \in V(G)} 4) \\ =& \frac{1}{2}[nZ_1(G) + nZ_1(G) + 2(2m)(2m) - 4n.2m - 4n.2m + 4m - 4Z_1(G) + 16m - 4n] \\ =& (n-2)Z_1(G) + 2m(2m-4n+5) - 2n. \end{aligned}$$

□

Corollary 3.7. If G is a regular graph on n vertices and m edges. Then,

$$\bar{RZ}(G) \geq 2m(\frac{4m(n-1)}{n} - 4n + 5) - 2n - RZ(G).$$

PROOF. By Lemma 2.3, it is seen that

$$\begin{aligned} RZ(G) + \bar{RZ}(G) \;&\geq\; (n-2)(\frac{4m^2}{n}) + 2m(2m-4n+5) - 2n \\ &= 8m^2\frac{n-1}{n} - 8mn + 10m - 2n. \end{aligned}$$

Hence,

$$\bar{RZ}(G) \geq 2m(\frac{4m(n-1)}{n} - 4n + 5) - 2n - RZ(G).$$

□

Corollary 3.8. Let G be a regular graph on n vertices and m edges. Then,

$$\begin{aligned} \bar{RZ}(\bar{G}) \;&\leq\; (n^2-n-2m)(-n^2+3n-2m-3) - 2n \\ &+(5n-10)(\frac{-4m^2}{n} + 2m(n-1)) - (\frac{-4m^2}{n} + 2m(n-1))^2. \end{aligned}$$

PROOF. By Theorem 3.7,

$$\bar{RZ}(\bar{G}) = (n-2)Z_1(\bar{G}) + 2\bar{m}(2\bar{m} - 4n + 5) - 2n - RZ(\bar{G}).$$

Since $\bar{m} = \dfrac{n^2 - n - 2m}{2}$ then,

$$\bar{RZ}(\bar{G}) = (n-2)Z_1(\bar{G}) + (n^2 - n - 2m)(n^2 - 5n - 2m + 5) - 2n - RZ(\bar{G}).$$

By Lemma 2.4 and Corollary 2.3,

$$\begin{aligned} \bar{RZ}(\bar{G}) \;\leq\; & (n^2 - n - 2m)(-n^2 + 3n - 2m - 3) - 2n \\ & + (5n - 10)\left(\frac{-4m^2}{n} + 2m(n-1)\right) - \left(\frac{-4m^2}{n} + 2m(n-1)\right)^2. \end{aligned}$$

□

4. Conclusion

In this paper, Reformulated Zagreb index which is one of the topological indices in graph theory is studied. New inequalities are formed for this index in terms of the degrees, edges and vertices. Indeed, Reformulated Zagreb coindex is defined and some bounds are obtained by the help of other Zagreb indices.

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