



Continuous dependence of solutions to double dispersive equation with dissipative term

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Abstract

This work is focused on the properties of solutions to the initial-boundary value problem of the double dispersive-dissipative equation. It is established that the solution depends continuously on the dispersive and dissipative coefficients.

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1. Introduction

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. We study the following initial boundary value problem for nonlinear hyperbolic equations containing fourth order dissipative term:

$$u_{tt} - \Delta u - a\Delta u_{tt} + b\Delta^2 u - d\Delta u_t + g(u_t) = f(u) \quad (x, t) \in \Omega \times [0, T], \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega \subset \mathbb{R}^n \quad (n \geq 3), \quad (1.2)$$

$$u = \Delta u \equiv 0 \quad (x, t) \in \partial\Omega \times [0, T], \quad T > 0, \quad (1.3)$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is an n -dimensional Laplace operator, Δ^2 denotes biharmonic operator, and $g(u_t) = \alpha|u_t|^{m-2}u_t$, $f(u) = -\beta|u|^{p-2}u$. For $n = 1, 2$, one has $2 < m, p \leq \infty$, but we consider here $n \geq 3$, so we have $2 < m, p \leq \frac{2n-2}{n-2}$. Because certain unknown parameters in the mathematical model are usually the results of an experimental work, they are actual physical parameters. In this equation, $a > 0$ and $b > 0$ are dispersive coefficients, $d > 0$ is a dissipative coefficient expressing energy loss in (1.1). There exist many examples of physical problems modeled by (1.1). For instance, the following equation is derived by the longitudinal displacement $u(x, t)$ of the elastic rod, in case of accepting the likelihood of energy exchange over the sides of the waveguide in the physical study of nonlinear wave propagation in waveguide;

$$u_{tt} - u_{xx} = \frac{1}{4}(6u^2 - bu_{xx} + au_{tt})_{xx}, \quad (1.4)$$

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which is derived by Hamiltonian principle [14, 15]. Likewise, the cubic double dispersive equation (CDDE) in general

$$u_{tt} - u_{xx} = \frac{1}{4}(cu^3 + 6u^2 + au_{tt} - bu_{xx} + du_t)_{xx} \tag{1.5}$$

can be achieved for $a, b, c, d > 0$ [14, 15]. In [6], the authors take into account the nonlinear function $g(u_t) = a|u_t|^{m-2}u_t$ and the source term $f(u) = -b|u|^{p-2}u$ in (1.1) under the initial condition and homogeneous Dirichlet and Neumann boundary conditions in the region $\Omega \times (0, \infty)$. They used the Galerkin method to demonstrate the existence of a global weak solution to this problem. In [9], taking $f(u) = u^2$ and $g(u_t) = 0$ in (1.1) and using Lie symmetries, invariant solutions are obtained to the following problem

$$u_{tt} - c^2u_{xx} = \left(u^2 + au_{xx} + bu_{tt}\right)_{xx} + \mu u_{xxt}. \tag{1.6}$$

In [18], the global existence of the weak solution to the problem

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + k\Delta u_t = \Delta f(u), \quad x \in \mathbb{R}^n, t > 0, \tag{1.7}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.8}$$

where $a = b = 1$ in (1.1) has examined under conditions related to nonlinear term and initial data. In [16], the existence and uniqueness of a global solution for the Cauchy problem in the following equation have studied:

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g(u)_{xx}, \tag{1.9}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \tag{1.10}$$

On the other hand, the existence of a global solution of the following IBVP is established

$$u_{tt} - u_{xx} - au_{xxtt} + bu_{xxxx} - du_{xxt} = f(u)_{xx}, \tag{1.11}$$

$$u(0, t) = u(0, l) = 0, \quad u_{xx}(0, l) = u_{xx}(l, t) = 0, \quad t \geq 0 \tag{1.12}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \bar{\Omega}, \tag{1.13}$$

where $\Omega = (0, l)$ [3]. In [8], blow-up phenomena of generalized double dispersion equations

$$u_{tt} - u_{xx} - u_{xxt} + u_{xxxx} - u_{xxtt} = f(u_x)_x \tag{1.14}$$

is surveyed. The authors establish a blow-up result for the solutions with arbitrary high initial energy, and give some upper bounds for blow-up time T^* , depending on sign and size of initial energy. In [7], the existence and nonexistence of global weak solutions to the Cauchy problem for a higher order generalized Boussinesq-type equation with hydrodynamical damped term in n -dimensional space is considered. It can be seen from previous studies that the problem (1.1)-(1.3) is solved in the proper spaces and sufficient conditions for existence and uniqueness of the solution to the problem are derived. In the current paper, we investigate the continuous dependence of the solution of (1.1)-(1.3) on the coefficients of that equation in $L^2(\Omega)$ space. These coefficients are usually determined via conditions of the experiments that are subject to variability and chance. Modeling physical phenomena mathematically with appropriate parameters determined from experiments, one tries to predict the outcome of certain physical processes [17]. With these difficulties in mind, one must raise the question that if the mathematical model is derived correctly? That is, do small changes of the set of parameters only result in small variations of the outcome? To answer this question, this paper aims at surveying how the results are affected when the small changes occur in parameters a, d, α , and β . Recently, many important works have been done on deriving estimates about structural stability of the equation. Those of works are examined for multidimensional equations in following main books [1, 2], the articles [4, 5, 10-13, 19], and references therein.

Definition 1.1. We call a weak solution of the problem a function $u \in L^\infty(0, T; H_0^2(\Omega) \cap L^p(\Omega))$, $u_t \in L^\infty(0, T; H_0^1(\Omega) \cap L^m(Q_T))$, satisfying (1.1) in the sense of distributions [7].

Remark 1.2. The interested reader is referred to Theorem 3.1. in [7] for the proof of the existence and uniqueness of a weak solution for the problem (1.1) – (1.3).

2. A priori estimates

Theorem 2.1. *Let (u_0, u_1) belong to $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. Then the solution u of the problem (1.1)-(1.3) satisfies the following inequalities:*

$$\|u_t(t)\|^2 \leq A_1, \|\nabla u(t)\|^2 \leq A_1, \|\Delta u(t)\|^2 \leq A_2, \|\nabla u_t(t)\|^2 \leq A_3, \tag{2.1}$$

where $A_1, A_2, A_3 > 0$ depend on the coefficients of the equation (1.1) and the initial data.

Proof. Multiplying (1.1) by u_t in $L^2(\Omega)$, we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\nabla u(t)\|^2 + \frac{b}{2} \|\Delta u(t)\|^2 + \frac{a}{2} \|\nabla u_t(t)\|^2 + \frac{\beta}{2} \|u(t)\|_p^p \right] \\ + d \|\nabla u_t(t)\|^2 + \alpha \|u_t(t)\|_m^m = 0. \end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$E_u(t) := \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\nabla u(t)\|^2 + \frac{b}{2} \|\Delta u(t)\|^2 + \frac{a}{2} \|\nabla u_t(t)\|^2 + \frac{\beta}{2} \|u(t)\|_p^p \leq E_u(0). \tag{2.3}$$

Hence, all of the estimates in (2.1) follow from (2.3). □

Theorem 2.2. *Suppose that $u_0 \in L^2(\Omega)$, and $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then the solution of the problem (1.1)-(1.3) satisfies the following regular estimates:*

$$\|\nabla u_{tt}(t)\|^2 \leq A_4 \quad \forall t \in [0, T], \tag{2.4}$$

where A_4 depends on the parameters of the system and initial data.

Proof. To get uniform estimate, (1.1) is differentiated with respect to t :

$$u_{ttt} - \Delta u_t - d \Delta u_{tt} + b \Delta^2 u_t - a \Delta u_{ttt} + \alpha(m-1)|u_t|^{m-2}u_{tt} + \beta(p-1)|u|^{p-2}u_t = 0. \tag{2.5}$$

Multiplying (2.5) by u_{tt} in $L^2(\Omega)$ gives

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \|u_{tt}(t)\|^2 + \frac{1}{2} \|\nabla u_t(t)\|^2 + \frac{b}{2} \|\Delta u_t(t)\|^2 + \frac{a}{2} \|\nabla u_{tt}(t)\|^2 \right] \\ + d \|\nabla u_{tt}(t)\|^2 + \alpha(m-1) \int_{\Omega} |u_t|^{m-2} u_{tt}^2 dx + \beta(p-1) \int_{\Omega} |u|^{p-2} u_t u_{tt} dx = 0. \end{aligned} \tag{2.6}$$

We estimate last two integral terms on the left hand side of the equation. First, we have

$$\begin{aligned} \alpha(m-1) \int_{\Omega} |u_t|^{m-2} |u_{tt}|^2 dx &\leq \alpha(m-1) \max |u_t|^{m-2} |\Omega| \int_{\Omega} |u_{tt}|^2 dx \\ &\leq \alpha(m-1) 2|\Omega| E_u(0)^{\frac{m-2}{2}} \|u_{tt}\|^2 \\ &\leq C \|u_{tt}(t)\|^2 \\ &\leq C_1 \|\nabla u_{tt}(t)\|^2, \end{aligned}$$

where $C_1 = 2C\alpha(m-1)E_u(0)^{\frac{m-2}{2}}|\Omega|$. For the last term, we get

$$\begin{aligned} \beta(p-1) \int_{\Omega} |u|^{p-2} u_t u_{tt} dx &\leq \beta(p-1) \max |u|^{p-2} \|u_t(t)\| \|u_{tt}(t)\| \\ &\leq \beta(p-1) C(\Omega) \|\nabla u(t)\|^{p-2} \|u_t(t)\| \|u_{tt}(t)\| \\ &\leq C_2 \left[\|\nabla u_t(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \right], \end{aligned}$$

where $C_2 = \frac{\beta(p-1)C(\Omega)A_1^{\frac{p-2}{2}}}{2}$. Combining these in (2.6), and simplifying the inequality, we have

$$\frac{d}{dt}E_{u_t}(t) \leq C_2\|\nabla u_t(t)\|^2 + (C_1 + C_2)\|\nabla u_{tt}(t)\|^2, \tag{2.7}$$

where $E_{u_t}(t) = \frac{1}{2}\|u_{tt}(t)\|^2 + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{b}{2}\|\Delta u_t(t)\|^2 + \frac{a}{2}\|\nabla u_{tt}(t)\|^2$. Therefore, we obtain

$$\frac{d}{dt}E_{u_t}(t) \leq M_1E_{u_t}(t), \tag{2.8}$$

where $M_1 = \max\{\frac{1}{2}, \frac{a}{2}, C_1 + C_2\}$. Integrating the last inequality, we get

$$E_{u_t}(t) \leq e^{M_1t}E_{u_t}(0). \tag{2.9}$$

Hence, we have

$$\|\nabla u_{tt}(t)\|^2 \leq \frac{2E_{u_t}(0)}{a}e^{M_1t}, \tag{2.10}$$

which yields (2.4) with $A_4 = \frac{2E_{u_t}(0)}{a}e^{M_1t} \forall t \in [0, T]$. □

3. Continuous dependence on all parameters

In general, the subject of the continuous dependence is examined on coefficients separately. However, in this section, we show that the solution of the problem (1.1)-(1.3) depends continuously on coefficients a, d, α, β under the same inequality.

Suppose that u is the solution of

$$u_{tt} - \Delta u - a_1\Delta u_{tt} + b\Delta^2u - d_1\Delta u_t + \alpha_1|u_t|^{m-2}u_t + \beta_1|u|^{p-2}u = 0, \tag{3.1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega \subset \mathbb{R}^n \ (n \geq 3) \tag{3.2}$$

$$u = \Delta u \equiv 0 \quad (x, t) \in \partial\Omega \times [0, T], \ T > 0, \tag{3.3}$$

and v is the solution of the following IBVP:

$$v_{tt} - \Delta v - a_2\Delta v_{tt} + b\Delta^2v - d_2\Delta v_t + \alpha_2|v_t|^{m-2}v_t + \beta_2|v|^{p-2}v = 0, \tag{3.4}$$

$$v(x, 0) = u_0(x), \quad v_t(x, 0) = u_1(x) \quad x \in \Omega \subset \mathbb{R}^n \ (n \geq 3), \tag{3.5}$$

$$v = \Delta v \equiv 0 \quad (x, t) \in \partial\Omega \times [0, T], \ T > 0. \tag{3.6}$$

Therefore w is a solution of the following IBVP:

$$\begin{aligned} w_{tt} - \Delta w - a_1\Delta w_{tt} - a\Delta v_{tt} + b\Delta^2w - d_1\Delta w_t - d\Delta v_t \\ + \alpha_1(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t) + \alpha|v_t|^{m-2}v_t \\ + \beta_1(|u|^{p-2}u - |v|^{p-2}v) + \beta|v|^{p-2}v = 0, \end{aligned} \tag{3.7}$$

$$w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad x \in \Omega \subset \mathbb{R}^n, \tag{3.8}$$

$$w = \Delta w \equiv 0 \quad (x, t) \in \partial\Omega \times [0, T], \ T > 0, \tag{3.9}$$

where $w = u - v, a = a_1 - a_2, \beta = \beta_1 - \beta_2, d = d_1 - d_2, \alpha = \alpha_1 - \alpha_2$.

Theorem 3.1. *The solution w of the problem (3.7)-(3.9) satisfies the inequality*

$$\|\nabla w_t(t)\|^2 + \frac{b}{2}\|\Delta w(t)\|^2 \leq \left\{ \frac{A_4}{2}a^2 + \frac{A_3}{2}d^2 + \left(d_0^{m-1}A_3^{\frac{m-1}{2}}\right)\alpha^2 + \left(d_1^{p-2}A_1^{\frac{p-1}{2}}\right)\beta^2 \right\} te^{M_2t},$$

where $A_1, A_3, A_4, M_2 \geq 0$ depend on the coefficients of the equation (1.1) and the initial data.

Proof. First, multiplying (3.7) by w_t we get

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \|w_t(t)\|^2 + \frac{1}{2} \|\nabla w(t)\|^2 + \frac{a_1}{2} \|\nabla w_t(t)\|^2 + \frac{b}{2} \|\Delta w(t)\|^2 \right] \\ & + a(\nabla w_t, \nabla v_{tt}) + d_1 \|\nabla w_t(t)\|^2 + d(\nabla w_t, \nabla v_t) \\ & + \alpha_1(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t, w_t) + \alpha(|v_t|^{m-2}v_t, w_t) \\ & + \beta_1(|u|^{p-2}u - |v|^{p-2}v, w_t) + \beta(|v|^{p-2}v, w_t) = 0. \end{aligned} \tag{3.10}$$

Rearranging (3.10), we have

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \|w_t(t)\|^2 + \frac{1}{2} \|\nabla w(t)\|^2 + \frac{a_1}{2} \|\nabla w_t(t)\|^2 + \frac{b}{2} \|\Delta w(t)\|^2 \right] + d_1 \|\nabla w_t(t)\|^2 \\ & + \alpha_1(|u_t|^{m-2}u_t - |v_t|^{m-2}v_t, w_t) \leq |a(\nabla w_t, \nabla v_{tt})| + |d(\nabla w_t, \nabla v_t)| \\ & + |\alpha(|v_t|^{m-2}v_t, w_t)| + |\beta_1(|u|^{p-2}u - |v|^{p-2}v, w_t)| + |\beta(|v|^{p-2}v, w_t)|. \end{aligned} \tag{3.11}$$

Now, we get the upper boundaries for the right hand side of (3.11). Due to Cauchy-Schwarz inequality, we have

$$|a(\nabla w_t, \nabla v_{tt})| \leq \frac{a^2 \|\nabla v_{tt}\|^2}{2} + \frac{\|\nabla w_t(t)\|^2}{2},$$

and

$$|d(\nabla w_t, \nabla v_t)| \leq \frac{d^2 \|\nabla v_t(t)\|^2}{2} + \frac{\|\nabla w_t(t)\|^2}{2}.$$

Thanks to ε -Young and Sobolev inequalities, with $m, p \leq \frac{2n-2}{n-2}$, we obtain

$$\begin{aligned} |(|v_t|^{m-2}v_t, w_t)| & \leq \int |v_t|^{m-2}|v_t||w_t|dx \\ & \leq \|v_t(t)\|_{2(m-1)}^{m-1} \|w_t(t)\| \\ & \leq d_0^{m-1} \|\nabla v_t(t)\|^{m-1} \|w_t(t)\| \\ & \leq \alpha d_0^{m-1} A_3^{\frac{m-1}{2}} + \frac{1}{4\alpha} \|w_t(t)\|^2. \end{aligned}$$

Similarly, we have

$$(|v|^{p-2}v, w_t) \leq \beta d_1^{p-2} A_1^{\frac{p-1}{2}} + \frac{1}{4\beta} \|w_t(t)\|^2.$$

Besides, it can be easily shown that $\int_{\Omega} (|u_t|^{m-2}u_t - |v_t|^{m-2}v_t) w_t dx \geq 0$.

Moreover, using the mean value theorem, Hölder's and Sobolev inequalities with $p \leq \frac{2n-2}{n-2}$ and ε -Young inequality with $\varepsilon = \beta_1$, we have

$$\begin{aligned} \left| \int_{\Omega} (|u|^{p-2}u - |v|^{p-2}v) w_t dx \right| & \leq (p-1) \int_{\Omega} |w||w_t| (|u|^{p-2} + |v|^{p-2}) dx \\ & \leq (p-1) \|w(t)\|_{\frac{2n}{n-2}} \|w_t(t)\| \left(\|u(t)\|_{(p-2)n}^{p-2} + \|v(t)\|_{(p-2)n}^{p-2} \right) \\ & \leq (p-1) \|w_t(t)\| C_1 \|\nabla w(t)\| C_2 \left(\|\nabla u(t)\|^{p-2} + \|\nabla v(t)\|^{p-2} \right) \\ & \leq (p-1) C_1 C_2 2 A_1^{\frac{p-2}{2}} \|w_t(t)\| \|\nabla w(t)\| \\ & \leq C_3 \|w_t(t)\| \|\nabla w(t)\| \\ & \leq \beta_1 C_3^2 \|\nabla w(t)\|^2 + \frac{1}{4\beta_1} \|w_t(t)\|^2. \end{aligned}$$

Combining all boundaries for the right hand side of (3.11), we get

$$\frac{d}{dt}E_w(t) \leq M_2 E_w(t) + B, \quad (3.12)$$

where

$$M_2 = \max \left\{ \beta_1^2 C_3^2, a_1/2, 1 + d_1 \right\},$$

and

$$B = \frac{a^2}{2} A_4 + \frac{d^2}{2} A_3 + \alpha^2 d_0^{m-1} A_3^{\frac{m-1}{2}} + \beta^2 d_1^{p-2} A_1^{\frac{p-1}{2}}.$$

Solving differential inequality (3.12), we obtain

$$E_w(t) \leq B t e^{M_2 t}. \quad (3.13)$$

It is proved that if $a \rightarrow 0$, $d \rightarrow 0$, $\alpha \rightarrow 0$, $\beta \rightarrow 0$ at the same time, then $w \rightarrow 0$, $\forall t \in [0, T]$, which is the desired result. \square

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