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Research Article

Representations and T^*-extensions of δ-Bihom-Jordan-Lie algebras

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Abstract

The purpose of this article is to study representations of δ-Bihom-Jordan-Lie algebras. In particular, adjoint representations, trivial representations, deformations, T^*-extensions of δ-Bihom-Jordan-Lie algebras are studied in detail. Derivations and central extensions of δ-Bihom-Jordan-Lie algebras are also discussed as an application.

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1. Introduction

The notion of Jordan-Lie algebras was introduced in [7], which is closely related to both Lie and Jordan superalgebras. Engel's theorem of Jordan-Lie algebras was proved, and some properties of Cartan subalgebras of Jordan-Lie algebras were given in [8].

Recently, the definition of δ-hom-Jordan-Lie algebras were introduced in [10], and their representations and T^*-extensions were studied in detail.

A Bihom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms α, β. This class of algebras was introduced from a categorical approach in [4] as an extension of the class of Hom-algebras. The origin of Hom-structures can be found in the physics literature around 1900, appearing in the study of quasi deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras in [5]. Since then, many authors have been interested in the study of Hom-algebras, mainly motivated by their applications in mathematical physics (see for instance the recent references [1, 6]). The fundamental for getting the basic notions, motivations, and results on Bihom-algebras is the reference [4].

More applications of the Bihom-Lie algebras, Bihom-algebras, Bihom-Lie superalgebras and Bihom-Lie admissible superalgebras can be found in [3, 9].

The notion of derivations, representations, and T^*-extensions of δ-Bihom-Jordan Lie algebras are not so well developed.

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The paper is organized as follows. In Section 2 we give the definition of δ-Bihom-Jordan-Lie algebras, and show that the direct sum of two δ-Bihom-Jordan-Lie algebras is still a δ-Bihom-Jordan-Lie algebra. A linear map between δ-Bihom-Jordan-Lie algebras is a morphism if and only if its graph is a Bihom subalgebra. In Section 3 we study derivations of multiplicative δ-Bihom-Jordan-Lie algebras. For any nonnegative integers k and l, we define $\alpha^k\beta^l$-derivations of multiplicative δ-Bihom-Jordan-Lie algebras. Considering the direct sum of the space of $\alpha^k\beta^l$-derivations, we prove that it is a Lie algebra. In particular, any $\alpha^0\beta^1$-derivation gives rise to a derivation extension of the multiplicative δ-hom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ (Theorem 3.3). In Section 4 we give the definition of representations of multiplicative δ-Bihom-Jordan-Lie algebras. We can obtain the semidirect product multiplicative δ-Bihom-Jordan-Lie algebra $(L\oplus M, [\cdot,\cdot]_\rho, \alpha+\alpha_M, \beta+\beta_M)$ associated to any representation ρ on M of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$. In Section 5 we study trivial representations of multiplicative δ-Bihom-Jordan-Lie algebras. We show that central extensions of a multiplicative δ-Bihom-Jordan-Lie algebra are controlled by the second cohomology with coefficients in the trivial representation. In Section 6 we study the adjoint representation of a regular δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$. For any integers s, t, we define the $\alpha^s\beta^t$-derivations. We show that a 1-cocycle associated to the $\alpha^s\beta^t$-derivation is exactly an $\alpha^{s+2}\beta^{t-1}$-derivation of the regular δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ in some conditions. We also give the definition of Bihom-Nijienhuis operators of regular δ-Bihom-Jordan-Lie algebras. We show that the deformation generated by a Bihom-Nijienhuis operator is trivial. In Section 7 we study T^*-extensions of δ-Bihom-Jordan-Lie algebras, show that T^*-extensions preserve many properties such as nilpotency, solvability and decomposition in some sense.

2. Definitions and proprieties of δ-Bihom-Jordan-Lie algebras

Definition 2.1 ([7])**.** A δ-Jordan Lie algebra is a couple $(L, [\cdot,\cdot]_L)$ consisting of a vector space L and a bilinear map (bracket) $[\cdot,\cdot]_L : L\times L \to L$ satisfying

$$[x,y] = -\delta[y,x], \qquad \delta = \pm 1,$$
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \quad \forall x,y,z \in L.$$

Definition 2.2 ([10])**.** A δ-hom-Jordan Lie algebra is a triple $(L, [\cdot,\cdot]_L, \alpha)$ consisting of a vector space L, a bilinear map (bracket) $[\cdot,\cdot]_L : L\otimes L \to L$ and a linear map $\alpha : L \to L$ satisfying

$$[x,y] = -\delta[y,x], \qquad \delta = \pm 1,$$
$$[\alpha(x),[y,z]] + [\alpha(y),[z,x]] + [\alpha(z),[x,y]] = 0, \quad \forall x,y,z \in L.$$

Especially, for $\delta = 1$ one has a hom-Lie algebra and for $\delta = -1$ a hom-Jordan Lie algebra.

Definition 2.3 ([3])**.** A Bihom-Lie algebra is a 4-tuple $(L, [\cdot,\cdot]_L, \alpha, \beta)$ consisting of vector space L, a bilinear map $[\cdot,\cdot] : L\times L \to L$ and two homomorphisms $\alpha, \beta : L \to L$ such that for all elements $x, y, z \in L$ we have

$$\alpha\circ\beta = \beta\circ\alpha,$$
$$[\beta(x),\alpha(y)] = -[\beta(y),\alpha(x)],$$
$$[\beta^2(x),[\beta(y),\alpha(z)]] + [\beta^2(y),[\beta(z),\alpha(x)]] + [\beta^2(z),[\beta(x),\alpha(y)]] = 0$$

(Bihom-Jacobi equation).

Definition 2.4. A δ-Bihom-Jordan Lie algebra is a 4-tuple $(L, [\cdot,\cdot]_L, \alpha, \beta)$ consisting of a vector space L, a bilinear map (bracket) $[\cdot,\cdot]_L : L\otimes L \to L$ and two linear maps

$\alpha, \beta : L \to L$ satisfying

$$\alpha \circ \beta = \beta \circ \alpha, \tag{2.1}$$
$$[\beta(x), \alpha(y)] = -\delta[\beta(y), \alpha(x)], \delta = \pm 1, \tag{2.2}$$
$$[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0, \forall x, y, z \in L. \tag{2.3}$$

Especially, for $\delta = 1$ one has a Bihom-Lie algebra and for $\delta = -1$ a Bihom-Jordan Lie algebra.

Definition 2.5. 1) A δ-Bihom-Jordan Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is multiplicative if α and β are algebra morphisms, i.e., for any $x, y \in L$, we have

$$\alpha([x, y]_L) = [\alpha(x), \alpha(y)]_L \quad \text{and} \quad \beta([x, y]_L) = [\beta(x), \beta(y)]_L.$$

2) A δ-Bihom-Jordan Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is regular if α and β are algebra automorphisms.
3) A subvector space $\eta \in L$ is a Bihom subalgebra of $(L, [\cdot,\cdot]_L, \alpha, \beta)$ if $\alpha(\eta) \in \eta$, $\beta(\eta) \in \eta$ and

$$[x, y]_L \in \eta, \quad \forall x, y \in \eta.$$

4) A subvector space $\eta \in L$ is a Bihom ideal of $(L, [\cdot,\cdot]_L, \alpha, \beta)$ if $\alpha(\eta) \in \eta$, $\beta(\eta) \in \eta$ and

$$[x, y]_L \in \eta, \quad \forall x \in \eta, y \in L.$$

Definition 2.6. A δ-Bihom associative algebra is a triple (L, α, β) consisting of a vector space L, a bilinear map on L, and two linear commuting maps $\alpha, \beta : L \to L$ satisfying

$$\alpha(x)(yz) = \delta(xy)\beta(z), \quad \forall x, y, z \in L. \tag{2.4}$$

Proposition 2.7. *Let (L, α, β) be a multiplicative δ-Bihom associative algebra. Define a bilinear map (bracket) $[\cdot,\cdot]_L : L \times L \to L$ satisfying*

$$[x, y]_L = xy - \delta\alpha^{-1}(\beta(y))\beta^{-1}(\alpha(x)), \forall x, y \in L. \tag{2.5}$$

Then $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is a δ-Bihom-Jordan-Lie algebra.

Proof. First we check that the bracket product $[\cdot,\cdot]$ is compatible with the structure maps α and β. For any $x, y \in L$, we have

$$\begin{aligned} [\alpha(x), \alpha(y)] &= \alpha(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))(\alpha\beta^{-1}(\alpha(x))) \\ &= \alpha(x)\alpha(y) - \delta\beta(y)(\alpha^2\beta^{-1}(x)) \\ &= \alpha([x, y]). \end{aligned}$$

Similarly, one can prove that $\beta([x, y]) = [\beta(x), \beta(y)]$.
And

$$\begin{aligned} [\beta(x), \alpha(y)] &= \beta(x)\alpha(y) - \delta(\alpha^{-1}\beta(\alpha(y)))(\alpha\beta^{-1}(\beta(x))) \\ &= \beta(x)\alpha(y) - \delta\beta(y)(\alpha(x)) \\ &= -\delta[\beta(y), \alpha(x)]. \end{aligned}$$

Now we prove the Bihom-Jacobi condition. For any elements $x, y \in L$, we have

$$\begin{aligned} [\beta^2(x), [\beta(y), \alpha(z)]] =& [\beta^2(x), \beta(y)\alpha(z) - \delta\alpha^{-1}\beta(\alpha(z))\alpha\beta^{-1}(\beta(y))] \\ =& [\beta^2(x), \beta(y)\alpha(z)] - \delta[\beta^2(x), \beta(z)\alpha(y)] \\ =& \Big(\beta^2(x)(\beta(y)\alpha(z)) - \delta(\alpha^{-1}(\beta^2(y))\beta(z))\alpha(\beta(x))\Big) \\ & - \delta\Big(\beta^2(x)(\beta(z)\alpha(y)) - \delta(\alpha^{-1}(\beta^2(z))\beta(y))\alpha(\beta(x))\Big). \end{aligned}$$

Similarly, we have

$$[\beta^2(y),[\beta(z),\alpha(x)]] = \Big(\beta^2(y)(\beta(z)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y))\Big) - \delta\Big(\beta^2(y)(\beta(x)\alpha(z)) - \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y))\Big).$$

$$[\beta^2(z),[\beta(x),\alpha(y)]] = \Big(\beta^2(z)(\beta(x)\alpha(y)) - \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z))\Big) - \delta\Big(\beta^2(z)(\beta(y)\alpha(x)) - \delta(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z))\Big).$$

Note that

$$\begin{aligned}
\beta^2(x)(\beta(y)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(y))\alpha(\beta(z)),\\
\beta^2(y)(\beta(x)\alpha(z)) &= \delta(\alpha^{-1}(\beta^2(y))\beta(x))\alpha(\beta(z)),\\
\beta^2(x)(\beta(z)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(x))\beta(z))\alpha(\beta(y)),\\
\beta^2(y)(\beta(z)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(y))\beta(z))\alpha(\beta(x)),\\
\beta^2(z)(\beta(x)\alpha(y)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(x))\alpha(\beta(y)),\\
\beta^2(z)(\beta(y)\alpha(x)) &= \delta(\alpha^{-1}(\beta^2(z))\beta(y))\alpha(\beta(x)).
\end{aligned}$$

Then we obtain $[\beta^2(x),[\beta(y),\alpha(z)]] + [\beta^2(y),[\beta(z),\alpha(x)]] + [\beta^2(z),[\beta(x),\alpha(y)]] = 0.$ □

Proposition 2.8. *Given two δ-Bihom-Jordan-Lie algebras $(L,[\cdot,\cdot]_L,\alpha_1,\beta_1)$ and $(L',[\cdot,\cdot]_{L'},\alpha_2,\beta_2)$, there is a δ-Bihom-Jordan-Lie algebra $(L\oplus L',[\cdot,\cdot]_{L\oplus L'},\alpha_1+\alpha_2,\beta_1+\beta_2)$, where the bilinear map $[\cdot,\cdot]_{L\oplus L'} : L\oplus L' \times L\oplus L' \to L\oplus L'$ is given by*

$$[u_1+v_1,u_2+v_2]_{L\oplus L'} = [u_1,v_1]_L + [u_2,v_2]_{L'}, \forall u_1,u_2\in L, v_1,v_2\in L',$$

and the two linear maps $\alpha_1+\alpha_2, \beta_1+\beta_2 : L\oplus L' \to L\oplus L'$ defined by

$$\begin{aligned}
(\alpha_1+\alpha_2)(u_1+v_1) &= \alpha_1(u_1)+\alpha_2(v_1),\\
(\beta_1+\beta_2)(u_1+v_1) &= \beta_1(u_1)+\beta_2(v_1).
\end{aligned}$$

Proof. For any $u_1,u_2,u_3\in L$ and $v_1,v_2,v_3\in L'$ we have:

$$\begin{aligned}
&[(\beta_1+\beta_2)(u_1+v_1),(\alpha_1+\alpha_2)(u_2+v_2)]_{L\oplus L'}\\
&= [\beta_1(u_1),\alpha_1(u_2)]_L + [\beta_2(v_1),\alpha_2(v_2)]_{L'} = -\delta[\beta_1(u_2),\alpha_1(u_1)]_L - \delta[\beta_2(v_2),\alpha_2(v_1)]_{L'}\\
&= -\delta([\beta_1(u_2),\alpha_1(u_1)]_L + [\beta_2(v_2),\alpha_2(v_1)]_{L'})\\
&= -\delta[(\beta_1+\beta_2)(u_2+v_2),(\alpha_1+\alpha_2)(u_1+v_1)]_{L\oplus L'}.
\end{aligned}$$

$$\begin{aligned}
&(\alpha_1+\alpha_2)\circ(\beta_1+\beta_2)(u_1+v_1)\\
&= (\alpha_1+\alpha_2)(\beta_1(u_1)+\beta_2(v_1)) = \alpha_1\circ\beta_1(u_1)+\alpha_2\circ\beta_2(v_1)\\
&= \beta_1\circ\alpha_1(u_1)+\beta_2\circ\alpha_2(v_1)\\
&= (\beta_1+\beta_2)\circ(\alpha_1+\alpha_2)(u_1+v_1).
\end{aligned}$$

Then, we have $(\alpha_1+\alpha_2)\circ(\beta_1+\beta_2) = (\beta_1+\beta_2)\circ(\alpha_1+\alpha_2)$.
By a direct computation, we have

$$\begin{aligned}
&\circlearrowleft_{(u_1+v_1),(u_2+v_2),(u_3+v_3)} [(\beta_1+\beta_2)^2(u_1+v_1),[(\beta_1+\beta_2)(u_2+v_2),(\alpha_1+\alpha_2)(u_3+v_3)]_{L\oplus L'}]_{L\oplus L'}\\
&= \circlearrowleft_{(u_1+v_1),(u_2+v_2),(u_3+v_3)} [\beta_1^2(u_1)+\beta_2^2(v_1),[\beta_1(u_2),\alpha_1(u_3)]_L + [\beta_2(v_2),\alpha_2(v_3)]_{L'}]_{L\oplus L'}\\
&= \circlearrowleft_{u_1,u_2,u_3} [\beta_1^2(u_1),[\beta_1(u_2),\alpha_1(u_3)]_L]_L + \circlearrowleft_{v_1,v_2,v_3} [\beta_1^2(v_1),[\beta_1(v_2),\alpha_1(v_3)]_{L'}]_{L'}\\
&= 0,
\end{aligned}$$

where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x,y,z. □

Definition 2.9. Let $(L,[\cdot,\cdot]_L,\alpha_1,\beta_1)$ and $(L',[\cdot,\cdot]_{L'},\alpha_2,\beta_2)$ be two δ-Bihom-Jordan-Lie algebras. A linear map $\phi : L \to L'$ is said to be a morphism of δ-Bihom-Jordan-Lie algebras if

$$\phi[u,v]_L = [\phi(u),\phi(v)]_{L'}, \forall u,v \in L, \tag{2.6}$$

$$\phi \circ \alpha_1 = \beta_1 \circ \phi, \tag{2.7}$$

$$\phi \circ \alpha_2 = \beta_2 \circ \phi. \tag{2.8}$$

Denote by $\mathcal{G}_\phi \in L \oplus L'$ is the graph of a linear map $\phi : L \to L'$.

Proposition 2.10. *A map $\phi : (L,[\cdot,\cdot]_L,\alpha_1,\beta_1) \to (L',[\cdot,\cdot]_{L'},\alpha_2,\beta_2)$ is a morphism of δ-Bihom-Jordan-Lie algebras if and only if the graph $\mathcal{G}_\phi \in L \oplus L'$ is a Bihom subalgebra of $(L \oplus L',[\cdot,\cdot]_{L\oplus L'},\alpha_1+\alpha_2,\beta_1+\beta_2)$.*

Proof. Let $\phi : (L,[\cdot,\cdot]_L,\alpha_1,\beta_1) \to (L',[\cdot,\cdot]_{L'},\alpha_2,\beta_2)$ be a morphism of δ-Bihom-Jordan-Lie algebras, then for any $u,v \in L$, we have

$$[u+\phi(u),v+\phi(v)]_{L\oplus L'} = [u,v]_L + [\phi(u),\phi(v)]_{L'} = [u,v]_L + \phi[u,v]_L.$$

Then the graph $\mathcal{G}_\phi$ is closed under the bracket operation $[\cdot,\cdot]_{L\oplus L'}$. So, we obtain

$$(\alpha_1+\alpha_2)(u+\phi(u)) = \alpha_1(u) + \alpha_2\circ\phi(u) = \alpha_1(u) + \phi\circ\alpha_2(u),$$

and

$$(\beta_1+\beta_2)(u+\phi(u)) = \beta_1(u) + \beta_2\circ\phi(u) = \beta_1(u) + \phi\circ\beta_2(u),$$

which implies that $(\alpha_1+\alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi$ and $(\beta_1+\beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi$. Then $\mathcal{G}_\phi$ is a Bihom subalgebra of $(L \oplus L',[\cdot,\cdot]_{L\oplus L'},\alpha_1+\alpha_2,\beta_1+\beta_2)$.

Now, suppose that the graph $\mathcal{G}_\phi \subset L\oplus L'$ is a Bihom subalgebra of $(L\oplus L',[\cdot,\cdot]_{L\oplus L'},\alpha_1+\alpha_2,\beta_1+\beta_2)$, then we have

$$[u+\phi(u),v+\phi(v)]_{L\oplus L'} = [u,v]_L + [\phi(u),\phi(v)]_{L'} \in \mathcal{G}_\phi,$$

which implies that

$$[\phi(u),\phi(v)]_{L'} = \phi[u,v]_L.$$

Furthermore, $(\alpha_1+\alpha_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi$ and $(\beta_1+\beta_2)(\mathcal{G}_\phi) \subset \mathcal{G}_\phi$ implies

$$(\alpha_1+\alpha_2)(u+\phi(u)) = \alpha_1(u)+\alpha_2\circ\phi(u) \in \mathcal{G}_\phi \quad \text{and} \quad (\beta_1+\beta_2)(u+\phi(u)) = \beta_1(u)+\beta_2\circ\phi(u) \in \mathcal{G}_\phi.$$

Which is equivalent to the condition $\alpha_1\circ\phi(u) = \phi\circ\beta_1(u)$, and $\alpha_2\circ\phi(u) = \phi\circ\beta_2(u)$ i.e.

$$\begin{aligned} \alpha_1\circ\phi &= \phi\circ\beta_1 \\ \text{and} \quad \alpha_2\circ\phi &= \phi\circ\beta_2. \end{aligned}$$

Therefore, ϕ is a morphism of δ-Bihom-Jordan-Lie algebras. □

Example 2.11. Let $(L,[\cdot,\cdot])$ be a δ-Jordan-Lie algebra and $\alpha,\beta : L \to L$ two commuting linear maps such that $\alpha([x,y]) = [\alpha(x),\alpha(y)]$ and $\beta([x,y]) = [\beta(x),\beta(y)]$, for all $x,y \in L$. Then $(L,[\cdot,\cdot]_L,\alpha,\beta)$, where $[x,y]_L = [\alpha(x),\beta(y)]$, is a δ-Bihom-Jordan-Lie algebra. Moreover, suppose that $(L',[\cdot,\cdot])$ is another δ-Jordan-Lie algebra and $\alpha',\beta' : L' \to L'$ be two algebra endomorphisms. If $f : L \to L'$ is a δ-Jordan-Lie algebra homomorphism that satisfies $f\circ\alpha = \alpha'\circ f$ and $f\circ\beta = \beta'\circ f$, then $f : (L,[\cdot,\cdot]_L,\alpha,\beta) \to (L',[\cdot,\cdot]_{L'},\alpha',\beta')$ is also a homomorphism of δ-Bihom-Jordan-Lie algebras.

Proof. It is easy to show that $(L,[\cdot,\cdot]_L,\alpha,\beta)$ satisfies $[\beta(x),\alpha(y)]_L = [\alpha\beta(x),\beta\alpha(y)]) = \alpha\beta([x,y]) = \alpha\beta(-\delta[y,x]) = -\delta[\alpha\beta(y),\alpha\beta(x)]) = -\delta[\beta(y),\alpha(x)]_L$, and

$$\begin{aligned}
&[\beta^2(x),[\beta(y),\alpha(z)]_L]_L + [\beta^2(y),[\beta(z),\alpha(x)]_L]_L + [\beta^2(z),[\beta(x),\alpha(y)]_L]_L \\
&= [\beta^2(x),[\alpha\beta(y),\beta\alpha(z)]]_L + [\beta^2(y),[\alpha\beta(z),\beta\alpha(x)]]_L + [\beta^2(z),[\alpha\beta(x),\beta\alpha(y)]_L]_L \\
&= [\alpha\beta^2(x),\beta[\alpha\beta(y),\beta\alpha(z)]]_L + [\alpha\beta^2(y),\beta[\alpha\beta(z),\beta\alpha(x)]]_L + [\alpha\beta^2(z),\beta[\alpha\beta(x),\beta\alpha(y)]_L]_L \\
&= \alpha\beta^2([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) \\
&= 0.
\end{aligned}$$

Then $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is a δ-Bihom-Jordan-Lie algebra.

The second assertion follows from

$$f([x,y]_L) = f([\alpha(x),\beta(y)]) = [f(\alpha(x)), f(\beta(y))]) = [\alpha(f(x)),\beta(f(y))] = [f(x),f(y)]_{L'}.$$

Then $f : (L, [\cdot,\cdot]_L, \alpha, \beta) \to (L', [\cdot,\cdot]_{L'}, \alpha', \beta',)$ is also a homomorphism of δ-Bihom-Jordan-Lie algebras. □

Example 2.12. A three dimensional linear space L has a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $(L, [\cdot,\cdot])$ is a δ-Jordan-Lie algebra with respect to the product:

$$\left[\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a' & b' \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{pmatrix}\right] = \delta \begin{pmatrix} 0 & 0 & ac' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & a'c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we define two algebra endomorphisms α and β by

$$\alpha(e_1) = \delta e_1, \alpha(e_2) = e_3, \alpha(e_3) = e_2,$$

and

$$\beta(e_1) = \delta e_1, \beta(e_2) = e_3, \beta(e_3) = e_2.$$

Then $(L, \alpha \otimes \beta([\cdot,\cdot]_L) = [\alpha(.),\beta(.)], \alpha, \beta)$ is a δ-Bihom-Jordan-Lie algebra.

3. Derivations of δ-Bihom-Jordan-Lie algebras

In this section, we will study derivations of δ-Bihom-Jordan-Lie algebras. Let $(L, [\cdot,\cdot]_L, \alpha, \beta)$ be a multiplicative δ-Bihom-Jordan-Lie algebra. For any nonnegative integers k, l, denote by α^k the k-times composition of α and β^l the l-times composition of β, i.e.

$$\alpha^k = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k-times)}, \quad \beta^l = \underbrace{\beta \circ \cdots \circ \beta}_{(l-times)}.$$

Since the maps α, β commute, we denote by

$$\alpha^k\beta^l = \underbrace{\alpha \circ \cdots \circ \alpha}_{(k-times)} \circ \underbrace{\beta \circ \cdots \circ \beta}_{(l-times)}.$$

In particular, $\alpha^0\beta^0 = Id, \alpha^1\beta^1 = \alpha\beta$, $\alpha^{-k}\beta^{-l}$ is the inverse of $\alpha^k\beta^l$. If $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is a regular δ-Bihom-Jordan-Lie algebra, we denote by α^{-k} the k-times composition of α^{-1}, the inverse of α.

Definition 3.1. For any nonnegative integers k, l, a linear map $D : L \to L$ is called an $\alpha^k\beta^l$-derivation of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$, if

$$[D,\alpha] = 0, \quad i.e. \quad D \circ \alpha = \alpha \circ D, \tag{3.1}$$

$$[D,\beta] = 0, \quad i.e. \quad D \circ \beta = \beta \circ D, \tag{3.2}$$

and

$$D[u,v]_L = \delta^k([D(u), \alpha^k\beta^l(v)]_L + [\alpha^k\beta^l(u), D(v)]_L), \forall u,v \in L. \tag{3.3}$$

For a regular δ-Bihom-Jordan-Lie algebra, $\alpha^{-k}\beta^{-l}$-derivations can be defined similarly.

Note first that if α and β are bijective, the skew-symmetry condition (2.3) implies

$$[u,v] = -\delta[\alpha^{-1}\beta(v), \alpha\beta^{-1}(u)]_L, \forall u,v \in L. \tag{3.4}$$

Denote by $\mathrm{Der}_{\alpha^s\beta^l}(L)$ is the set of $\alpha^s\beta^l$-derivations of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$. For any $u \in L$ satisfying $\alpha(u) = u$, and $\beta(u) = u$, define $D_{k,l}(u) : L \to L$ by

$$D_{k,l}(u)(v) = -\delta[\alpha^k\beta^l(v), u]_L,\ \delta^k = 1, \quad \forall v \in L.$$

By Equation (3.4),

$$\begin{aligned} D_{k,l}(u)(v) &= -\delta[\alpha^k\beta^l(v), u]_L \\ &= \delta[\alpha^{-1}\beta(u), \alpha\beta^{-1}(\alpha^k\beta^l(v))]_L \\ &= \delta[u, \alpha^{k+1}\beta^{l-1}(v)]_L. \end{aligned}$$

Then $D_{k,l}(u)$ is an $\alpha^{k+1}\beta^l$-derivation. We call an inner $\alpha^{k+1}\beta^l$-derivation. In fact, we have

$$D_{k,l}(u)(\alpha(v)) = -\delta[\alpha^{k+1}\beta^l(v), u]_L = -\alpha(\delta[\alpha^k\beta^l(v), u]_L) = \alpha \circ D_{k,l}(u)(v).$$
$$D_{k,l}(u)(\beta(v)) = -\delta[\alpha^k\beta^{l+1}(v), u]_L = -\beta(\delta[\alpha^k\beta^l(v), u]_L) = \beta \circ D_{k,l}(u)(v).$$

On the other hand, we have

$$\begin{aligned} & D_{k,l}(u)([v,w]_L) \\ &= -\delta[\alpha^k\beta^l([v,w]_L), u]_L \\ &= -\delta[[\beta\alpha^k\beta^{l-1}(v), \alpha\alpha^k\beta^{l-1}(w)]_L, \beta^2(u)]_L \\ &= \delta[\beta^2(u), [\beta\alpha^k\beta^{l-1}(v), \alpha\alpha^k\beta^{l-1}(w)]_L] \\ &= -\delta([\alpha^{k+1}\beta^l(v), [\alpha^k\beta^l(w), \alpha(u)]_L]_L + [\alpha^k\beta^{l+1}(w), [\beta(u), \alpha^{k+2}\beta^{l-2}(v)]_L]_L) \\ &= -\delta[\alpha^{k+1}\beta^l(v), [\alpha^k\beta^l(w), \alpha(u)]_L]_L - \delta[\alpha^k\beta^{l+1}(w), [\beta(u), \alpha^{k+2}\beta^{l-2}(v)]_L]_L \\ &= -\delta^{k+1}[\alpha^{k+1}\beta^l(v), \delta[\alpha^k\beta^l(w), u]_L]_L - \delta^{k+1}[\delta[u, \alpha^{k+1}\beta^{l-1}(v)]_L, \alpha^{k+1}\beta^l(w)]_L \\ &= \delta^{k+1}[\alpha^{k+1}\beta^l(v), D_{k,l}(u)(w)]_L + [D_{k,l}(u)(v), \alpha^{k+1}\beta^l(v)]_L. \end{aligned}$$

Therefore, $D_{k,l}(u)$ is an $\alpha^{k+1}\beta^l$-derivation. Denote by $\mathrm{Inn}_{\alpha^k\beta^l}(L)$ the set of inner $\alpha^k\beta^l$-derivations, i.e.

$$\mathrm{Inn}_{\alpha^k,\beta^l}(L) = \{-\delta[\alpha^{k-1}\beta^l(\cdot), u]_L | u \in L, \alpha(u) = u, \beta(u) = u, \delta^k = 1\}. \tag{3.5}$$

For any $D \in \mathrm{Der}_{\alpha^k\beta^l}(L)$ and $D' \in \mathrm{Der}_{\alpha^s\beta^t}(L)$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - D' \circ D. \tag{3.6}$$

Lemma 3.2. *For any $D \in \mathrm{Der}_{\alpha^k,\beta^l}(L)$ and $D' \in \mathrm{Der}_{\alpha^s,\beta^t}(L)$, we have*

$$[D, D'] \in \mathrm{Der}_{\alpha^{k+s},\beta^{l+t}}(L).$$

Proof. For any $u, v \in L$, we have

$$\begin{aligned} [D,D']([u,v]_L) &= D \circ D'([u,v]_L) - D' \circ D([u,v]_L) \\ &= \delta^s D([D'(u), \alpha^s\beta^t(v)]_L + [\alpha^s\beta^t(u), D'(v)]_L) \\ &\quad -\delta^k D'([D(u), \alpha^k\beta^l(v)]_L + [\alpha^k\beta^l(u), D(v)]_L) \\ &= \delta^s D([D'(u), \alpha^s\beta^t(v)]_L) + \delta^s D([\alpha^s\beta^t(u), D'(v)]_L) \\ &\quad -\delta^k D'([D(u), \alpha^k\beta^l(v)]_L) - \delta^k D'([\alpha^k\beta^l(u), D(v)]_L) \\ &= \delta^{k+s}([D \circ D'(u), \alpha^{k+s}\beta^{l+t}(v)]_L + [\alpha^k\beta^l \circ D'(u), D \circ \alpha^s\beta^t(v)]_L \\ &\quad +[D \circ \alpha^s\beta^t(u), \alpha^k\beta^l \circ D'(v)]_L + [\alpha^{k+s} \circ D(u), D \circ D'(v)]_L \\ &\quad -[D' \circ D(u), \alpha^{k+s}(v)]_L - [\alpha^s \circ D(u), D' \circ \alpha^k(v)]_L \\ &\quad -[D' \circ \alpha^k(u), \alpha^s \circ D(v)]_L - [\alpha^{k+s}\beta^{l+t}(u), D' \circ D(v)]_L). \end{aligned}$$

Since any two of maps $D, D^{'}$, α, β commute, we have

$$\begin{aligned} D\circ\alpha^s = \alpha^s\circ D \quad &, \qquad D^{'}\circ\alpha^k = \alpha^k\circ D^{'}, \\ D\circ\beta^t = \beta^t\circ D \quad &, \qquad D^{'}\circ\beta^l = \beta^l\circ D^{'} \ . \end{aligned}$$

Therefore, we have

$$\begin{aligned} [D, D^{'}]([u,v]_L) &= \delta^{k+s}([D\circ D^{'}(u) - D^{'}\circ D(u), \alpha^{k+s}\beta^{l+t}(v)]_L \\ &\quad +[\alpha^{k+s}\beta^{l+t}(u), D\circ D^{'}(u) - D^{'}\circ D(v)]_L) \\ &= \delta^{k+s}([[D, D^{'}](u), \alpha^{k+s}\beta^{l+t}(v)]_L + [\alpha^{k+s}\beta^{l+t}(u), [D, D^{'}](v),]_L). \end{aligned}$$

Furthermore, it is straightforward to see that

$$\begin{aligned} [D, D^{'}]\circ\alpha &= D\circ D^{'}\circ\alpha - D\circ D^{'}\circ\alpha \\ &= \alpha\circ D\circ D^{'} - \alpha\circ D\circ D^{'} \\ &= \alpha\circ[D, D^{'}], \end{aligned}$$

and

$$\begin{aligned} [D, D^{'}]\circ\beta &= D\circ D^{'}\circ\beta - D\circ D^{'}\circ\beta \\ &= \beta\circ D\circ D^{'} - \beta\circ D\circ D^{'} \\ &= \beta\circ[D, D^{'}]. \end{aligned}$$

Therefore, $[D, D^{'}] \in \mathrm{Der}_{\alpha^{k+s}\beta^{l+t}}(L)$. □

For any integer k, l, denote by $\mathrm{Der}(L) = \oplus_{k\geq 0, l\geq 0}\mathrm{Der}_{\alpha^k\beta^l}(L)$. Obviously, $\mathrm{Der}(L)$ is a Lie algebra, in which the Lie bracket is given by equation (3.6).

In the end, we consider the derivation extension of the regular δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ and give an application of the $\alpha^0\beta^1$-derivation $\mathrm{Der}_{\alpha^0\beta^1}(L)$.

For any linear map $D, \alpha, \beta : L \to L$, where α and β are inverse, consider the vector space $L \oplus RD$. Define a skew-symmetric bilinear bracket operation $[\cdot,\cdot]_D$ on $L \oplus RD$ by

$$[u,v]_D = [u,v]_L, [D,u]_D = -\delta[\alpha^{-1}\beta(u), \alpha\beta^{-1}D]_D = D(u), \forall u, v \in L.$$

Define two linear maps by $\alpha_D, \beta_D : L \oplus RD \to L \oplus RD$ by

$$\alpha_D(u, D) = (\alpha(u), D), \quad \text{ and } \ \beta_D(u, D) = (\beta(u), D).$$

And the linear maps α, β involved in the definition of the bracket operation $[\cdot,\cdot]_D$ are required to be multiplicative, that is

$$\alpha\circ[D,u]_D = [\alpha\circ D, \alpha(u)]_D, \quad \beta\circ[D,u]_D = [\beta\circ D, \beta(u)]_D.$$

Then, we have

$$\begin{aligned} [u, D]_D &= -\delta[\alpha^{-1}\beta D, \alpha\beta^{-1}(u)]_D \\ &= -\delta\alpha^{-1}\beta[D, \alpha^2\beta^{-2}(u)]_D \\ &= -\delta\alpha^{-1}\beta D(\alpha^2\beta^{-2}(u)) \\ &= -\delta\alpha\beta^{-1}D(u). \end{aligned}$$

Theorem 3.3. *With the above notations, $(L \oplus RD, [\cdot,\cdot]_D, \alpha_D, \beta_D)$ is a multiplicative δ-Bihom-Jordan-Lie algebra if and only if D is an $\alpha^0\beta^1$-derivation of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$.*

Proof. For any $u, v \in L, m, n \in R$, we have

$$\alpha_D\circ\beta_D(u, mD) = \alpha_D(\beta(u), mD) = (\alpha\circ\beta(u), mD),$$

and

$$\beta_D\circ\alpha_D(u, mD) = \beta_D(\alpha(u), mD) = (\beta\circ\alpha(u), mD).$$

Hence, we have

$$\alpha_D \circ \beta_D = \beta_D \circ \alpha_D \quad \Longleftrightarrow \alpha \circ \beta = \beta \circ \alpha.$$

On the other hand,

$$\begin{aligned}\alpha_D[(u,mD),(v,nD)]_D &= \alpha_D([u,v]_L + [u,nD]_D + [mD,v]_D)\\ &= \alpha_D([u,v]_L - \delta nD\circ\alpha\beta^{-1}(u) + mD(v))\\ &= \alpha([u,v]_L) - \delta n\alpha\circ D\circ\alpha\beta^{-1}(u) + m\alpha\circ D(v)),\end{aligned}$$

$$\begin{aligned}[\alpha_D(u,mD),\alpha_D(v,nD)]_D &= [(\alpha(u),mD),(\alpha(v),nD)]_D\\ &= [\alpha(u),\alpha(v)]_L + [\alpha(u),nD]_D + [mD,\alpha(v)]_D\\ &= [\alpha(u),\alpha(v)]_L - \delta nD\circ\alpha\beta^{-1}(\alpha(u)) + mD(\alpha(v)).\end{aligned}$$

Since $\alpha([u,v]_L) = [\alpha(u),\alpha(v)]_L$,

$$\alpha_D[(u,mD),(v,nD)]_D = [\alpha_D(u,mD),\alpha_D(v,nD)]_D$$

if and only if

$$D\circ\alpha = \alpha\circ D, \quad D\circ\beta = \beta\circ D.$$

Similarly

$$\beta_D[(u,mD),(v,nD)]_D = [\beta_D(u,mD),\beta_D(v,nD)]_D$$

if and only if

$$D\circ\alpha = \alpha\circ D, \quad D\circ\beta = \beta\circ D.$$

Next, we have

$$\begin{aligned}[\beta_D(v,nD),\alpha_D(u,mD)]_D &= [(\beta(v),nD),(\alpha(u),mD)]_D\\ &= [\beta(v),\alpha(u)]_L + [\beta(v),mD]_D + [nD,\alpha(u)]_D\\ &= [\beta(v),\alpha(u)]_L - \delta m\alpha\beta^{-1}\circ D\circ(\beta(v)) + nD(\alpha(u))\\ &= -\delta([\beta(u),\alpha(v)]_L + m\alpha\beta^{-1}\circ D\circ(\beta(v)) - \delta nD(\alpha(u))),\end{aligned}$$

$$\begin{aligned}[\beta_D(u,mD),\alpha_D(v,nD)]_D &= [(\beta(u),mD),(\alpha(v),nD)]_D\\ &= [\beta(u),\alpha(v)]_L + [\beta(u),nD]_D + [mD,\alpha(v)]_D\\ &= [\beta(u),\alpha(v)]_L - \delta n\alpha\beta^{-1}\circ D\circ(\beta(u)) + mD(\alpha(v)),\end{aligned}$$

thus

$$[\beta_D(v,nD),\alpha_D(u,mD)]_D = -\delta[\beta_D(u,mD),\alpha_D(v,nD)]_D$$

if and only if

$$D\circ\alpha = \alpha\circ D, \quad D\circ\beta = \beta\circ D.$$

On the other hand, we have

$$\begin{aligned}
&[\beta_D^2(u,mD),[\beta_D(v,nD),\alpha(w,lD)]_D]_D+[\beta_D^2(v,nD),[\beta_D(w,lD),\alpha_D(u,mD)]_D]_D\\
&\quad+[\beta_D^2(w,lD),[\beta_D(u,mD),\alpha_D(v,nD)]_D]_D\\
&=[(\beta^2(u),mD),[(\beta(v),nD),(\alpha(w),lD)]_D]_D+[(\beta^2(v),nD),[(\beta(w),lD),(\alpha(u),mD)]_D]_D\\
&\quad+[(\beta^2(w),lD),[(\beta(u),mD),(\alpha(v),nD)]_D]_D\\
&=[(\beta^2(u),mD),([\beta(v),\alpha(w)]-\delta l\alpha\circ D(v)+nD\circ\alpha(w))]_D\\
&\quad+[(\beta^2(v),nD),([\beta(w),\alpha(u)]-\delta n\alpha\circ D(w)+mD\circ\alpha(u))]_D\\
&\quad+[(\beta^2(w),lD),([\beta(u),\alpha(v)]-\delta m\alpha\circ D(u)+mD\circ\alpha(v))]_D\\
&=[\beta^2(u),[\beta(v),\alpha(w)]]-\delta[\beta^2(u),l\alpha\circ D(v)]+[\beta^2(u),nD\circ\alpha(w)]\\
&\quad+[mD,[\beta(v),\alpha(w)]]-\delta[mD,l\alpha\circ D(v)]+[mD,nD\circ\alpha(w)]\\
&\quad+[\beta^2(v),[\beta(w),\alpha(u)]]-\delta[\beta^2(v),m\alpha\circ D(w)]+[\beta^2(v),lD\circ\alpha(u)]\\
&\quad+[nD,[\beta(w),\alpha(u)]]-\delta[nD,m\alpha\circ D(w)]+[nD,lD\circ\alpha(u)]\\
&\quad+[\beta^2(w),[\beta(u),\alpha(v)]]-\delta[\beta^2(w),n\alpha\circ D(u)]+[\beta^2(w),mD\circ\alpha(v)]\\
&\quad+[lD,[\beta(u),\alpha(w)]]-\delta[lD,n\alpha\circ D(u)]+[lD,mD\circ\alpha(v)]\\
&=[\beta^2(u),[\beta(v),\alpha(w)]]-\delta[mD,l\alpha\circ D(v)]+[\beta^2(u),nD\circ\alpha(w)]\\
&\quad+[mD,[\beta(v),\alpha(w)]]-\delta ml\alpha\circ D^2(v)+mnD^2\circ\alpha(w)\\
&\quad+[\beta^2(v),[\beta(w),\alpha(u)]]-\delta[\beta^2(v),m\alpha\circ D(w)]+[\beta^2(v),lD\circ\alpha(u)]\\
&\quad+[nD,[\beta(w),\alpha(v)]]-\delta mn\alpha\circ D^2(w)+nlD^2\circ\alpha(w)]\\
&\quad+[\beta^2(w),[\beta(u),\alpha(v)]]-\delta[\beta^2(w),n\alpha\circ D(u)]+[\beta^2(w),mD\circ\alpha(v)]\\
&\quad+[lD,[\beta(u),\alpha(w)]]-\delta ln\alpha\circ D^2(u)+lmD^2\circ\alpha(v)].
\end{aligned}$$

If D is an $\alpha^0\beta^1$-derivation of the multiplicative δ-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$, then

$$\begin{aligned}
[mD,[\beta(v),\alpha(w)]]_D&=mD[\beta(v),\alpha(w)]\\
&=\delta[mD\circ\beta(v),\alpha^0\beta^1(\alpha(w))]+[\alpha^0\beta^2(v),mD\circ\alpha(w)]\\
&=-\delta[\alpha^0\beta^2(w),mD\circ\alpha(v)]+[\alpha^0\beta^2(v),mD\circ\alpha(w)]\\
&=-\delta[\beta^2(w),mD\circ\alpha(v)]+[\beta^2(v),mD\circ\alpha(w)].
\end{aligned}$$

Similarly

$$[nD,[\beta(w),\alpha(u)]]_D=-\delta[\beta^2(u),nD\circ\alpha(w)]+[\beta^2(w),n\alpha\circ D(u)].$$

And

$$[lD,[\beta(u),\alpha(v)]]_D=-\delta[\beta^2(v),lD\circ\alpha(u)]+[\beta^2(w),l\alpha\circ D(w)].$$

Therefore, the δ-Bihom-Jacobi identity is satisfied if and only if D is an $\alpha^0\beta^1$-derivation of $(L,[\cdot,\cdot]_L,\alpha,\beta)$. Thus $(L\oplus RD,[\cdot,\cdot]_D,\alpha_D,\beta_D)$ is a multiplicative δ-Bihom-Jordan-Lie algebra if and only if D is an $\alpha^0\beta^1$-derivation of $(L,[\cdot,\cdot]_L,\alpha,\beta)$. □

4. Representations of δ-Bihom-Jordan-Lie algebras

In this section we study representations of δ-Bihom-Jordan-Lie algebras and give the corresponding coboundary operators. We can also construct the semidirect product of δ-Bihom-Jordan-Lie algebras. Let $A\in End(V)$ be an arbitrary linear transformation from V to V.

Definition 4.1. Let $(L,[\cdot,\cdot]_L,\alpha,\beta)$ be a multiplicative δ-Bihom-Jordan-Lie algebra. A representation of L is a 4-tuple $(M,\rho,\alpha_M,\beta_M)$, where M is a linear space, $\alpha_M,\beta_M:M\rightarrow M$ are two commuting linear maps and $\rho:L\rightarrow End(M)$ is a linear map such that, for all $u,v\in L$, we have

$$\rho(\alpha(u))\circ\alpha_M=\alpha_M\circ\rho(u), \tag{4.1}$$
$$\rho(\beta(u))\circ\beta_M=\beta_M\circ\rho(u), \tag{4.2}$$
$$\rho([\beta(u),v]_L)\circ\beta_M=\rho(\alpha\beta(u))\circ\rho(v)-\delta\rho(\beta(v))\circ\rho(\alpha(u)). \tag{4.3}$$

Let $(L,[\cdot,\cdot]_L,\alpha,\beta)$ be a regular δ-Bihom-Jordan-Lie algebra. The set of k-cochains on L with values in M, which we denote by $C^k(L;M)$, is the set of k-linear maps from $L\times\cdots\times L$ (k-times) to M:

$$C^k(L;M)\triangleq\{f:L\times\cdots\times L(k-times)\rightarrow M \textit{ is a linear map}\}.$$

A k-Bihom-cochain on L with values in M is defined to be a k-cochain $f\in C^k(L;M)$ such that it is compatible with α, β and α_M, β_M in the sense that $\alpha_M\circ f=f\circ\alpha$, $\beta_M\circ f=f\circ\beta$, i.e.

$$\begin{aligned}\alpha_M(f(u_1,\ldots,u_k)) &= f(\alpha(u_1),\ldots,\alpha(u_k)),\\ \beta_M(f(u_1,\ldots,u_k)) &= f(\beta(u_1),\ldots,\beta(u_k)).\end{aligned}$$

Denote by $C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)$ the set of k-Bihom-cochains:

$$C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)\triangleq\{f\in C^k(L,M)|\alpha_M\circ f=f\circ\alpha,\beta_M\circ f=f\circ\beta\}.$$

Define the linear map $d^k_\rho:C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)\rightarrow C^{k+1}(L,M)(k=1,2)$ as follows: we set

$$d^1_\rho f(u_1,u_2)=\rho(\alpha(u_1))f(u_2)-\delta\rho\alpha(u_2))f(u_1)-\delta f([\alpha^{-1}\beta(u_1),u_2]_L),$$

$$\begin{aligned}d^2_\rho f(u_1,u_2,u_3) &= \rho(\alpha\beta(u_1))f(u_2,u_3)-\delta\rho(\alpha\beta(u_2))f(u_1,u_3)+\rho(\alpha\beta(u_3))f(u_1,u_2)\\ &\quad -f([\alpha^{-1}\beta(u_1),u_2]_L,\beta(u_3))+\delta f([\alpha^{-1}\beta(u_1),u_3]_L,\beta(u_2))\\ &\quad -f([\alpha^{-1}\beta(u_2),u_3]_L,\beta(u_1)).\end{aligned}$$

Lemma 4.2. *With the above notations, for any $f\in C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)$, we have*

$$\begin{aligned}(d^k_\rho\circ f)\circ\alpha &= \alpha_M\circ d^k_\rho f,\\ (d^k_\rho\circ f)\circ\beta &= \beta_M\circ d^k_\rho f.\end{aligned}$$

Thus we obtain a well-defined map

$$d^k_\rho:C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)\rightarrow C^{k+1}_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L,M)$$

with $k=1,2$.

Proposition 4.3. *With the above notations, we have $d^2_\rho\circ d^1_\rho=0$.*

Proof. By straightforward computations, we have

$$\begin{aligned}&d^2_\rho\circ d^1_\rho f(u_1,u_2,u_3)\\ &=\rho(\alpha\beta(u_1))d^1_\rho f(u_2,u_3)-\delta\rho(\alpha\beta(u_2))d^1_\rho f(u_1,u_3)+\rho(\alpha\beta(u_3))d^1_\rho f(u_1,u_2)\\ &\quad -d^1_\rho f([\alpha^{-1}\beta(u_1),u_2]_L,\beta(u_3))+\delta d^1_\rho f([\alpha^{-1}\beta(u_1),u_3]_L,\beta(u_2))\\ &\quad -d^1_\rho f([\alpha^{-1}\beta(u_2),u_3]_L,\beta(u_1))\\ &=\rho(\alpha\beta(u_1))(\rho(\alpha(u_2))f(u_3)-\delta\rho\alpha(u_3))f(u_2)-\delta f([\alpha^{-1}\beta(u_2),u_3]_L))\end{aligned}$$

$$
\begin{aligned}
&- \delta\rho(\alpha\beta(u_2))(\rho(\alpha(u_1))f(u_3) - \delta\rho\alpha(u_3))f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_3]_L))\\
&+ \rho(\alpha\beta(u_3))(\rho(\alpha(u_1))f(u_2) - \delta\rho\alpha(u_2)f(u_1) - \delta f([\alpha^{-1}\beta(u_1), u_2]_L))\\
&- \rho(\alpha([\alpha^{-1}\beta(u_1), u_2]_L))f(\beta(u_3)) + \delta\rho\alpha(\beta(u_3))f(u_1)\\
&+ \delta f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_1), u_2]_L), \beta(u_3)]_L)\\
&- \delta\rho(\alpha([\alpha^{-1}\beta(u_1), u_3]_L))f(\beta(u_2)) + \rho\alpha(\beta(u_2))f(u_1)\\
&+ f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_1), u_3]_L), \beta(u_2)]_L)\\
&+ \delta\rho(\alpha([\alpha^{-1}\beta(u_2), u_3]_L))f(\beta(u_1)) - \rho\alpha(\beta(u_1))f(u_2)\\
&- f([\alpha^{-1}\beta([\alpha^{-1}\beta(u_2), u_3]_L), \beta(u_1)]_L)\\
=\ & \rho(\alpha\beta(u_1))\rho(\alpha(u_2))f(u_3) - \delta\rho(\alpha\beta(u_1))\rho\alpha(u_3))f(u_2) - \delta\rho(\alpha\beta(u_1))f([\alpha^{-1}\beta(u_2), u_3]_L)\\
&- \delta\rho(\alpha\beta(u_2))\rho(\alpha(u_1))f(u_3) + \rho(\alpha\beta(u_2))\rho\alpha(u_3))f(u_1) + \rho(\alpha\beta(u_2))f([\alpha^{-1}\beta(u_1), u_3]_L)\\
&+ \rho(\alpha\beta(u_3))\rho(\alpha(u_1))f(u_2) - \delta\rho(\alpha\beta(u_3))\rho\alpha(u_2)f(u_1) - \delta\rho(\alpha\beta(u_3))f([\alpha^{-1}\beta(u_1), u_2]_L))\\
&- \rho([\beta(u_1), \alpha(u_2)]_L)f(\beta(u_3)) + \delta\rho(\alpha(\beta(u_3)))f(u_1)\\
&+ \delta f([([\alpha^{-2}\beta^2(u_1), \alpha^{-1}\beta(u_2)]_L), \beta(u_3)]_L)\\
&- \delta\rho([\beta(u_1), \alpha(u_3)]_L))f(\beta(u_2)) + \rho(\alpha(\beta(u_2)))f(u_1)\\
&+ f([([\alpha^{-2}\beta^2(u_1), \alpha^{-1}\beta(u_3)]_L), \beta(u_2)]_L)\\
&+ \delta\rho([\beta(u_2), \alpha(u_3)]_L)f(\beta(u_1)) - \rho(\alpha(\beta(u_1)))f(u_2)\\
&- f([([\alpha^{-2}\beta^2(u_2), \alpha^{-1}\beta(u_3)]_L), \beta(u_1)]_L)\\
=\ & 0.
\end{aligned}
$$

Then $d^2_\rho \circ d^1_\rho f(u_1, u_2, u_3) = 0$. □

Associated to the representation ρ, we obtain the complex $(C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L, M), d_\rho)$. Denote the set of closed k-Bihom-cochains by $Z^k_{\alpha,\beta}(L;\rho)$ and the set of exact k-Bihom-cochains by $B^k_{\alpha,\beta}(L, \rho)$, $k = 1, 2$.

Denote the corresponding cohomology by

$$H^k_{\alpha,\beta}(L, \rho) = Z^k_{\alpha,\beta}(L;\rho)/B^k_{\alpha,\beta}(L, \rho),$$

where

$$
\begin{aligned}
Z^k_{\alpha,\beta}(L;\rho) &= \{f \in C^k_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L, M) \ \mid \ d^k_\rho f = 0\},\\
B^k_{\alpha,\beta}(L, \rho) &= \{d^k_\rho g \ \mid \ g \in C^{k-1}_{\substack{(\alpha,\alpha_M)\\(\beta,\beta_M)}}(L, M)\}.
\end{aligned}
$$

In the case of Lie algebras, we can form semidirect products when given representations. Similarly, we have

Proposition 4.4. *Let $(L, [\cdot,\cdot]_L, \alpha, \beta)$ be a multiplicative δ-Bihom-Jordan-Lie algebra and $(M, \rho, \alpha_M, \beta_M)$ a representation of L. Assume that the maps α_M and β_M are bijective. Then $L \ltimes M = (L \oplus M, [\cdot,\cdot]_\rho, \alpha \oplus \alpha_M, \beta \oplus \beta_M)$ is a δ-Bihom-Jordan-Lie algebra, where $\alpha \oplus \alpha_M, \beta \oplus \beta_M : L \oplus M \to L \oplus M$ are defined by $(\alpha \oplus \alpha_M)(u + x) = \alpha(u) + \alpha_M(x)$ and $(\beta \oplus \beta_M)(u+x) = \beta(u)+\beta_M(x)$, for all $u, v \in L$ and $x, y \in M$, the bracket $[\cdot,\cdot]_\rho$ is defined by*

$$[u + x, v + y]_\rho = [u, v]_L + \delta\rho(u)(y) - \rho(\alpha^{-1}\beta(v))(\alpha_M\beta_M^{-1}(x)). \tag{4.4}$$

We call $L \ltimes M$ the semidirect product of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ and M.

Proof. First we show that $[\cdot,\cdot]_\rho$ satisfies antisymmetry,

$$\begin{aligned}
&[(\beta\oplus\beta_M)(v+y),(\alpha\oplus\alpha_M)(u+x)]_\rho\\
&=[\beta(v)+\beta_M(y),\alpha(u)+\alpha_M(x)]_\rho\\
&=[\beta(v),\alpha(u)]_L+\delta\rho(\beta(v))(\alpha_M(x))-\rho(\alpha^{-1}\beta(\alpha(u)))(\alpha_M\beta_M^{-1}(\beta_M(v)))\\
&=[\beta(v),\alpha(u)]_L+\delta\rho(\beta(v))(\alpha_M(x))-\rho(\beta(u))(\alpha_M(v))\\
&=-\delta([\beta(u),\alpha(v)]_L+\delta\rho(\beta(u))(\alpha_M(y))-\rho(\beta(v))(\alpha_M(u)))\\
&=-\delta[(\beta\oplus\beta_M)(u+x),(\alpha\oplus\alpha_M)(v+y)]_\rho.
\end{aligned}$$

Next we show that $(\alpha\oplus\alpha_M)$ and $(\beta\oplus\beta_M)$ are algebra morphisms. On the one hand, we have

$$\begin{aligned}
&(\alpha\oplus\alpha_M)[u+x,v+y]_\rho\\
&=((\alpha\oplus\alpha_M)([u,v]_L+\delta\rho(u)(y)-\rho(\alpha^{-1}\beta(v))(\alpha_M\beta_M^{-1}(x)))\\
&=\alpha([u,v]_L)+\delta\alpha_M\circ\rho(u)(y)-\alpha_M\circ\rho(\alpha^{-1}\beta(v))(\alpha_M\beta_M^{-1}(x)))\\
&=[\alpha(u),\alpha(v)]_L+\delta\rho(\alpha(u))(\alpha_M(y))-\rho(\alpha(\alpha^{-1}\beta(v)))(\alpha_M\circ(\alpha_M\beta_M^{-1}(x))))\\
&=[\alpha(u),\alpha(v)]_L+\delta\rho(\alpha(u))(\alpha_M(y))-\rho(\beta(v))(\alpha_M^2\beta_M^{-1}(x))\\
&=[(\alpha\oplus\alpha_M)(u+x),(\alpha\oplus\alpha_M)(v+y)]_\rho.
\end{aligned}$$

Similarly, we obtain

$$(\beta\oplus\beta_M)[u+x,v+y]_\rho=[(\beta\oplus\beta_M)(u+x),(\beta\oplus\beta_M)(v+y)]_\rho.$$

Furthermore

$$\begin{aligned}
&[(\beta\oplus\beta_M)^2(u+x),[(\beta\oplus\beta_M)(v+y),(\alpha\oplus\alpha_M)(w+z)]_\rho]_\rho\\
&=[\beta^2(u)+\beta_M^2(x),[\beta(v)+\beta_M(y),\alpha(w)+\alpha_M(z)]_\rho]_\rho\\
&=[\beta^2(u)+\beta_M^2(x),[\beta(v),\alpha(w)]_L+\delta\rho(\beta(v))(\alpha_M(z))-\rho(\alpha^{-1}\beta(\alpha(w)))(\alpha_M\beta_M^{-1}(\beta_M(y)))]_\rho\\
&=[\beta^2(u)+\beta_M^2(x),[\beta(v),\alpha(w)]_L+\delta\rho(\beta(v))(\alpha_M(z))-\rho(\beta(w))(\alpha_M(y))]_\rho\\
&=[\beta^2(u),[\beta(v),\alpha(w)]_L]_L+\delta^2\rho(\beta^2(u))\rho(\beta(v))(\alpha_M(z))-\delta\rho(\beta^2(u))\rho(\beta(w))(\alpha_M(y))\\
&\quad-\rho(\alpha^{-1}\beta([\beta(v),\alpha(w)])(\alpha_M\beta_M^{-1}(\beta_M^2(x))))\\
&=[\beta^2(u),[\beta(v),\alpha(w)]_L]_L+\rho(\beta^2(u))\rho(\beta(v))(\alpha_M(z))-\delta\rho(\beta^2(u))\rho(\beta(w))(\alpha_M(y))\\
&\quad-\rho([\alpha^{-1}\beta^2(v),\beta(w)])(\alpha_M\beta_M(x)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[(\beta\oplus\beta_M)^2(v+y),[(\beta\oplus\beta_M)(w+z),(\alpha\oplus\alpha_M)(u+x)]_\rho]_\rho\\
&=[\beta^2(v),[\beta(w),\alpha(u)]_L]_L+\rho(\beta^2(v))\rho(\beta(w))(\alpha_M(x))-\delta\rho(\beta^2(v))\rho(\beta(u))(\alpha_M(z))\\
&\quad-\rho([\alpha^{-1}\beta^2(w),\beta(u)])(\alpha_M\beta_M(y)).
\end{aligned}$$

And

$$\begin{aligned}
&[(\beta\oplus\beta_M)^2(w+z),[(\beta\oplus\beta_M)(u+x),(\alpha\oplus\alpha_M)(v+y)]_\rho]_\rho\\
&=[\beta^2(w),[\beta(u),\alpha(v)]_L]_L+\rho(\beta^2(w))\rho(\beta(u))(\alpha_M(y))-\delta\rho(\beta^2(w))\rho(\beta(v))(\alpha_M(x))\\
&\quad-\rho([\alpha^{-1}\beta^2(u),\beta(v)])(\alpha_M\beta_M(z)).
\end{aligned}$$

By (4.3), the δ-Bihom-Jacobi identity is satisfied. Thus, $(L\oplus M,[\cdot,\cdot]_\rho,\alpha\oplus\alpha_M,\beta\oplus\beta_M)$ is a multiplicative δ-Bihom-Jordan-Lie algebra. □

5. The trivial representation of δ-Bihom-Jordan-Lie algebras

In this section, we study the trivial representation of multiplicative δ-hom-Jordan-Lie algebras. As an application, we show that the central extension of a multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ is controlled by the second cohomology of L with coefficients in the trivial representation.

Now let $M = \mathbb{R}$, Then we have $End(M) = \mathbb{R}$. Any $\alpha_M, \beta_M \in End(M)$ is exactly two real numbers, which we denote by r_1, r_2 respectively. Let $\rho : L \to End(M) = \mathbb{R}$ be the zero map. Obviously, ρ is a representation of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$ with respect to any $r_1, r_2 \in \mathbb{R}$. We will always assume that $r_1 = r_2 = 1$. We call this representation the trivial representation of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$.

Associated to the trivial representation, the set of k-cochains on L, which we denote by $C^k(V) = \wedge^k L^*$, is the set of skew-symmetric k-linear maps from $V \times \cdots \times V$ to $\mathbb{R}$.
The set of k-Bihom-cochains is given by

$$C^k_{\alpha,\beta}(L) = \{f \in C^k(L) | f \circ \alpha = f, f \circ \beta = f\}.$$

The corresponding coboundary operator $d_T : C^k_{\alpha,\beta}(L) \to C^{k+1}_{\alpha,\beta}(L)(k = 1, 2)$ is given by

$$d^1_T f(u_1, u_2) = -\delta f([\alpha^{-1}\beta(u_1), u_2]_L), \tag{5.1}$$

$$\begin{aligned} d^2_T f(u_1, u_2, u_3) \quad = \quad & -f([\alpha^{-1}\beta(u_1), u_2]_L, \beta(u_3)) + \delta f([\alpha^{-1}\beta(u_1), u_3]_L, \beta(u_2)) \\ & -f([\alpha^{-1}\beta(u_2), u_3]_L, \beta(u_1)). \end{aligned}$$

Denote $Z^k_{\alpha,\beta}(L)$ and $B^k_{\alpha,\beta}(L)(k = 1, 2)$ similarly.

In the following we consider central extensions of the multiplicative δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$. Obviously, $(\mathbb{R}, 0, 1, 1)$ is an abelian multiplicative δ-Bihom-Jordan-Lie algebra with the trivial bracket and the identity morphism. Let $\theta \in C^2_{\alpha,\beta}(L)$, we have $\theta \circ \alpha = \theta$, $\theta \circ \beta = \theta$ and $\theta(u, v) = -\delta\theta(v, u), \forall u, v \in L$. We consider the direct sum $\mathfrak{g} = L \oplus \mathbb{R}$ with the following bracket

$$[u + s, v + t]_\theta = [u, v]_L + \theta(\alpha\beta^{-1}(u), v), \quad \forall u, v \in L, s, t \in \mathbb{R}. \tag{5.2}$$

Define $\alpha_\mathfrak{g}, \beta_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ by $\alpha_\mathfrak{g}(u + s) = \alpha(u) + s$, and $\beta_\mathfrak{g}(u + s) = \beta(u) + s$.

Theorem 5.1. *With the above notations, the 4-tuple $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ is a multiplicative δ-Bihom-Jordan-Lie algebra if and only if $\theta \in C^2_{\alpha,\beta}(L)$ is a 2-cocycle associated to the trivial representation, i.e.*

$$d_T\theta = 0.$$

We call the multiplicative δ-Bihom-Jordan-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ the central extension of $(L, [\cdot,\cdot]_L, \alpha, \beta)$ by the abelian δ-Bihom-Jordan-Lie algebra $(\mathbb{R}, 0, 1, 1)$.

Proof. Obviously, since $\alpha \circ \beta = \beta \circ \alpha$, we have $\alpha_\mathfrak{g} \circ \beta_\mathfrak{g} = \beta_\mathfrak{g} \circ \alpha_\mathfrak{g}$. Then we show that $\alpha_\mathfrak{g}$ is an algebra morphism with the respect to the bracket $[\cdot,\cdot]_\theta$. On one hand, we have

$$\begin{aligned} \alpha_\mathfrak{g}([u + s, v + t]_\theta &= \alpha_\mathfrak{g}([u, v]_L + \theta(\alpha\beta^{-1}(u), v)) \\ &= \alpha([u, v]_L) + \theta(\alpha\beta^{-1}(u), v)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [\alpha_\mathfrak{g}(u + s), \alpha_\mathfrak{g}(v + t)]_\theta &= [\alpha(u) + s, \alpha(v) + t]_\theta \\ &= [\alpha(u), \alpha(v)]_L + \theta(\alpha\beta^{-1}(\alpha(u)), \alpha(v)). \end{aligned}$$

Since α is an algebra morphism and $\theta(\alpha\beta^{-1}(\alpha(u)), \alpha(v)) = \theta\circ\alpha(\alpha\beta^{-1}(u), v) = \theta(\alpha\beta^{-1}(u), v)$. Then $\alpha_\mathfrak{g}$ is an algebra morphism.

Similarly, we have $\beta_\mathfrak{g}$ is also an algebra morphism.

Furthermore, we have

$$\begin{aligned}[\beta_{\mathfrak{g}}(u+s),\alpha_{\mathfrak{g}}(v+t)]_\theta &= [\beta(u)+s,\alpha(v)+t]_\theta\\ &= [\beta(u),\alpha(v)]_L+\theta(\alpha\beta^{-1}(\beta(u)),\alpha(v))\\ &= [\beta(u),\alpha(v)]_L+\theta(\alpha(u),\alpha(v))\\ &= [\beta(u),\alpha(v)]_L+\theta(u,v)\end{aligned}$$

and

$$\begin{aligned}[\beta_{\mathfrak{g}}(v+t),\alpha_{\mathfrak{g}}(u+s)]_\theta &= [\beta(v)+t,\alpha(u)+s]_\theta\\ &= [\beta(v),\alpha(u)]_L+\theta(\alpha\beta^{-1}(\beta(v)),\alpha(u))\\ &= [\beta(v),\alpha(u)]_L+\theta(\alpha(v),\alpha(u))\\ &= [\beta(v),\alpha(u)]_L+\theta(v,u)\\ &= -\delta([\beta(u),\alpha(v)]_L+\theta(u,v)).\end{aligned}$$

Then $[\beta_{\mathfrak{g}}(u+s),\alpha_{\mathfrak{g}}(v+t)]_\theta = -\delta[\beta_{\mathfrak{g}}(v+t),\alpha_{\mathfrak{g}}(u+s)]_\theta$.

By direct computations, we have

$$\begin{aligned}&[\beta_{\mathfrak{g}}^2(u+s),[\beta_{\mathfrak{g}}(v+t),\alpha_{\mathfrak{g}}(w+r)]_\theta]_\theta+[\beta_{\mathfrak{g}}^2(v+t),[\beta_{\mathfrak{g}}(w+r),\alpha_{\mathfrak{g}}(u+s)]_\theta]_\theta\\ &\quad+[\beta_{\mathfrak{g}}^2(w+r),[\beta_{\mathfrak{g}}(u+s),\alpha_{\mathfrak{g}}(v+t)]_\theta]_\theta\\ &=[\beta^2(u)+s,[\beta(v)+t,\alpha(w)+r]_\theta]_\theta+[\beta^2(v)+t,[\beta(w)+r,\alpha(u)+s]_\theta]_\theta\\ &\quad+[\beta^2(w)+r,[\beta(u)+s,\alpha(v)+t]_\theta]_\theta\\ &=[\beta^2(u)+s,[\beta(v)\alpha(w)]_L+\theta(\alpha\beta^{-1}(\beta(v)),\alpha(w))]_\theta\\ &\quad+[\beta^2(v)+t,[\beta(w)\alpha(u)]_L+\theta(\alpha\beta^{-1}(\beta(w)),\alpha(u))]_\theta\\ &\quad+[\beta^2(w)+r,[\beta(u)\alpha(v)]_L+\theta(\alpha\beta^{-1}(\beta(u)),\alpha(v))]_\theta)\\ &=[\beta^2(u),[\beta(v)\alpha(w)]_L]_L+\theta(\alpha\beta^{-1}(\beta^2(u))),[\beta(v),\alpha(w)]_L\\ &\quad+[\beta^2(v),[\beta(w)\alpha(u)]_L]_L+\theta(\alpha\beta^{-1}(\beta^2(v))),[\beta(w),\alpha(u)]_L\\ &\quad+[\beta^2(w),[\beta(u)\alpha(v)]_L]_L+\theta(\alpha\beta^{-1}(\beta^2(w))),[\beta(u),\alpha(v)]_L\\ &=[\beta^2(u),[\beta(v)\alpha(w)]_L]_L+\theta(\alpha\beta(u),[\beta(v),\alpha(w)]_L)\\ &\quad+[\beta^2(v),[\beta(w)\alpha(u)]_L]_L+\theta(\alpha\beta(v),[\beta(w),\alpha(u)]_L)\\ &\quad+[\beta^2(w),[\beta(u)\alpha(v)]_L]_L+\theta(\alpha\beta(w),[\beta(u),\alpha(v)]_L).\end{aligned}$$

Thus by the Bihom-Jacobi identity of L, $[\cdot,\cdot]_\theta$ satisfies the δ-Bihom-Jacobi identity if and only if

$$\theta(\alpha\beta(u)),[\beta(v),\alpha(w)]_L+\theta(\alpha\beta(v)),[\beta(w),\alpha(u)]_L+\theta(\alpha\beta(w)),[\beta(u),\alpha(v)]_L=0.$$

Namely,

$$\theta(\beta(u),[\alpha^{-1}\beta(v),w]_L)+\theta(\beta(v),[\alpha^{-1}\beta(w),u]_L)+\theta(\beta(w),[\alpha^{-1}\beta(u),v]_L)=0.$$

On the other hand,

$$\begin{aligned}&d_T\theta(u,v,w)\\ &=\delta^3(-\delta\theta([\alpha^{-1}\beta(u),v]_L,\beta(w))+\theta([\alpha^{-1}\beta(w),u]_L,\beta(v))-\delta\theta([\alpha^{-1}\beta(v),w]_L,\beta(u)))\\ &=-(\theta([\alpha^{-1}\beta(u),v]_L,\beta(w))+\theta([\alpha^{-1}\beta(w),u]_L,\beta(v))+\theta([\alpha^{-1}\beta(v),w]_L,\beta(u)))\\ &=\delta([\beta_{\mathfrak{g}}^2(u+s),[\beta_{\mathfrak{g}}(v+t),\alpha_{\mathfrak{g}}(w+r)]_\theta]_\theta+[\beta_{\mathfrak{g}}^2(v+t),[\beta_{\mathfrak{g}}(w+r),\alpha_{\mathfrak{g}}(u+s)]_\theta]_\theta)\\ &=0.\end{aligned}$$

Then the 4-tuple $(\mathfrak{g}, [\cdot,\cdot]_\theta, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ is a multiplicative δ-Bihom-Jordan-Lie algebra if and only if $\theta \in C^2_{\alpha,\beta}(L)$ satisfies $d_T\theta = 0$. □

Proposition 5.2. *For $\theta_1, \theta_2 \in Z^2(V)$, if $\delta(\theta_1 - \theta_2)$ is exact, the corresponding two central extensions $(\mathfrak{g}, [\cdot,\cdot]_{\theta_1}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ and $(\mathfrak{g}, [\cdot,\cdot]_{\theta_2}, \alpha_\mathfrak{g}, \beta_\mathfrak{g})$ are isomorphic.*

Proof. Assume that $\theta_1 - \theta_2 = \delta d_T f$, $f \in C^1_{\alpha,\beta}(L)$. Thus we have

$$\theta_1(\alpha\beta^{-1}(u), v) - \theta_2(u, v) = \delta d^1_T f(\alpha\beta^{-1}(u), v) = -f([\alpha^{-1}\beta \circ \alpha\beta^{-1}(u), v]) = -f([u, v]).$$

Define $\varphi_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ by

$$\varphi_\mathfrak{g}(u + s) = u + s + f(u).$$

Obviously, $\varphi_\mathfrak{g}$ is an isomorphism of vector spaces. The fact that $\varphi_\mathfrak{g}$ is a morphism of the δ-Bihom-Jordan-Lie algebra follows from the fact $\theta \circ \alpha = \theta, \theta \circ \beta = \theta$. More precisely, we have

$$\varphi_\mathfrak{g} \circ \alpha_\mathfrak{g}(u + s) = \varphi_\mathfrak{g}(\alpha(u) + s) = \alpha(u) + s + f(\alpha(u)) = \alpha(u) + s + f(u).$$

On the other hand, we have

$$\alpha_\mathfrak{g} \circ \varphi_\mathfrak{g}(u + s) = \alpha_\mathfrak{g}(u + s + f(u)) = \alpha(u) + s + f(u).$$

Thus, we obtain that $\varphi_\mathfrak{g} \circ \alpha_\mathfrak{g} = \alpha_\mathfrak{g} \circ \varphi_\mathfrak{g}$. Similarly

$$\varphi_\mathfrak{g} \circ \beta_\mathfrak{g} = \beta_\mathfrak{g} \circ \varphi_\mathfrak{g}.$$

We also have

$$\begin{aligned}\varphi_\mathfrak{g}[u + s, v + t]_{\theta_1} &= \varphi_\mathfrak{g}([u, v]_L + \theta_1(\alpha\beta^{-1}(u), v))\\ &= [u, v]_L + \theta_1(\alpha\beta^{-1}(u), v) + f([u, v]_L) = ([u, v]_L, \theta_2(\alpha\beta^{-1}(u), v)\\ &= [\varphi_\mathfrak{g}(u + s), \varphi_\mathfrak{g}(v + t)]_{\theta_2}.\end{aligned}$$

Therefore, $\varphi_\mathfrak{g}$ is also an isomorphism of multiplicative δ-Bihom-Jordan-Lie algebras. □

6. The adjoint representation of δ-Bihom-Jordan-Lie algebras

Let $(L, [\cdot,\cdot]_L, \alpha, \beta)$ be a regular δ-Bihom-Jordan-Lie algebra. We consider that L represents on itself via the bracket with respect to the morphisms α, β. A very interesting phenomenon is that the adjoint representation of a δ-Bihom-Jordan-Lie algebra is not unique as one will see in sequel.

Definition 6.1. For any integer s, t, the $\alpha^s\beta^t$-adjoint representation of the regular δ-Bihom-Jordan-Lie algebra $(L, [\cdot,\cdot]_L, \alpha, \beta)$, which we denote by $\mathrm{ad}_{s,t}$, is defined by

$$\mathrm{ad}_{s,t}(u)(v) = \delta[\alpha^s\beta^t(u), v]_L, \forall u, v \in L.$$

Lemma 6.2. *With the above notations, we have*

$$\begin{aligned}\mathrm{ad}_{s,t}(\alpha(u)) \circ \alpha &= \alpha \circ \mathrm{ad}_{s,t}(u);\\ \mathrm{ad}_{s,t}(\beta(u)) \circ \beta &= \beta \circ \mathrm{ad}_{s,t}(u);\\ \mathrm{ad}_{s,t}([\beta(u), v]_L) \circ \beta &= \mathrm{ad}_{s,t}(\alpha\beta(u)) \circ \mathrm{ad}_{s,t}(v) - \delta \mathrm{ad}_{s,t}(\alpha(v)) \circ \mathrm{ad}_{s,t}(\beta(u)).\end{aligned}$$

Thus the definition of $\alpha^s\beta^t$-adjoint representation is well defined.

Proof. For any $u, v, w \in L$, first we show that $\mathrm{ad}_{s,t}(\alpha(u)) \circ \alpha = \alpha \circ \mathrm{ad}_{s,t}(u)$

$$\begin{aligned}\mathrm{ad}_{s,t}(\alpha(u))(\alpha(v)) &= \delta[\alpha^{s+1}\beta^t(u), \alpha(v)]_L\\ &= \alpha(\delta[\alpha^s\beta^t(u), v]_L) = \alpha \circ \mathrm{ad}_{s,t}(u)(v).\end{aligned}$$

Similarly, we have

$$\mathrm{ad}_{s,t}(\beta(u)) \circ \beta = \beta \circ \mathrm{ad}_{s,t}(u).$$

Note that the skew-symmetry condition implies

$$\begin{aligned}\mathrm{ad}_s(u)(v) &= \delta[\alpha^s\beta^t(u), v]_L\\ &= \delta[\beta(\alpha^s\beta^{t-1}(u)), \alpha(\alpha^{-1}(v))]_L\\ &= -\delta^2[\alpha^{-1}\beta(v), \alpha^{s+1}\beta^{t-1}(u)]_L\\ &= -[\alpha^{-1}\beta(v), \alpha^{s+1}\beta^{t-1}(u)]_L, \forall u, v \in L.\end{aligned}$$

On one hand, we have

$$\begin{aligned}\mathrm{ad}_{s,t}([\beta(u), v]_L) \circ \beta(w) &= \mathrm{ad}_{s,t}([\beta(u), v]_L)(\beta(w))\\ &= -[\alpha^{-1}\beta(\beta(w)), \alpha^{s+1}\beta^{t-1}([\beta(u), v]_L)]_L\\ &= -[\alpha^{-1}\beta^2(w), [\alpha^{s+1}\beta^t(u), \alpha^{s+1}\beta^{t-1}(v)]_L]_L.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathrm{ad}_{s,t}(\alpha\beta(u)) \circ \mathrm{ad}_{s,t}(v)(w) - \delta\mathrm{ad}_{s,t}(\alpha(v)) \circ \mathrm{ad}_{s,t}(\beta(u))(w)\\ &= \mathrm{ad}_{s,t}(\alpha\beta(u))(-[\alpha^{-1}\beta(w), \alpha^{s+1}\beta^{t-1}(v)]_L)\\ &\quad - \delta\mathrm{ad}_{s,t}(\alpha(v))(-[\alpha^{-1}\beta(w), \alpha^{s+2}\beta^{t-1}(u)]_L)\\ &= [\alpha^{-1}\beta([\alpha^{-1}\beta(w), \alpha^{s+1}\beta^{t-1}(v)]_L), \alpha^{s+1}\beta^{t-1}(\alpha\beta(u))]_L)\\ &\quad - \delta[\alpha^{-1}\beta([\alpha^{-1}\beta(w), \alpha^{s+2}\beta^{t-1}(u)]_L), \alpha^{s+1}\beta^{t-1}(\beta(v))]_L)\\ &= [\beta([\alpha^{-2}\beta(w), \alpha^{s}\beta^{t-1}(v)]_L), \alpha^{s+2}\beta^{t}(u)]_L\\ &\quad - \delta[\beta([\alpha^{-2}\beta(w), \alpha^{s+1}\beta^{t-1}(u)]_L), \alpha^{s+1}\beta^{t}(v)]_L)\\ &= -\delta[\beta(\alpha^{s+1}\beta^t(u)), \alpha[\alpha^{-2}\beta(w), \alpha^s\beta^{t-1}(v)]_L]_L\\ &\quad + [\beta(\alpha^s\beta^t(v)), \alpha[\alpha^{-2}\beta(w), \alpha^{s+1}\beta^{t-1}(u)]_L]_L)\\ &= -\delta[\beta(\alpha^{s+1}\beta^t(u)), [\alpha^{-1}\beta(w), \alpha^{s+1}\beta^{t-1}(v)]_L]_L\\ &\quad + [\alpha^s\beta^{t+1}(v), [\alpha^{-1}\beta(w), \alpha^{s+2}\beta^{t-1}(u)]_L]_L)\\ &= [\alpha^{s+1}\beta^{t+1}(u)), [\alpha^s\beta^t(v), w]_L]_L\\ &\quad + [\alpha^s\beta^{t+1}(v), [\alpha^{-1}\beta(w), \alpha^{s+2}\beta^{t-1}(u)]_L]_L)\\ &= [\beta^2(\alpha^{s+1}\beta^{t-1}(u)), [\beta(\alpha^s\beta^{t-1}(v)), \alpha(\alpha^{-1}(w))]_L]_L\\ &\quad + [\beta^2(\alpha^s\beta^{t-1}(v)), [\beta(\alpha^{-1}(w)), \alpha(\alpha^{s+1}\beta^{t-1}(u))]_L]_L)\\ &= -[\beta^2(\alpha^{-1}(w)), [\beta(\alpha^{s+1}\beta^{t-1}(u)), \alpha(\alpha^s\beta^{t-1}(v))]_L]_L\\ &= -[\alpha^{-1}\beta^2(w), [\alpha^{s+1}\beta^t(u), \alpha^{s+1}\beta^{t-1}(v)]_L]_L.\end{aligned}$$

Thus, the definition of $\alpha^s\beta^t$-adjoint representation is well defined. The proof is completed. □

The set of k-Bihom-cochains on L with coefficients in L, which we denote by $C^k_{\alpha,\beta}(L; L)$, is given by

$$C^k_{\alpha,\beta}(L; L) = \{f \in C^k(L; L) | \alpha \circ f = f \circ \alpha, \beta \circ f = f \circ \beta\}.$$

In particular, the set of 0-Bihom-cochains is given by:

$$C^0_{\alpha,\beta}(L; L) = \{u \in L | \alpha(u) = u, \beta(u) = u\}.$$

Associated to the $\alpha^s\beta^t$-adjoint representation, the corresponding operator

$$d_{s,t} : C^k_{\alpha,\beta}(L; L) \to C^{k+1}_{\alpha,\beta}(L; L)(k = 1, 2)$$

is given by

$$d_{s,t}f(u_1, u_2) = \delta[\alpha^{1+s}\beta^t(u_1), f(u_2)] - [\alpha^{1+s}\beta^t(u_2), f(u_1)] - \delta f([\alpha^{-1}\beta(u_1), u_2]); \quad (6.1)$$

$$d_{s,t}f(u_1,u_2,u_3) = \qquad \delta[\alpha^{1+s}\beta^{t+1}(u_1), f(u_2,u_3)] - [\alpha^{1+s}\beta$$
$$+ \delta[\alpha^{1+s}\beta^{t+1}(u_3), f(u_1,u_2)] - f([\alpha^{-1}\beta(u_1), u_2], \beta(u_3))$$
$$+ \delta f([\alpha^{-1}\beta(u_1), u_3], \beta(u_2)) - f([\alpha^{-1}\beta(u_2), u_3], \beta(u_1)).$$

For the $\alpha^s\beta^t$-adjoint representation $\mathrm{ad}_{s,t}$, we obtain the $\alpha^s\beta^t$-adjoint complex $(C^k_{\alpha,\beta}(L;L), d_{s,t})$.

We have known that a 1-cocycle associated to the adjoint representation is a derivation for Lie algebras and Hom-Lie algebras. Similarly, we have

Proposition 6.3. *Associated to the $\alpha^s\beta^t$-adjoint representations $\mathrm{ad}_{s,t}$ of the regular δ-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$, it satisfies $\delta^{s+1} = 1$, $D \in C^1_{\alpha,\beta}(L,L)$ is a 1-cocycle if and only if D is an $\alpha^{s+2}\beta^{t-1}$-derivation, i.e. $D \in \mathrm{Der}_{\alpha^{s+2}\beta^{t-1}}(\mathrm{L})$.*

Proof. The conclusion follows directly from the definition of the operator $d_{s,t}$. D is closed if and only if

$$d_{s,t}(D)(u,v) = \delta[\alpha^{s+1}\beta^t(u), D(v)]_L - [\alpha^{s+1}\beta^t(v), D(u)]_L - \delta D[\alpha^{-1}\beta(u), v]_L = 0.$$

D is an $\alpha^{s+2}\beta^{t-1}$-derivation if and only if

$$\begin{aligned} D[\alpha^{-1}\beta(u), v]_L &= -\delta[\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(v), D(u)]_L + [\alpha^{s+2}\beta^{t-1}\alpha^{-1}\beta(u), D(v)]_L \\ &= \delta^{s+1}([D(u), \alpha^{s+1}\beta^t(v)]_L + [\alpha^{s+1}\beta^t(u), D(v)]_L). \end{aligned}$$

Then, $D \in C^1_{\alpha,\beta}(L,L)$ is a 1-cocycle if and only if D is an $\alpha^{s+2}\beta^{t-1}$-derivation, i.e. $D \in \mathrm{Der}_{\alpha^{s+2}\beta^{t-1}}(\mathrm{L})$. □

Let $\psi \in C^2_{\alpha,\beta}(L;L)$ be a bilinear operator commuting with α and β, also $\psi(u,v) = -\delta\psi(v,u)$. Consider a t-parameterized family of bilinear operations

$$[u,v]_t = [u,v]_L + t\psi(u,v). \tag{6.2}$$

Since ψ commutes with α,β, then α,β are morphisms with respect to the bracket $[\cdot,\cdot]_t$ for every t. If all the brackets $[\cdot,\cdot]_t$ endow $(L,[\cdot,\cdot]_t,\alpha,\beta)$ with regular δ-Bihom-Jordan-Lie algebra structures, we say that ψ generates a deformation of the regular δ-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$. The anti-symmetry of $[\cdot,\cdot]_t$ means that

$$\begin{aligned} [\beta(v),\alpha(u)]_t &= [\beta(v),\alpha(u)]_L + t\psi(\beta(v),\alpha(u)) \\ \text{and}\ [\beta(u),\alpha(v)]_t &= [\beta(u),\alpha(v)]_L + t\psi(\beta(u),\alpha(v)). \end{aligned}$$

Then $[\beta(v),\alpha(u)]_t = -\delta[\beta(u),\alpha(v)]_t$ if and only if

$$\psi(\beta(v),\alpha(u)) = -\delta\psi(\beta(u),\alpha(v)). \tag{6.3}$$

By computing the Bihom-Jacobi identity of $[\cdot,\cdot]_t$

$$\begin{aligned} &[\beta^2(u), [\beta(v),\alpha(w)]_t]_t + [\beta^2(v), [\beta(w),\alpha(u)]_t]_t + [\beta^2(w), [\beta(u),\alpha(v)]_t]_t \\ &= [\beta^2(u), [\beta(v),\alpha(w)]_L + t\psi(\beta(v),\alpha(w))]_t \\ &\quad + [\beta^2(v), [\beta(w),\alpha(u)]_L + t\psi(\beta(w),\alpha(u))]_t \\ &\quad + [\beta^2(w), [\beta(u),\alpha(v)]_L + t\psi(\beta(u),\alpha(v))]_t \\ &= [\beta^2(u), [\beta(v),\alpha(w)]_L]_t + [\beta^2(u), t\psi(\beta(v),\alpha(w))]_t \\ &\quad + [\beta^2(v), [\beta(w),\alpha(u)]_L]_t + [\beta^2(v), t\psi(\beta(w),\alpha(u))]_t \\ &\quad + [\beta^2(w), [\beta(u),\alpha(v)]_L]_t + [\beta^2(w), t\psi(\beta(u),\alpha(v))]_t \\ &= [\beta^2(u), [\beta(v),\alpha(w)]_L]_L + t\psi(\beta^2(u), [\beta(v),\alpha(w)]_L) \\ &\quad + [\beta^2(u), t\psi(\beta(v),\alpha(w))]_L + t\psi(\beta^2(u), t\psi(\beta(v),\alpha(w))) \\ &\quad + [\beta^2(v), [\beta(w),\alpha(u)]_L]_L + t\psi(\beta^2(v), [\beta(w),\alpha(u)]_L) \end{aligned}$$

$$
\begin{aligned}
&+ [\beta^2(v), t\psi(\beta(w), \alpha(u))]_L + t\psi(\beta^2(v), t\psi(\beta(w), \alpha(u)))\\
&+ [\beta^2(w), [\beta(u), \alpha(v)]_L]_L + t\psi(\beta^2(w), [\beta(u), \alpha(v)]_L)\\
&+ [\beta^2(w), t\psi(\beta(u), \alpha(v))]_L + t\psi(\beta^2(w), t\psi(\beta(u), \alpha(v))).
\end{aligned}
$$

This is equivalent to the conditions

$$\psi(\beta^2(u), \psi(\beta(v), \alpha(w))) + \psi(\beta^2(v), \psi(\beta(w), \alpha(u))) + \psi(\beta^2(w), \psi(\beta(u), \alpha(v))) = 0, \quad (6.4)$$

$$
\begin{aligned}
&\psi(\beta^2(u), [\beta(v), \alpha(w)]_L) + [\beta^2(u), \psi(\beta(v), \alpha(w))]_L\\
&+\psi(\beta^2(v), [\beta(w), \alpha(u)]_L) + [\beta^2(v), \psi(\beta(w), \alpha(u))]_L\\
&+\psi(\beta^2(w), [\beta(u), \alpha(v)]_L) + [\beta^2(w), \psi(\beta(u), \alpha(v))]_L = 0. \quad (6.5)
\end{aligned}
$$

Obviously, (6.4) and (6.3) means that ψ must itself define a δ-Bihom-Jordan-Lie algebra structure on L. Furthermore, (6.5) means that ψ is closed with respect to the $\alpha^{-1}\beta$-adjoint representation $\mathrm{ad}_{-1,1}$, i.e. $d_{-1,1}\psi = 0$.

$$
\begin{aligned}
&d_{-1,1}\psi(u, v, w)\\
&= \delta[\beta^2(u), \psi(v, w)]_L - [\beta^2(v), \psi(u, w)]_L + \delta[\beta^2(w), \psi(u, v)]_L\\
&\quad - \psi([\alpha^{-1}\beta(u), v]_L, \beta(w)) + \delta\psi([\alpha^{-1}\beta(u), w]_L, \beta(v)) - \psi([\alpha^{-1}\beta(v), w]_L, \beta(u))\\
&= \delta[\beta^2(u), \psi(v, w)]_L + \delta[\beta^2(v), \psi(w, u)]_L + \delta[\beta^2(w), \psi(u, v)]_L\\
&\quad + \delta\psi(\beta(w), [\alpha^{-1}\beta(u), v]_L) + \delta\psi(\beta(v), [\alpha^{-1}\beta(w), u]_L) + \delta\psi(\beta(u), [\alpha^{-1}\beta(v), w]_L)\\
&= 0.
\end{aligned}
$$

A deformation is said to be trivial if there is a linear operator $N \in C^1_{\alpha,\beta}(L; L)$ such that for $T_t = \mathrm{id} + tN$, there holds

$$T_t[u, v]_t = [T_t(u), T_t(v)]_L. \quad (6.6)$$

Definition 6.4. A linear operator $N \in C^1_{\alpha,\beta}(L, L)$ is called a Bihom-Nijienhuis operator if we have

$$[Nu, Nv]_L = N[u, v]_N, \quad (6.7)$$

where the bracket $[\cdot, \cdot]_N$ is defined by

$$[u, v]_N \triangleq [Nu, v]_L + [u, Nv]_L - N[u, v]_L. \quad (6.8)$$

Theorem 6.5. *Let $N \in C^1_\alpha(L, L)$ be a Bihom-Nijienhuis operator. Then a deformation of the regular δ-Bihom-Jordan-Lie algebra $(L, [\cdot, \cdot]_L, \alpha, \beta)$ can be obtained by putting*

$$\psi(u, v) = \delta d_{-1,1}N(u, v) = [u, v]_N.$$

Furthermore, this deformation is trivial.

Proof. Since $\psi = \delta d_{-1,1}N$, $d_{-1,1}\psi = 0$ is valid. To see that ψ generates a deformation, we need to check the Bihom-Jacobi identity for ψ. Using the explicit expression of ψ, and we denote $\circlearrowleft_{u,v,w}$ the summation over the cyclic permutation on u, v, w. We have

$$
\begin{aligned}
&\circlearrowleft_{u,v,w} \psi(\beta^2(u), \psi(\beta(v), \alpha(w))\\
&=\circlearrowleft_{u,v,w} \psi(\beta^2(u), [N\beta(v), \alpha(w)] + [\beta(v), N\alpha(u)] - N[\beta(v), \alpha(w)])\\
&=\circlearrowleft_{u,v,w} \psi(\beta^2(u), [N\beta(v), \alpha(w)]) + \psi(\beta^2(u), [\beta(v), N\alpha(u)]) - \psi(\beta^2(u), N[\beta(v), \alpha(w)])\\
&=\circlearrowleft_{u,v,w} [N\beta^2(u), [N\beta(v), \alpha(w)]] + [N\beta^2(v), [\beta(w), N\beta(u)]] + [\beta^2(w), N[\beta(u), \alpha(v)]_N]\\
&\quad + \circlearrowleft_{u,v,w} N[\beta^2(v), N[\beta(w), \alpha(u)]] - [N\beta^2(v), N[\beta(w), \alpha(u)]]\\
&\quad \circlearrowleft_{u,v,w} -N[\beta^2(u), [N\beta(v), \alpha(w)]] + N[\beta^2(w), [\beta(u), N\alpha(v)]]
\end{aligned}
$$

Since N commutes with α and β, by the Bihom-Jacobi identity of L, we have

$$[N\beta^2(u),[N\beta(v),\alpha(w)]]+[N\beta^2(v),[\beta(w),N\alpha(u)]]=[[N\beta(u),N\alpha(v)],\beta^2(w)].$$

Since N is a Bihom-Nijenhuis operator, the last equation becomes

$$\circlearrowleft_{u,v,w}[N\beta^2(u),[N\beta(v),\alpha(w)]]+[N\beta^2(v),[\beta(w),N\alpha(u)]]+[\beta^2(w),N[\beta(u),\alpha(v)]_N]=0.$$

Furthermore, also by the fact that N is a Bihom-Nijenhuis operator and we take in (6.7) and (6.8), $u=\beta^2(v)$ and $v=[\beta(w),\alpha(u)]$, we have $N[\beta^2(v),N[\beta(w),\alpha(u)]]-[N\beta^2(v),N[\beta(w),\alpha(u)]]=-N[N\beta^2(v),[\beta(w),\alpha(u)]]+N^2[\beta^2(v),[\beta(w),\alpha(u)]]$.

By the Bihom-Jacobi identity of L, we have

$$\begin{aligned}&\circlearrowleft_{u,v,w}N[\beta^2(v),N[\beta(w),\alpha(u)]]-[N\beta^2(v),N[\beta(w),\alpha(u)]]\\&=\circlearrowleft_{u,v,w}N[N\beta^2(v),[\beta(w),\alpha(u)]].\end{aligned}$$

Then,

$$\begin{aligned}&\circlearrowleft_{u,v,w}\psi(\beta^2(u),\psi(\beta(v),\alpha(w))\\&=-N[N\beta^2(v),[\beta(w),\alpha(u)]]-N[\beta^2(u),[N\beta(v),\alpha(w)]]+[\beta^2(w),[\beta(u),N\beta(v)]]\\&=-N[\beta^2(Nv),[\beta(w),\alpha(u)]]+[\beta^2(u),[N\beta(v),\alpha(w)]]+[\beta^2(w),[\beta(u),N\beta(v)]]\\&=0.\end{aligned}$$

Thus ψ generates a deformation of the δ-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$.

Let $T_t=\mathrm{id}+tN$, then we have

$$\begin{aligned}T_t[u,v]_t&=(\mathrm{id}+tN)([u,v]_L+t\psi(u,v))\\&=(\mathrm{id}+tN)([u,v]_L+t[u,v]_N)\\&=[u,v]_L+t([u,v]_N+N[u,v]_L)+t^2N[u,v]_N.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[T_t(u),T_t(v)]_L&=[u+tNu,v+tNv]_L\\&=[u,v]_L+t([Nu,v]_L+[u,Nv]_L)+t^2[Nu,Nv]_L.\end{aligned}$$

By the equations (6.7) and (6.8), we have

$$T_t[u,v]_t=[T_t(u),T_t(v)]_L,$$

which implies that the deformation is trivial. □

7. T^*-extensions of δ-Bihom-Jordan-Lie algebras

The last part deals with T^*-extension. We provide in this section, for δ-Bihom-Jordan-Lie algebras, characterizations of T^*-extensions and observations about T^*-extensions of nilpotent and solvable δ-Bihom-Jordan-Lie algebras. This method was introduced by Martin Bordemann in [2].

Definition 7.1. Let $(L,[\cdot,\cdot]_L,\alpha,\beta)$ be a δ-Bihom-Jordan-Lie algebra. A bilinear form f on L is said to be nondegenerate if

$$L^\perp=\{x\in L|f(x,y)=0,\forall y\in L\}=0;$$

$\alpha\beta$-invariant if

$$f([\beta(x),\alpha(y)],\alpha(z))=f(\alpha(x),[\beta(y),\alpha(z)]),\forall x,y,z\in L;$$

symmetric if

$$f(x,y)=f(y,x).$$

A subspace I of L is called isotropic if $I\subseteq I^\perp$.

Definition 7.2. Let $(L,[\cdot,\cdot]_L,\alpha,\beta)$ be a δ-Bihom-Jordan-Lie algebra over a field $\mathbb{K}$. If L admits a nondegenerate invariant symmetric bilinear form f, then we call (L,f,α,β) a quadratic δ-Bihom-Jordan-Lie algebra. In particular, a quadratic vector space V is a vector space admitting a nondegenerate symmetric bilinear form.

Let $(L',[\cdot,\cdot]'_L,\alpha_1,\beta_1)$ be another δ-Bihom-Jordan-Lie algebra. Two quadratic δ-Bihom-Jordan-Lie algebras (L,f,α,β) and (L',f',α_1,β_1) are said to be isometric if there exists a δ-Bihom-Jordan-Lie algebra isomorphism $\phi: L\to L'$ such that

$$f(x,y)=f'(\phi(x),\phi(y)),\forall x,y\in L.$$

Lemma 7.3. *Let* ad *be the adjoint representation of a δ-Bihom-Jordan-Lie algebra $(L,[\cdot,\cdot]_L,\alpha,\beta)$. Let us consider L^* the dual space of L, $\tilde{\alpha},\tilde{\beta}: L^*\to L^*$ two homomorphisms defined by*

$$\tilde{\alpha}(f)=f\circ\alpha,\tilde{\beta}(f)=f\circ\beta,\forall f\in L^*.$$

Then the linear map $\pi: L\to \mathrm{End}(L^*)$ defined by, $\pi(x)(f)(y)=-\delta f\circ\mathrm{ad}(x)(y),\forall x,y\in L$, is a representation of L on $(L^*,\tilde{\alpha},\tilde{\beta})$ if and only if

$$\alpha\circ\mathrm{ad}\alpha(x)=\mathrm{ad}x\circ\alpha; \tag{7.1}$$

$$\beta\circ\mathrm{ad}\beta(x)=\mathrm{ad}x\circ\beta; \tag{7.2}$$

$$\mathrm{ad}(\alpha(x))\circ\mathrm{ad}\beta(y)-\delta\mathrm{ad}y\circ\mathrm{ad}(\alpha\beta(x))=\beta\circ\mathrm{ad}[\beta(x),y]_L. \tag{7.3}$$

We call the representation π the coadjoint representation of L.

Proof. Firstly, we have

$$(\pi(\alpha(x))\circ\tilde{\alpha})(f)=-\delta\tilde{\alpha}(f)\circ\mathrm{ad}\alpha(x)=-\delta f\circ\alpha\circ\mathrm{ad}\alpha(x),$$

and

$$\tilde{\alpha}(\pi(x))(f)=-\delta\tilde{\alpha}(f\circ\mathrm{ad}x)=-\delta f\circ\mathrm{ad}x\circ\alpha.$$

Similarly,

$$(\pi(\beta(x))\circ\tilde{\beta})(f)=-\delta\tilde{\beta}(f)\circ\mathrm{ad}\beta(x)=-\delta f\circ\beta\circ\mathrm{ad}\beta(x),$$

and

$$\tilde{\beta}(\pi(x))(f)=-\delta\tilde{\beta}(f\circ\mathrm{ad}x)=-\delta f\circ\mathrm{ad}x\circ\beta.$$

Therefore,

$$(\pi([\beta(x),y])\circ\tilde{\beta})(f)=-\delta f\circ\beta\circ\mathrm{ad}[\beta(x),y];$$

$$\begin{aligned}&(\pi(\alpha\beta(x))\circ\pi(y)-\delta\pi(\beta(y))\circ\pi(\alpha(x)))(f)\\&=-\delta\pi(\alpha\beta(x))(f\circ ady)+\pi(\beta(y))(f\circ ad\alpha(x))\\&=f\circ ady\circ\mathrm{ad}\alpha\beta(x)-\delta f\circ ad\alpha(x)\circ\mathrm{ad}\beta(y)\\&=-\delta f\circ(\mathrm{ad}\alpha(x)\circ\mathrm{ad}\beta(y)-\delta\mathrm{ad}y\circ\mathrm{ad}\alpha\beta(x)).\end{aligned}$$

Then we have

$$\begin{aligned}\pi(\alpha(x))\circ\tilde{\alpha}&=\tilde{\alpha}(\pi(x));\\\pi(\beta(x))\circ\tilde{\beta}&=\tilde{\beta}(\pi(x));\\\pi([\beta(x),y])\circ\tilde{\beta}&=\pi(\alpha\beta(x))\circ\pi(y)-\delta\pi(\beta(y))\circ\pi(\alpha(x)).\end{aligned}$$

Then π is a representation of L on $(L^*,\tilde{\alpha},\tilde{\beta})$. □

Lemma 7.4. *Under the above notations, let $(L, [\cdot,\cdot]_L, \alpha, \beta)$ be a δ-Bihom-Jordan-Lie algebra, and $\omega : L \times L \to L^*$ be a bilinear map. Assume that the coadjoint representation exists. The space $L \oplus L^*$, provided with the following bracket and a linear map defined respectively by*

$$[x+f, y+g]_{L\oplus L^*} = [x,y]_L + \omega(x,y) + \delta\pi(x)g - \pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f), \tag{7.4}$$

$$\alpha^{'}(x+f) = \alpha(x) + f\circ\alpha, \tag{7.5}$$

$$\beta^{'}(x+f) = \beta(x) + f\circ\beta. \tag{7.6}$$

Then $(L\oplus L^*, [\cdot,\cdot]_{L\oplus L^*}, \alpha^{'}, \beta^{'})$ is a δ-Bihom-Jordan-Lie algebra if and only if ω is a 2-cocycle: $L\times L \to L^*$, i.e. $\omega \in Z^2(L, L^*)$.

Proof. For any elements $x+f, y+g, z+h \in L\oplus L^*$. We have

$$\begin{aligned}
&[\beta^{'}(x+f), \alpha^{'}(y+g)]\\
&= [\beta(x)+f\circ\beta, \alpha(y)+g\circ\alpha]\\
&= [\beta(x),\alpha(y)]_L + w(\beta(x),\alpha(y)) + \delta\pi(\beta(x))(g\circ\alpha) - \pi(\alpha^{-1}\beta(\alpha(y)))\tilde{\alpha}\tilde{\beta}^{-1}(f\circ\beta)\\
&= [\beta(x),\alpha(y)]_L + w(\beta(x),\alpha(y)) + \delta\pi(\beta(x))(g\circ\alpha) - \pi(\beta(y))(f\circ\alpha).
\end{aligned}$$

Similarly, we have

$$[\beta^{'}(y+g), \alpha^{'}(x+f)] = [\beta(y),\alpha(x)]_L + w(\beta(y),\alpha(x)) + \delta\pi(\beta(y))(f\circ\alpha) - \pi(\beta(x))(g\circ\alpha).$$

Then, we have $[\beta^{'}(x+f), \alpha^{'}(y+g)] = -\delta[\beta^{'}(y+g), \alpha^{'}(x+f)]$ if and only if

$$w(\beta(x),\alpha(y)) = -\delta w(\beta(y),\alpha(x)).$$

Therefore,

$$\begin{aligned}
&[\beta^{'2}(x+f), [\beta^{'}(y+g), \alpha^{'}(z+h)]]\\
&= [\beta^2(x)+f\circ\beta^2, [\beta(y)+g\circ\beta, \alpha(z)+h\circ\alpha]]\\
&= [\beta^2(x)+f\circ\beta^2, [\beta(y),\alpha(z)]_L + w(\beta(y),\alpha(z))\\
&\quad + \delta\pi(\beta(y))(h\circ\alpha) - \pi(\alpha^{-1}\beta(\alpha(z)))(\tilde{\alpha}\tilde{\beta}^{-1}(g\circ\beta))]\\
&= [\beta^2(x)+f\circ\beta^2, [\beta(y),\alpha(z)]_L + w(\beta(y),\alpha(z)) + \delta\pi(\beta(y))(h\circ\alpha) - \pi(\beta(z))(g\circ\alpha)]\\
&= [\beta^2(x), [\beta(y),\alpha(z)]_L]_L + w(\beta^2(x), [\beta(y),\alpha(z)]_L)\\
&\quad + \delta\pi(\beta^2(x))w(\beta(y),\alpha(z)) + \pi(\beta^2(x))\pi(\beta(y))(h\circ\alpha)\\
&\quad - \delta\pi(\beta^2(x))\pi(\beta(z))(g\circ\alpha) - \pi(\alpha^{-1}\beta[\beta(y),\alpha(z)])(f\circ\beta\circ\alpha)).
\end{aligned}$$

And

$$\begin{aligned}
&[\beta^{'2}(y+g), [\beta^{'}(z+h), \alpha^{'}(x+f)]]\\
&= [\beta^2(y), [\beta(z),\alpha(x)]_L]_L + w(\beta^2(y), [\beta(z),\alpha(x)]_L)\\
&\quad + \delta\pi(\beta^2(y))w(\beta(z),\alpha(x)) + \pi(\beta^2(y))\pi(\beta(z))(f\circ\alpha)\\
&\quad - \delta\pi(\beta^2(y))\pi(\beta(x))(h\circ\alpha) - \pi(\alpha^{-1}\beta[\beta(z),\alpha(x)])(g\circ\beta\circ\alpha),
\end{aligned}$$

$$\begin{aligned}
&[\beta^{'2}(z+h), [\beta^{'}(x+f), \alpha^{'}(y+g)]]\\
&= [\beta^2(z), [\beta(x),\alpha(y)]_L]_L + w(\beta^2(z), [\beta(x),\alpha(y)]_L)\\
&\quad + \delta\pi(\beta^2(z))w(\beta(x),\alpha(y)) + \pi(\beta^2(z))\pi(\beta(x))(g\circ\alpha)\\
&\quad - \delta\pi(\beta^2(z))\pi(\beta(y))(f\circ\alpha) - \pi(\alpha^{-1}\beta[\beta(x),\alpha(y)])(h\circ\beta\circ\alpha).
\end{aligned}$$

Since π is the coadjoint representation of L, we have

$$\pi(\alpha^{-1}\beta[\beta(x),\alpha(y)]_L)h\circ\beta\circ\alpha$$

$$\begin{aligned}
&= \pi([\beta(\alpha^{-1}\beta(x)),\beta(y)]_L)\circ\tilde{\beta}(h\circ\alpha)\\
&= \pi(\alpha\beta(\alpha^{-1}\beta(x)))\pi(\beta(y))(h\circ\alpha)-\delta\pi(\beta(\beta(y)))\pi(\alpha(\alpha^{-1}\beta(x)))(h\circ\alpha)\\
&= \pi(\beta^2(x))\pi(\beta(y))(h\circ\alpha)-\delta\pi(\beta^2(y)))\pi(\beta(x))(h\circ\alpha).
\end{aligned}$$

Similarly,

$$\pi(\alpha^{-1}\beta[\beta(y),\alpha(z)]_L)f\circ\beta\circ\alpha=\pi(\beta^2(y))\pi(\beta(z))(f\circ\alpha)-\delta\pi(\beta^2(z)))\pi(\beta(y))(f\circ\alpha),$$

and

$$\pi(\alpha^{-1}\beta[\beta(z),\alpha(x)]_L)g\circ\beta\circ\alpha=\pi(\beta^2(z))\pi(\beta(x))(g\circ\alpha)-\delta\pi(\beta^2(x)))\pi(\beta(z))(g\circ\alpha).$$

Consequently, $[\beta'^2(x+f),[\beta'(y+g),\alpha'(z+h)]]+[\beta'^2(y+g),[\beta'(z+h),\alpha'(x+f)]]+[\beta'^2(z+h),[\beta'(x+f),\alpha'(y+g)]]=0$ if and only if

$$\begin{aligned}
0 &= w(\beta^2(x),[\beta(y),\alpha(z)]_L)+\delta\pi(\beta^2(x))w(\beta(y),\alpha(z))\\
&\quad +w(\beta^2(y),[\beta(z),\alpha(x)]_L)-\delta\pi(\beta^2(y))w(\beta(z),\alpha(x))\\
&\quad +\delta w(\beta^2(z),[\beta(x),\alpha(y)]_L)+\delta\pi(\beta^2(z))w(\beta(x),\alpha(y))\\
&= [\beta^2(x),w(\beta(y),\alpha(z))]-\delta[\beta^2(y),w(\beta(z),\alpha(x))]+\delta[\beta^2(z),w(\beta(x),\alpha(y))]\\
&\quad -\delta w([\beta(y),\alpha(z)]_L,\beta^2(x))+w(\beta^2(y),[\beta(x),\alpha(y)]_L)-\delta w([\beta(x),\alpha(y)]_L,\beta^2(z))\\
&= \delta d_{-1,1}\omega(x,y,z).
\end{aligned}$$

That is $\omega\in Z^2_{\alpha,\beta}(L,L^*)$. Then confirmation holds if and only if $\omega\in Z^2(L,L^*)$. Consequently, we prove the lemma. □

Clearly, L^* is an abelian Bihom-ideal of $(L\oplus L^*,[\cdot,\cdot],\alpha',\beta')$ and L is isomorphic to the factor δ-Bihom-Jordan-Lie algebra $(L\oplus L^*)/L^*$. Moreover, consider the following symmetric bilinear form q_L on $L\oplus L^*$ for all $x+f,y+g\in L\oplus L^*$,

$$q_L(x+f,y+g)=f(y)+g(x).$$

Then we have the following lemma.

Lemma 7.5. *Let L, L^*, ω and q_L be as above. Then the 4-tuple $(L\oplus L^*,q_L,\alpha',\beta')$ is a quadratic δ-Bihom-Jordan-Lie algebra if and only if ω is Jordancyclic in the following sense:*

$$\omega(\beta(x),\alpha(y))(\alpha(z))=\omega(\beta(y),\alpha(z))(\alpha(x))\quad \textit{for all } x,y,z\in L.$$

Proof. If $x+f$ is orthogonal to all elements of $L\oplus L^*$, then $f(y)=0$ and $g(x)=0$, which implies that $x=0$ and $f=0$. So the symmetric bilinear form q_L is nondegenerate.

Now suppose that $x+f,y+g,z+h\in L\oplus L^*$, then

$$\begin{aligned}
&q_L([\beta'(x+f),\alpha'(y+g)]_{L\oplus L^*},\alpha'(z+h))\\
&= q_L([\beta(x)+f\circ\beta,\alpha(y)+g\circ\alpha]_{L\oplus L^*},\alpha(z)+h\circ\alpha)\\
&= q_L([\beta(x),\alpha(y)]_L+\omega(\beta(x),\alpha(y))+\delta\pi(\beta(x))g\circ\alpha-\pi(\alpha^{-1}\beta\alpha(y))\tilde{\alpha}\tilde{\beta}^{-1}(f\circ\beta),\alpha(z)\\
&\quad +h\circ\alpha)\\
&= q_L([\beta(x),\alpha(y)]_L+\omega(\beta(x),\alpha(y))+\delta\pi(\beta(x))g\circ\alpha-\pi(\beta(y))(f\circ\alpha),\alpha(z)+h\circ\alpha)\\
&= \omega(\beta(x),\alpha(y))(\alpha(z))+\delta(\pi(\beta(x))g\circ\alpha)(\alpha(z))-\pi(\beta(y))(f\circ\alpha)(\alpha(z))\\
&\quad +h\circ\alpha([\beta(x),\alpha(y)]_L)\\
&= \omega(\beta(x),\alpha(y))(\alpha(z))-\delta g\circ\alpha([\beta(x),\alpha(z)]_L)+f\circ\alpha([\beta(y),\alpha(z)]_L)+h\circ\alpha([\beta(x),\alpha(y)]_L)\\
&= \omega(\beta(x),\alpha(y))(\alpha(z))+g\circ\alpha([\beta(z),\alpha(x)]_L)+f\circ\alpha([\beta(y),\alpha(z)]_L)-\delta h\circ\alpha([\beta(y),\alpha(x)]_L).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&q_L(\alpha'(x+f),[\beta'(y+g),\alpha'(z+h)]_{L\oplus L^*})\\
&= q_L(\alpha(x)+f\circ\alpha,[\beta(y)+g\circ\beta,\alpha(z)+h\circ\alpha]_{L\oplus L^*})
\end{aligned}$$

$$= q_L(\alpha(x) + f\circ\alpha, [\beta(y),\alpha(z)]_L + \omega(\beta(y),\alpha(z)) + \delta\pi(\beta(y))h\circ\alpha$$
$$- \pi(\alpha^{-1}\beta\alpha(z))\tilde{\alpha}\tilde{\beta}^{-1}(g\circ\beta))$$
$$= q_L(\alpha(x) + f\circ\alpha, [\beta(y),\alpha(z)]_L + \omega(\beta(y),\alpha(z)) + \delta\pi(\beta(y))h\circ\alpha - \pi(\beta(z))(g\circ\alpha))$$
$$= f\circ\alpha([\beta(y),\alpha(z)]_L) + \omega(\beta(y),\alpha(z))(\alpha(x)) + \delta\pi(\beta(y))h\circ\alpha(\alpha(x))$$
$$- \pi(\beta(z))(g\circ\alpha))(\alpha(x))$$
$$= \omega(\beta(y),\alpha(z))(\alpha(x)) + g\circ\alpha([\beta(z),\alpha(x)]_L) + f\circ\alpha([\beta(y).\alpha(z)]_L) - \delta h\circ\alpha([\beta(y),\alpha(x)]_L).$$

Hence the lemma follows. □

Now, for a Jordancyclic 2-cocycle ω we shall call the quadratic δ-Bihom-Jordan-Lie algebra $(L\oplus L^*, q_L, \alpha', \beta')$ the T^*-extension of L (by ω) and denote the δ-Bihom-Jordan-Lie algebra $(L\oplus L^*, [\cdot,\cdot], \alpha', \beta')$ by $T^*_\omega L$.

Definition 7.6. Let L be a δ-Bihom-Jordan-Lie algebra over a field $\mathbb{K}$. We inductively define a derived series

$$(L^{(n)})_{n\geq 0} : L^{(0)} = L,\ L^{(n+1)} = [L^{(n)}, L^{(n)}],$$

and a central descending series

$$(L^n)_{n\geq 0} : L^0 = L,\ L^{n+1} = [L^n, L].$$

L is called solvable and nilpotent(of length k) if and only if there is a (smallest) integer k such that $L^{(k)} = 0$ and $L^k = 0$, respectively.

In the following theorem we discuss some properties of $T^*_\omega L$.

Theorem 7.7. *Let $(L, [\cdot,\cdot]_L, \alpha, \beta)$ be a δ-Bihom-Jordan-Lie algebra over a field $\mathbb{K}$.*

1. *If L is solvable (nilpotent) of length k, then the T^*-extension $T^*_\omega L$ is solvable (nilpotent) of length r, where $k\leq r\leq k+1$ ($k\leq r\leq 2k-1$).*
2. *If L is decomposed into a direct sum of two Bihom-ideals of L, so is the trivial T^*-extension $T^*_0 L$.*

Proof. (1) Firstly we suppose that L is solvable of length k. Since $(T^*_\omega L)^{(n)}/L^* \cong L^{(n)}$ and $L^{(k)} = 0$, we have $(T^*_\omega L)^{(k)} \subseteq L^*$, which implies $(T^*_\omega L)^{(k+1)} = 0$ because L^* is abelian, and it follows that $T^*_\omega L$ is solvable of length k or $k+1$.

Suppose now that L is nilpotent of length k. Since $(T^*_\omega L)^n/L^* \cong L^n$ and $L^k = 0$, we have $(T^*_\omega L)^k \subseteq L^*$. Let $g\in (T^*_\omega L)^k \subseteq L^*, b\in L,\ x_1+f_1,\cdots,x_{k-1}+f_{k-1}\in T^*_\omega L$, $1\leq i\leq k-1$, we have

$$[[\cdots[g, x_1+f_1]_{L\oplus L^*},\cdots]_{L\oplus L^*}, x_{k-1}+f_{k-1}]_{L\oplus L^*}(b)$$
$$= \delta^{k-1} g\mathrm{ad}(x_1)\mathrm{ad}(\beta^{-1}\alpha(x_2))\cdots\mathrm{ad}(x_{k-1})\beta^{-(k-1)}\alpha^{k-1}(b)$$
$$= g([x_1, [\beta^{-1}\alpha(x_2), [\cdots, [\beta^{-(k-2)}\alpha^{k-2}(x_{k-1}), \beta^{-(k-1)}\alpha^{k-1}(b)]_L\cdots]_L]_L]_L)$$
$$\in g(L^k) = 0.$$

This proves that $(T^*_\omega L)^{2k-1} = 0$. Hence $T^*_w L$ is nilpotent of length at least k and at most $2k-1$.

(2) Suppose that $0\neq L = I\oplus J$, where I and J are two nonzero Bihom-ideals of $(L[\cdot,\cdot]_L, \alpha, \beta)$. Let I^* (resp. J^*) denote the subspace of all linear forms in L^* vanishing on J (resp. I). Clearly, I^* (resp. J^*) can canonically be identified with the dual space of I (resp. J) and $L^*\cong I^*\oplus J^*$.

Since $[I^*, L]_{L\oplus L^*}(J) = I^*([L, \beta^{-1}\alpha(J)]_L) \subseteq I^*([L,J]_L) \subseteq I^*(J) = 0$ and $[I, L^*]_{L\oplus L^*}(J) = L^*([I,J]_L) \subseteq L^*(I\cap J) = 0$, we have $[I^*, L]_{L\oplus L^*}\subseteq I^*$ and $[I, L^*]_{L\oplus L^*}\subseteq I^*$. Then

$$\begin{aligned}[T^*_0 I, T^*_0 L]_{L\oplus L^*} &= [I\oplus I^*, L\oplus L^*]_{L\oplus L^*}\\ &= [I,L]_L + [I,L^*]_{L\oplus L^*} + [I^*,L]_{L\oplus L^*} + [I^*,L^*]_{L\oplus L^*} \subseteq I\oplus I^* = T^*_0 I.\end{aligned}$$

T_0^*I is a Bihom-ideal of L and so is T_0^*J in the same way. Hence T_0^*L can be decomposed into the direct sum $T_0^*I \oplus T_0^*J$ of two nonzero Bihom-ideals of T_0^*L. □

In the proof of a criterion for recognizing T^*-extensions of a δ-Bihom-Jordan-Lie algebra, we will need the following result.

Lemma 7.8. *Let (L, q_L, α, β) be a quadratic δ-Bihom-Jordan-Lie algebra of even dimension n over a field $\mathbb{K}$ and I be an isotropic $n/2$-dimensional subspace of L. If I is a Bihom-ideal of $(L, [\cdot,\cdot]_L, \alpha, \beta)$, then $[\beta(I), \alpha(I)] = 0$.*

Proof. Since $\dim I + \dim I^\perp = n/2 + \dim I^\perp = n$ and $I \subseteq I^\perp$, we have $I = I^\perp$. If I is a ideal of $(L, [\cdot,\cdot]_L, \alpha, \beta)$, then $q_L(\alpha(L), [\beta(I), \alpha(I^\perp)]) = q_L([\beta(L), \alpha(I)], \alpha(I^\perp)) \subseteq q_L([\beta(L), I], \alpha(I^\perp)) \subseteq q_L(I, I^\perp) = 0$, which implies $[\beta(I), \alpha(I)] = [\beta(I), \alpha(I^\perp)] \subseteq \alpha(L)^\perp = 0$. □

Theorem 7.9. *Let (L, q_L, α, β) be a quadratic regular δ-Bihom-Jordan-Lie algebra of even dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then (L, q_L, α, β) is isometric to a T^*-extension $(T_\omega^*B, q_B, \alpha', \beta')$ if and only if n is even and $(L, [\cdot,\cdot]_L, \alpha, \beta)$ contains an isotropic Bihom-ideal I of dimension $n/2$. In particular, $B \cong L/I$, with B^* satisfying $\alpha(B^*) \subseteq B^*$and $\beta(B^*) \subseteq B^*$.*

Proof. ($\Longrightarrow$) Since $\dim B$=$\dim B^*$, $\dim T_\omega^*B$ is even. Moreover, it is clear that B^* is a Bihom-ideal of half the dimension of T_ω^*B and by the definition of q_B, we have $q_B(B^*, B^*) = 0$, i.e., $B^* \subseteq (B^*)^\perp$ and so B^* is isotropic.

($\Longleftarrow$) Suppose that I is an $n/2$-dimensional isotropic Bihom-ideal of L. By Lemma 7.8, $[\beta(I), \alpha(I)] = 0$. Let $B = L/I$ and $p : L \to B$ be the canonical projection. Since $\mathrm{ch}\mathbb{K} \neq 2$, we can choose an isotropic complement subspace B_0 to I in L, i.e., $L = B_0 \dotplus I$ and $B_0 \subseteq B_0^\perp$. Then $B_0^\perp = B_0$ since $\dim B_0 = n/2$.

Denote by p_0 (resp. p_1) the projection $L \to B_0$ (resp. $L \to I$) and let q_L^* denote the homogeneous linear map $I \to B^* : i \mapsto q_L^*(i)$, where $q_L^*(i)(p(x)) := q_L(i, x), \forall x \in L$. We claim that q_L^* is a linear isomorphism. In fact, if $p(x) = p(y)$, then $x - y \in I$, hence $q_L(i, x - y) \in q_L(I, I) = 0$ and so $q_L(i, x) = q_L(i, y)$, which implies q_L^* is well-defined and it is easily seen that q_L^* is linear. If $q_L^*(i) = q_L^*(j)$, then $q_L^*(i)(p(x)) = q_L^*(j)(p(x)), \forall x \in L$, i.e., $q_L(i, x) = q_L(j, x)$, which implies $i - j \in L^\perp = 0$, hence q_L^* is injective. Note that $\dim I = \dim B^*$, then q_L^* is surjective.

In addition, q_L^* has the following property:

$$\begin{aligned}
&q_L^*([\beta(x), \alpha(i)])(p(\alpha(y)))\\
&= q_L([\beta(x), \alpha(i)]_L, \alpha(y)) = -\delta q_L([\beta(i), \alpha(x)]_L, \alpha(y))\\
&= -\delta q_L(\alpha(i), [\beta(x), \alpha(y)]_L) = -\delta q_L^*(\alpha(i))p([\beta(x), \alpha(y)]_L)\\
&= -\delta q_L^*(\alpha(i))[p(\beta(x)), p(\alpha(y))]_L = -q_L^*(\alpha(i))(\mathrm{ad}p(\beta(x))(p(\alpha(y))))\\
&= \delta(\pi(p(\beta(x)))q_L^*(\alpha(i)))(p(\alpha(y))) = [p(\beta(x)), q_L^*(\alpha(i))]_{L\oplus L^*}(p(\alpha(y))),
\end{aligned}$$

where $x, y \in L$, $i \in I$. A similar computation shows that

$$q_L^*([\beta(x), \alpha(i)]) = [p(\beta(x)), q_L^*(\alpha(i))]_{L\oplus L^*}, \quad q_L^*([\beta(i), \alpha(x)]) = [q_L^*(\beta(i)), p(\beta(x))]_{L\oplus L^*}.$$

Define a homogeneous bilinear map

$$\begin{aligned}
\omega : \quad & B \times B && \longrightarrow && B^*\\
& (p(b_0), p(b_0')) && \longmapsto && q_L^*(p_1([b_0, b_0'])),
\end{aligned}$$

where $b_0, b_0' \in B_0$. Then w is well-defined since the restriction of the projection p to B_0 is a linear isomorphism.

Let φ be the linear map $L \to B \oplus B^*$ defined by $\varphi(b_0 + i) = p(b_0) + q_L^*(i), \forall b_0 + i \in B_0 \dotplus I = L$. Since the restriction of p to B_0 and q_L^* are linear isomorphisms, φ is also a linear isomorphism. Note that

$$\begin{aligned}
&\varphi([\beta(b_0+i),\alpha(b_0'+i')]_L)\\
&=\varphi([\beta(b_0),\alpha(b_0')]_L+[\beta(b_0),\alpha(i')]_L+[\beta(i),\alpha(b_0')]_L)\\
&=\varphi(p_0(\beta(b_0),\alpha(b_0')]_L)+p_1([\beta(b_0),\alpha(b_0')]_L)+[\beta(b_0),\alpha(i')]_L+[\beta(i),\alpha(b_0')]_L)\\
&=p(p_0([\beta(b_0),\alpha(b_0')]_L))+q_L^*(p_1([\beta(b_0),\alpha(b_0')]_L)+[\beta(b_0),\alpha(i')]_L+[\beta(i),\alpha(b_0')]_L)\\
&=[p(\beta(b_0)),p(\alpha(b_0'))]_L+\omega(p(\beta(b_0)),p(\alpha(b_0')))+[p(\beta(b_0)),q_L^*(\alpha(i'))]_L\\
&\quad+[q_L^*(\beta(i)),p(\alpha(b_0'))]_L\\
&=[p(\beta(b_0)),p(\alpha(b_0'))]_L+\omega(p(\beta(b_0)),p(\alpha(b_0')))+\delta\pi(p(\beta(b_0))(q_L^*(\alpha(i')))\\
&\quad-\pi(p(\beta(b_0'))(q_L^*(\alpha(i)))\\
&=[p(\beta(b_0))+q_L^*(\beta(i)),p(\alpha(b_0'))+q_L^*(\alpha(i'))]_{B\oplus B^*}\\
&=[\varphi\beta((b_0+i)),\varphi\alpha((b_0'+i'))]_{L\oplus L^*}.
\end{aligned}$$

Then φ is an isomorphism of algebras, and so $(B\oplus B^*,[\cdot,\cdot]_{B\oplus B^*},\alpha,\beta)$ is a δ-Bihom-Jordan-Lie algebra. Furthermore, we have

$$\begin{aligned}
q_B(\varphi(b_0+i),\varphi(b_0'+i')) &= q_B(p(b_0)+q_L^*(i),p(b_0')+q_L^*(i'))\\
&= q_L^*(i)(p(b_0'))+q_L^*(i')(p(b_0))\\
&= q_L(i,b_0')+q_L(i',b_0)\\
&= q_L(b_0+i,b_0'+i'),
\end{aligned}$$

then φ is isometric. The relation

$$\begin{aligned}
&q_B([\beta^{'}(\varphi((x)),\alpha^{'}(\varphi(\alpha(y)))],\alpha^{'}(\varphi(\alpha(z))))\\
&=q_B([\varphi(\beta(x)),\varphi(\alpha(y))],\varphi(\alpha(z)))=q_B(\varphi([\beta(x),\alpha(y)]),\varphi(\alpha(z)))=q_L([\beta(x),\alpha(y)],\alpha(z))\\
&=q_L(\alpha(x),[\beta(y),\alpha(z)])=q_B(\varphi(\alpha(x)),[\varphi(\beta(y)),\varphi(\alpha(z))])\\
&=q_B(\alpha^{'}(\varphi(x)),[\beta^{'}(\varphi(y)),\alpha^{'}(\varphi(z))])
\end{aligned}$$

which implies that q_B is a nondegenerate invariant symmetric bilinear form, and so $(B\oplus B^*,q_B,\alpha',\beta^{'})$ is a quadratic δ-Bihom-Jordan-Lie algebra. In this way, we get a T^*-extension $T^*_\omega B$ of B and consequently, (L,q_L,α,β) and $(T^*_\omega B,q_B,\alpha',\beta^{'})$ are isometric as required. □

Let$(L,[\cdot,\cdot]_L,\alpha,\beta)$ be a δ-Bihom-Jordan-Lie algebra over a field $\mathbf{K}$, and let $\omega_1: L\times L\to L^*$ and $\omega_2: L\times L\to L^*$ be two different Jordancyclic 2-cocycles. The T^*-extensions $T^*_{\omega_1}L$ and $T^*_{w_2}L$ of L are said to be equivalent if there exists an isomorphism of δ-Bihom-Jordan-Lie algebras $\phi: T^*_{\omega_1}L\to T^*_{\omega_2}L$ which is the identity on the Bihom-ideal L^* and which induces the identity on the factor δ-Bihom-Jordan-Lie algebra $T^*_{\omega_1}L/L^*\cong L\cong T^*_{\omega_2}L/L^*$. The two T^*-extensions $T^*_{\omega_1}L$ and $T^*_{\omega_2}L$ are said to be isometrically equivalent if they are equivalent and ϕ is an isometry.

Proposition 7.10. *Let L be a δ-Bihom-Jordan-Lie algebra over a field $\mathbf{K}$ of characteristic not equal to 2, and ω_1, ω_2 be two Jordan cyclic 2-cocycles $L\times L\to L^*$. Then we have*

(i) *$T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$ if and only if there is $z\in C^1(L,L^*)$ such that*

$$\omega_1(x,y)-\omega_2(x,y)=\delta\pi(x)z(y)-\pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}z(x)-z([x,y]_L),\forall x,y\in L. \quad (7.7)$$

If this is the case, then the symmetric part z_s of z, defined by $z_s(x)(y):=\frac{1}{2}(z(x)(y)+z(y)(x))$, for all $x,y\in L$, induces a symmetric invariant bilinear form on L.

(ii) *$T^*_{\omega_1}L$ is isometrically equivalent to $T^*_{\omega_2}L$ if and only if there is $z\in C^1(L,L^*)$ such that* (29) *holds for all $x,y\in L$ and the symmetric part z_s of z vanishes.*

Proof. (i) $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$ if and only if there is an isomorphism of δ-Bihom-Jordan-Lie algebras $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ satisfying $\Phi|_{L^*} = 1_{L^*}$ and $x - \Phi(x) \in L^*, \forall x \in L$.

Suppose that $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ is an isomorphism of δ-hom-Jordan-Lie algebra and define a linear map $z : L \to L^*$ by $z(x) := \Phi(x) - x$, then $z \in C^1(L, L^*)$ and for all $x + f, y + g \in T^*_{\omega_1}L$, we have

$$\begin{aligned}
&\Phi([x+f, y+g]_\Omega)\\
&= \Phi([x,y]_L + \omega_1(x,y) + \delta\pi(x)g - \pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f)\\
&= [x,y]_L + z([x,y]_L) + \omega_1(x,y) + \delta\pi(x)g - \pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&[\Phi(x+f), \Phi(y+g)]\\
&= [x + z(x) + f, y + z(y) + g]\\
&= [x,y]_L + \omega_2(x,y) + \delta\pi(x)g + \delta\pi(x)z(y) - \pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}z(x) - \pi(\alpha^{-1}\beta(y)\tilde{\alpha}\tilde{\beta}^{-1}(f).
\end{aligned}$$

Since Φ is an isomorphism, (7.7) holds.

Conversely, if there exists $z \in C^1(L, L^*)$ satisfying (7.7), then we can define $\Phi : T^*_{\omega_1}L \to T^*_{\omega_2}L$ by $\Phi(x+f) := x + z(x) + f$. It is easy to prove that Φ is an isomorphism of δ-Bihom-Jordan-Lie algebras such that $\Phi|_{L^*} = \mathrm{id}_{L^*}$ and $x - \Phi(L) \in L^*, \forall x \in L$, i.e. $T^*_{\omega_1}L$ is equivalent to $T^*_{\omega_2}L$.

Consider the symmetric bilinear form $q_L : L \times L \to \mathbf{K}, (x,y) \mapsto z_s(x)(y)$ induced by z_s. Note that

$$\begin{aligned}
&\omega_1(\beta(x), \alpha(y))(\alpha(m)) - \omega_2(\beta(x), \alpha(y))(\alpha(m))\\
&= \delta\pi(\beta(x))z(\alpha(y))(\alpha(m)) - \pi(\alpha^{-1}\beta(\alpha(y)))\tilde{\alpha}\tilde{\beta}^{-1}z(\beta(x))(\alpha(m)) - z([\beta(x), \alpha(y)]_L)(\alpha(m))\\
&= \delta\pi(\beta(x))z(\alpha(y))(\alpha(m)) - \pi(\alpha(y))z(\alpha(x))(\alpha(m)) - z([\beta(x), \alpha(y)]_L)(\alpha(m))\\
&= -\delta z(\alpha(y))([\beta(x), \alpha(m)]_L) + z(\alpha(x))([\beta(y), \alpha(m)]_L) - z([\beta(x), \alpha(y)]_L)(\alpha(m)),
\end{aligned}$$

and

$$\begin{aligned}
&\omega_1(\beta(y), \alpha(m))(\alpha(x)) - \omega_2(\beta(y), \alpha(m))(\alpha(x))\\
&= \delta\pi(\beta(y))z(\alpha(m))(\alpha(x)) - \pi(\alpha(m))z(\alpha(y))(\alpha(x)) - z([\beta(y), \alpha(m)]_L)(\alpha(x))\\
&= -\delta z(\alpha(m))([\beta(y), \alpha(x)]_L) + z(\alpha(y))([\beta(m), \alpha(x)]_L) - z([\beta(y), \alpha(m)]_L)(\alpha(x))\\
&= z(\alpha(m))([\beta(x), \alpha(y)]_L) - \delta z(\alpha(y))([\beta(x), \alpha(m)]_L) - z([\beta(y), \alpha(m)]_L)(\alpha(x)).
\end{aligned}$$

Since both ω_1 and ω_2 are Jordancyclic, the right hand sides of above two equations are equal. Hence

$$\begin{aligned}
&-\delta z(\alpha(y))([\beta(x), \alpha(m)]_L) + z(\alpha(x))([\beta(y), \alpha(m)]_L) - z([\beta(x), \alpha(y)]_L)(\alpha(m))\\
&= z(\alpha(m))([\beta(x), \alpha(y)]_L) - \delta z(\alpha(y))([\beta(x), \alpha(m)]_L) - z([\beta(y), \alpha(m)]_L)(\alpha(x)).
\end{aligned}$$

That is

$$\begin{aligned}
&z(\alpha(x))([\beta(y), \alpha(m)]_L) + z([\beta(y), \alpha(m)]_L)(\alpha(x))\\
&= z([\beta(x), \alpha(y)]_L)(\alpha(m)) + z(\alpha(m))([\beta(x), \alpha(y)]_L).
\end{aligned}$$

Since $\mathrm{ch}\mathbf{K} \neq 2$, $q_L(\alpha(x), [\beta(y), \alpha(m)]) = q_L([\beta(x), \alpha(y)], \alpha(m))$, which proves the invariance of the symmetric bilinear form q_L induced by z_s.

(ii) Let the isomorphism Φ be defined as in (i). Then for all $x + f, y + g \in L \oplus L^*$, we have

$$\begin{aligned}
&q_B(\Phi(x+f), \Phi(y+g)) = q_B(x + z(x) + f, y + z(y) + g)\\
&= z(x)(y) + f(y) + z(y)(x) + g(x)\\
&= z(x)(y) + z(y)(x) + f(y) + g(x)
\end{aligned}$$

$$= 2z_s(x)(y) + q_B(x + f, y + g).$$

Thus, Φ is an isometry if and only if $z_s = 0$. □

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