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On Invariants of m-Vector in Lorentzian Geometry

İdris Ören

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ABSTRACT

Let G be the group $M(n,1)$ generated by all pseudo-orthogonal transformations and translations of Lorentzian space E_1^n or $G = SM(n,1)$ is the subgroup of $M(n,1)$ generated by rotations and translations of E_1^n. We describe the correlations between Gram determinant $detG(x_1,\ldots,x_m)$ of the system $\{x_1,\ldots,x_m\}$ and the number of linearly independent null vectors in the system $\{x_1,\ldots,x_m\}$. Using methods of invariant theory and these results, the system of generators of the polynomial ring of all G-invariant polynomial functions of vectors $x_1,x_2,\ldots,x_m$ in E_1^n is obtained for groups $G = M(n,1)$ and $G = SM(n,1)$.

Keywords: *Invariant; Lorentz group.*

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1. Introduction

Let R be real numbers field. The ring of polynomials $f(x_1,x_2,\ldots,x_m)$ in m variables with real coefficients is denoted $R[x_1,\ldots,x_m]$(shortly, $R[x]$). Let G be a subgroup of the group $GL(n,R)$ of invertible $n\times n$-matrices. Given a polynomial function $f\in R[x_1,\ldots,x_m]$. We are interested the set $R[x_1,\ldots,x_m]^G$(shortly, $R[x]^G$) of all polynomials which are invariant $f(x_1,x_2,\ldots,x_m) = f(gx_1,gx_2,\ldots,gx_m)$ for all $g\in G$. We call $R[x_1,\ldots,x_n]^G$ the invariant subring of G. One of important and fundamental problems of invariant theory is finding a set $I_1,\ldots,I_m$ of generators for the invariant subring $R[x_1,x_2,\ldots,x_m]^G$ under the group G.

All geometric magnitudes and properties are invariant with respect to the underlying transformation group. Properties in Euclidean geometry are invariant under the Euclidean group of rotations, reflections and translations, properties in projective geometry are invariant under the projective transformations,etc. Similarly, properties in Lorentzian space(that is n-dimensional pseudo-Euclidean geometry of index 1) are invariant under the Lorentz transformations.For the classical group, the following problem is given in [13, pp.15] :
"Given a geometric property P, find the corresponding invariants and vice versa". This problem is also important for Lorentz group.

First, finding a system generator invariants of Lorentz group is suggested for Lorentz group by [14, pp.66]. The first comprehensive treatment of Euclidean geometry is given in the fundamental work of [12] and [14, pp.52] . Fundamental theorems for invariants in orthogonal group are obtained by [3] and [14]. Recently, all m-points invariants for different geometries is determined by a characterization of orbits of m-tuples of vectors in paper [4]. All scalar concomitants of vectors and all biscalars of a system of $s\leq n$ linearly independent contravariant vectors in n-dimensional Lorentz space is determined in papers [6, 11]. Let U be a subspace generated by vectors $x_1,x_2,\ldots,x_m$. All subspaces U in Euclidean space E_0^n are nondegenerate(or regular). But for the Lorentzian space E_1^n, therefore mentioned subspace U can not be a non-degenerate subspace. Therefore,the classification of subspaces in Lorentzian space is given by [9]. By using methods and results in [9] and [14], we will give the system generator invariants of Lorentz group(or pseudo-orthogonal group of

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**** Corresponding author***
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index 1) in terms of inner products and determinant of m-vector.

Applications of the invariant theory and invariants, transformations and invariants of curves, surfaces and graphical objects appear in many areas Computer Aided Geometric Design, the computer vision, etc. The important problem is to find simple but efficient method for the equivalence check of two m-uples $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_m\}$ vectors in E_1^n. This problem can be solve by invariants in this paper for groups $G = M(n,1), SM(n,1)$. In [10], a solution of the problem of G-equivalence of a system of vectors for groups $G = O(1,1), SO(1,1), L(1,1)$ in terms of invariants of vectors $\{x_1, x_2, \ldots, x_m\}$ in the two dimensional Minkowski spacetime geometry and its an application of control invariants of Bézier curves are given. In [5, 10],the complete systems of G-invariants of m-tuples and describe complete systems relations between elements of obtained complete systems of G-invariants.For solutions of these problems in paper [5] , hyperbolic numbers are used.

This paper is organized as follows: In Section 2, we give some known definitions and propositions, which we use in the next sections. In Section 3, using results of the Section 2, correlations between gram determinant $detG(x_1, \ldots, x_m)$ of the system $\{x_1, \ldots, x_m\}$ and the number of linearly independent null vectors in the system $\{x_1, \ldots, x_m\}$ is given (Theorem3.1, Corollaries3.1 and 3.2). In Section 4, using methods of invariant theory, we prove that a system of generators of $R[x]^{O(n,1)}$, $R[x]^{SO(n,1)}$, $R[x]^{M(n,1)}$ and $R[x]^{SM(n,1)}$ is given(Theorems 4.1-4.4).

This paper is devoted to the study of a system generator invariants of m-vectors for the groups $G = O(n,1)$, $G = SO(n,1)$, $G = M(n,1)$ and $G = SM(n,1)$.

2. Preliminaries

Let E_1^n be the n-dimensional Lorentzian space(or pseudo-Euclidean space R^n of index 1) with the scalar product(or Lorentz inner product) $g(x,y) = \langle x, y\rangle = x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n$, where R is the field of real numbers and $n > 0$. In particularly, E_1^4 is the Minkowski spacetime. Denote the group of all pseudo-orthogonal transformations(that is the set of all linear transformations $g : E_1^n \to E_1^n$ such that $\langle gx, gy\rangle = \langle x, y\rangle$ for all $x, y \in E_1^n$) by $O(n,1)$.

Then the group $M(n,1)$ of all pseudo-Euclidean motions of an n-dimensional pseudo-Euclidean space has the form
$M(n,1) = \{F : E_1^n \to E_1^n : Fx = gx + b, g \in O(n,1), b \in E_1^n\}$, where gx is the multiplication of a matrix g and a column vector $x \in E_1^n$.

The group of all proper pseudo-orthogonal transformations of E_1^n is denoted by $SO(n,1)$. It is a subgroup of $O(n,1)$. That is, $SO(n,1) = \{g \in O(n,1) : detg = 1\}$.
Put $SM(n,1) = \{F \in M(n,1) : Fx = gx + b, g \in SO(n,1), b \in E_1^n\}$.

Remark 2.1. In [7, pp.14-16], the groups $O(n,1)$ and $SO(n,1)$ are named general Lorentz group and proper Lorentz group , respectively.

The following definition is known in [7, pp.10,12].

Definition 2.1. (i) A vector x in E_1^n is called timelike, if $\langle x, x\rangle < 0$.

(ii) A vector x in E_1^n is called spacelike, if $\langle x, x\rangle > 0$.

(iii) A non-zero vector x in E_1^n is called null, if $\langle x, x\rangle = 0$.

A subspace U of E_1^n is called spacelike (or timelike) if $\langle u, u\rangle > 0$ (or $\langle u, u\rangle < 0$) for any nonzero vector u in U. We denote a restriction of Lorentz inner product g to U by $g \downarrow U$.

Definition 2.2. Let U be a subspace of E_1^n. A subspace U will be called regular if $rank\,(g \downarrow U) = dim(U)$.

If a subspace U is non-regular, then U is called singular subspace.

Remark 2.2. Regular space is called as non-degenerate space. In this opposite case the singular space is called degenerate space.

A subspace U of E_1^n is called regular if $g \downarrow U$ is regular. When g is referred to as a Euclidean inner product, every subspace of Euclidean space E_0^n is regular. But, when g is referred as a Lorentz inner product, there will always be singular subspaces.

Definition 2.3. Let U be a subspace of E_1^n. A subspace U will be called a null if it contains a null vector, but no timelike vector.

Proposition 2.1. *Let U be a subspace of E_1^n. Then U is a null subspace if and only if U is a singular.*

Proof. The proof of the proposition is easy, so it is omitted or see [1]. □

Let $S(n,1)$ be the set of all subspaces of E_1^n. We consider the following action of the group $O(n,1)$ on $S(n,1) : \alpha(F,V) = F(V)$, where $F \in O(n,1)$ and $V \in S(n,1)$. Let Z be the ring of all integers.

Definition 2.4. A function $f : S(n,1) \to Z$ will be called $O(n,1)$-invariant if $f(F(V)) = f(V)$ for all $F \in O(n,1)$ and $V \in S(n,1)$.

Let $U \in S(n,1)$. Denote the dimension of U by $\dim(U)$. It is obvious that the function $\dim(U)$ is $O(n,1)$-invariant function on $S(n,1)$. Denote the number of linearly independent null vectors in U by $\kappa(U)$.
The following propositions is given in [9].

Proposition 2.2. *The function $\kappa(U)$ is $O(n,1)$-invariant.*

Proposition 2.3. *Let U be a subspace of E_1^n such that $\dim(U) = 1$. Then only the following three cases hold:*

1. *$\kappa(U) = 0$ and $index(U) = 0$ that is U is spacelike;*
2. *$\kappa(U) = 0$ and $index(U) = 1$ that is U is timelike;*
3. *$\kappa(U) = 1$ and $index(U) = 0$.*

Proposition 2.4. *Let U be a subspace of E_1^n such that $\dim(U) > 1$. Then $\kappa(U) = 0$ if and only if U is a spacelike subspace.*

Proposition 2.5. *Let U be a subspace of E_1^n such that $1 \le \dim(U) < n$. Then $\kappa(U) = 1$ if and only if U is a null subspace.*

Proof. It follows from Proposition 2.3 and [9, Theorem 4.4]. □

Corollary 2.1. *Let U be a subspace of E_1^n such that $1 \le \dim(U) < n$. Then $\kappa(U) = 1$ if and only if U is a singular subspace.*

Proof. It follows from Propositions 2.1 and 2.5. □

3. Gram determinant and its properties

Let $x_1, x_2, \ldots, x_m$ be real vectors in E_1^n.

Definition 3.1. The matrix $\|\langle x_i, x_j\rangle\|_{i,j=1,2,\ldots,m}$ is called the Gram matrix of $x_1, x_2, \ldots, x_m \in E_1^n$ and it is denoted by $Gr\,(x_1, x_2, \ldots, x_m)$.
The determinant of it will be called the Gram determinant of $x_1, x_2, \ldots, x_m$ and denoted by $\det Gr\,(x_1, x_2, \ldots, x_m)$.

Proposition 3.1. *Vectors $x_1, x_2, \ldots, x_m \in E_1^n$ are linearly depended if and only if $\det Gr\,(x_1, x_2, \ldots, x_m) = 0$.*

Proof. It is similar to the proof of [14, pp.75]. □

We denote the matrix of column-vectors $x_1, x_2, \ldots, x_m \in E_1^n$ by
$\|x_1 x_2 \ldots x_m\|$. Denote by $[x_1 \ldots x_n]$ determinant of the matrix $\|x_1 \ldots x_n\|$. Denote by $\|x_1 \ldots x_n\|^T$ the transpose matrix $\|x_1 \ldots x_n\|$.

Theorem 3.1. *Let $x_1, x_2, \ldots, x_m$ be linearly independent vectors in E_1^n and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U such that for $1 \le m < n$. Then $\kappa(U) = 1$ if and only if $\det Gr\,(x_1, x_2, \ldots, x_m) = 0$.*

Proof. $\Rightarrow$. Assume that $\kappa(U) = 1$.
(a) Let $m = 1$. Then there exists vector x_1 such that $\langle x_1, x_1\rangle = 0$. Clearly, $\det Gr\,(x_1) = 0$.
(b) Let $1 < m < n$ and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U. From [9, Proposition3.8], there exist $F \in O(n,1)$ such that
$F(U) = \{\bar{x}_1 = (1, 0, \ldots, 0), \bar{x}_i = (0, \bar{x}_{i2}, \ldots, \bar{x}_{in-1}, 0), i = 2, 3, \ldots, m\}$. Hence

$\det Gr\left(Fx_1, Fx_2, \ldots, Fx_m\right) = \det Gr\left(x_1, x_2, \ldots, x_m\right) = \det Gr\left(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\right) = 0.$

$\Leftarrow$. Let $\det Gr\left(x_1, x_2, \ldots, x_m\right) = 0$ and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U. Assume that $\kappa(U) \neq 1$. Then $\kappa(U) = 0$ or $\kappa(U) > 1$.
(i) Assume that $\kappa(U) = 0$.
$(i.1)$ Let $m = 1$. From Proposition2.3, we have $\det Gr\left(x_1\right) > 0$. This is a contradiction by $\det Gr\left(x_1\right) = 0$.
$(i.2)$ Let $1 < m < n$ and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U. From Proposition2.4, we have U is a spacelike subspace. That is, $\det Gr\left(x_1, x_2, \ldots, x_m\right) > 0$.This is a contradiction by $\det Gr\left(x_1\right) = 0$.
(ii) Assume that $\kappa(U) > 1$.
From [9, Proposition3.13], we have $\kappa(U) = dim(U) > 1$. From [9, Proposition3.6], there exist $F \in O(n,1)$ such that
$F(U) = \{\bar{x}_1 = (1, 0, \ldots, 0), \bar{x}_i = (\bar{x}_{i1}, \bar{x}_{i2}, \ldots, \bar{x}_{in-1}, 0), i = 2, 3, \ldots, m\}$. For example, assume that $\bar{x}_{21} \neq 0$ and $\bar{x}_{i1} = 0$ for all $3 \leq i \leq m$. Then, $\det Gr\left(Fx_1, Fx_2, \ldots, Fx_m\right) = \det Gr\left(x_1, x_2, \ldots, x_m\right) = \det Gr\left(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\right) = -\bar{x}_{21}^2 \det Gr\left(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_m\right)$. Since vectors $\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_m$ are linearly independent and $\langle x_i, x_i\rangle > 0$ for all $3 \leq i \leq m$, we have $\det Gr\left(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_m\right) \neq 0$. So, $\det Gr\left(x_1, x_2, \ldots, x_m\right) \neq 0$. This is a contradiction by $\det Gr\left(x_1, x_2, \ldots, x_m\right) = 0$. □

Corollary 3.1. *Let $x_1, x_2, \ldots, x_m$ be linearly independent vectors in E_1^n and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U such that for $1 \leq m < n$. Then $\kappa(U) = 0$ if and only if $\det Gr\left(x_1, x_2, \ldots, x_m\right) > 0$.*

Proof. It follows from Theorem3.1. □

Corollary 3.2. *Let $x_1, x_2, \ldots, x_m$ be linearly independent vectors in E_1^n and $\{x_1, x_2, \ldots, x_m\}$ be a basis of U such that $1 < m \leq n$. Then $\kappa(U) > 1$ if and only if $\det Gr\left(x_1, x_2, \ldots, x_m\right) < 0$.*

Proof. It follows from Theorem3.1. □

4. The generating system of the ring of invariants polynomials of m-vector

Let $x_1, x_2, ..., x_m$ be real vectors(or points) in E_1^n.

Definition 4.1. A polynomial $p(x_1, x_2, ..., x_m)$ of $x_1, x_2, ..., x_m$ will be called a polynomial of $x_1, x_2, ..., x_m$. It will be denoted by $p\left\{x\right\}$.

We denote the set of all polynomials of $x_1, x_2, ..., x_m$ by $R\left[x_1, x_2, ..., x_m\right]$ (shortly, $R\left[x\right]$). It is a R-algebra.
Let G be a subgroup of $O\left(n, 1\right)$.

Definition 4.2. A polynomial $p\left\{x\right\}$ will be called G-invariant if $p\left\{gx\right\} = p\left\{x\right\}$ for all $g \in G$.

The set of all G-invariant polynomials of $x_1, x_2, ..., x_m$ will be denoted by $R\left[x_1, x_2, ..., x_m\right]^G$(shortly, $R\left[x\right]^G$). It is a R-subalgebra of $R\left[x\right]$.

Definition 4.3. A subset S of $R\left[x\right]^G$ will be called a generating system of $R\left[x\right]^G$ if the smallest R-subalgebra with the unit containing S is $R\left[x\right]^G$.

The following lemma is similar to [14, Theorem2.9.A,pp.53].

Lemma 4.1. (i) *Every even pseudo-orthogonal invariant depending on m-vectors $x_1, x_2, \ldots, x_m \in E_1^n$ is expressible in terms of $< x_i, x_j >, 1 \leq i, j \leq m$.*

(ii) *Every odd pseudo-orthogonal is a sum of terms $[x_{i_1} x_{i_2} \ldots x_{i_n}] F\left\{x\right\}$, where $x_{i_j} \in E_1^n$ for all $j = 1, 2, \ldots, n$ are selected from the row $x_1, x_2, \ldots, x_m \in E_1^n$ and $F\left\{x\right\}$ is an even pseudo-orthogonal invariant.*

Proof. We denote every even pseudo-orthogonal invariant depending on m-vectors $x_1, x_2, \ldots, x_m$ in E_1^n by T_n^m. By using Capelli's general and special identities [8, Theorem 5,pp.56], the theorem $T_n^m (m \geq n)$ is reduced to the theorem T_n^{n-1} concerning $n-1$ argument vectors. When $n-1$ vectors $x_1, x_2, \ldots, x_{n-1}$ are numerically given and linearly independent, one may introduce a new pseudo-orthogonal coordinate system such that they lie in the $(n-1)$-dimensional space spanned by the first $n-1$ fundamental vectors(non-formal part). Thus one has reduced the question to the study of pseudo-orthogonal invariants in $n-1$-dimensions ,or more precisely, since they depend on exactly $n-1$ vectors, to T_{n-1}^{n-1}. In view of this situation , it seems best first to pass from

$$T_{n-1}^{n-1} \to T_{n}^{n-1} \to T_{n}^{n} \tag{4.1}$$

and then to generalize T_n^n to T_n^m. We prove first the step $T_n^n \to T_{n+1}^n$. The two steps into which the transition T_{n-1}^{n-1} to T_n^n breaks up according to (4.1) are performed by the "non-formal" argument and Capelli's special identity respectively, whereas the transition T_n^n to $T_n^m, (n > m)$ rest on Capelli's general identity. As it is obvious how to carry out the second part, we turn to the inductive proof of T_n^n according to the scheme (4.1). Let us first restate. An even invariant depending on n vectors $x_1, x_2, \ldots, x_n$ in E_1^n is expressible in terms of their Lorentzian inner products and its denoted T_n^n. We prove first the step T_{n-1}^{n-1} to T_n^{n-1}.

Let $f(x_1, x_2, \ldots, x_{n-1})$ be an even invariant depending on $n-1$ vectors $x_1, x_2, \ldots, x_{n-1} \in E_1^n$.

Let $x_i = (x_{i1}, \ldots, x_{in-1}, x_{in}) \in E_1^n$ for all $i = 1, \ldots, n-1$ and $\{x_1, x_2, \ldots, x_{n-1}\}$ be a basis of U. There is two situations:

(a) According to Corollaries3.1 and 3.2, we have $\kappa(U) \neq 1$ if and only if $\det Gr(x_1, x_2, \ldots, x_{n-1}) \neq 0$.

(b) According to Theorem3.1, we have $\kappa(U) = 1$ if and only if $\det Gr(x_1, x_2, \ldots, x_{n-1}) = 0$.

(a) Assume that $\det Gr(x_1, x_2, \ldots, x_{n-1}) \neq 0$.
Then there exists $g \in O(n,1)$ such that $gx_i = (0, y_{i2} \ldots, y_{in-1}, y_{in}) = y_i$ for all $i = 1, \ldots, n-1$ and so $\det Gr(y_1, y_2, \ldots, y_{n-1}) \neq 0$. (That is, g can be rewrite $g = \begin{pmatrix} h & 0_{(n-1)1} \\ 0_{1(n-1)} & h \end{pmatrix}$, where $0_{1(n-1)}$ is the zero $1 \times (n-1)$-matrix, $0_{(n-1)1}$ is the zero $(n-1) \times 1$-matrix and $h \in O(n-1,1)$).

We have the function $f(y_1, y_2, \ldots, y_{n-1})$ is a pseudo-orthogonal invariant in E_1^{n-1}, and hence according to T_{n-1}^{n-1} is expressible as a polynomial F in the Lorentzian inner products $< y_i, y_j >$ for all $i, j = 1, 2, \ldots, n-1$, where $< y_i, y_j >= y_{i2}y_{j2} + \ldots + y_{in-1}y_{jn-1} - y_{in}y_{jn}$.

(b) Assume that $\det Gr(x_1, x_2, \ldots, x_{n-1}) = 0$ and vectors $x_1, x_2, \ldots, x_{n-1}$ are linearly independent.
Then, by using the principle of the irrelevance of algebraic inequalities [14, Lemma 1.1.A,pp.4], the proof seen to clear.

If f were odd, there exist $n \times n$-matrix $\sigma = \begin{pmatrix} -1 & 0_{1(n-1)} \\ 0_{(n-1)1} & I_{n-1} \end{pmatrix}$,where $0_{1(n-1)}$ is the zero $1 \times (n-1)$-matrix, $0_{(n-1)1}$ is the zero $(n-1) \times 1$-matrix and $I_{(n-1)}$ is the identity $(n-1) \times (n-1)$-matrix such that $\sigma x_i = (-x_{i1}, \ldots, x_{in-1}, x_{in}) \in E_1^n$ for all $i = 1, \ldots, n-1$ and so
$\det Gr(x_1, x_2, \ldots, x_{n-1}) = \det Gr(\sigma x_1, \sigma x_2, \ldots, \sigma x_{n-1}) \neq 0$.
Then, we have the function $f(x_1, x_2, \ldots, x_{n-1})$ would show that
$f(x_1, x_2, \ldots, x_{n-1}) = f(\sigma x_1, \sigma x_2, \ldots, \sigma x_{n-1}) = det(\sigma) f(x_1, x_2, \ldots, x_{n-1})$, hence $f(x_1, x_2, \ldots, x_{n-1}) = 0$.
Invariance of f with respect to the proper pseudo-orthogonal transformation which we have thus performed results in the equation
$f(x_1, x_2, \ldots, x_{n-1}) = f_0(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1})$, where $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}$ are the $(n-1)$-dimensional vectors with the components $\bar{x}_i = (0, \bar{x}_{i2}, \bar{x}_{i3}, \ldots, \bar{x}_{in-1}, \bar{x}_{in}) = y_i$ for all $i = 1, \ldots, n-1$.
If f be odd we obtain at once

$$f = 0; \tag{4.2}$$

if f be even we apply T_{n-1}^{n-1} to the even pseudo-orthogonal $(n-1)$-dimensional invariant f_0 as mentioned above and thus find $f_0(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n-1}) = F(< x_i, x_j >)$ for all $i, j = 1, 2, \ldots, n-1$.
Since our transformation was pseudo-orthogonal, $< x_i, x_j >=< \bar{x}_i, \bar{x}_j >$ and therefore, as we claimed,

$$f(x_1, x_2, \ldots, x_{n-1}) = F(< x_i, x_j >) \tag{4.3}$$

for all $i, j = 1, 2, \ldots, n-1$.
The equations (4.2) and (4.3), one for the odd and the other for the even invariants,hold numerically irrespective of the values of the vectors $x_1, x_2, \ldots, x_{n-1}$ and consequently also as identities in the formal sense.

Our results is T_n^{n-1}. That is, there does not exist any odd invariant form of $n-1$ vectors in n-dimensions ,while every even invariant of $n-1$ vectors is expressible by their inner products.

The other step $T_n^{n-1} \to T_n^n$ is taken care of by Capelli's special identity applied to invariants $f(x_1, x_2, \dots, x_n)$ depending on n vectors. Its right side,

$$[x_1 x_2 \dots x_n]\Omega f, \tag{4.4}$$

contains the factor Ωf of lower rank than f. From [8, Proposition 37,pp.74], if f is even, Ωf is odd, and by hypothesis for induction can be expressed as the product of $[x_1 x_2 \dots x_n]$ with a polynomial of inner products. One the resorts to the equation $[x_1 x_2 \dots x_n]^2 = -detG(x_1, x_2, \dots, x_n)$, in order to express the even invariant (4.4) in terms of inner products only. It should be noticed that merely this special case of the equation $detG(x_1, x_2, \dots, x_n)$ enters into our proof. □

Theorem 4.1. *The system* $\{< x_i, x_j >, 1 \leq i \leq j \leq m\}$ *is a system of generators of* $R[x_1, x_2, \dots, x_m]^{O(n,1)}$.

Proof. It follows from the proof of the first part of Lemma 4.1. □

Theorem 4.2. *The systems* $\{[x_{i_1} x_{i_2} \dots x_{i_n}], 1 \leq i_1 < \dots < i_n \leq m\}$ *and* $\{< x_i, x_j >, 1 \leq i \leq j \leq m,\}$ *are a system of generators of* $R[x_1, x_2, \dots, x_m]^{SO(n,1)}$.

Proof. It follows from the proof of the second part of Lemma 4.1 and [8, Proposition 38,pp.76]. □

Example 4.1. Since $< gx_i, gx_j > = < x_i, x_j >$ for all $g \in O(n,1)$, we obtain that the inner products $< x_i, x_j >$ of vectors $x_i \in E_1^n$ is $O(n,1)$ -invariant.

Example 4.2. Let $x_1, x_2, \dots, x_n$ be vectors in E_1^n. We denote the the matrix of column-vectors $x_1, x_2, \dots, x_n$ by $U = \|x_1 x_2 \dots x_n\|$ and its determinant by $\det U$. Then $\det U$ is $SO(n,1)$-invariant. In fact, $det \|gx_1 gx_2 \dots gx_n\| = \det g \det U = \det U$ for all $g \in SO(n,1)$. Similarly, since $< gx_i, gx_j > = < x_i, x_j >$ for all $g \in SO(n,1)$, the inner products $< x_i, x_j >$ are $SO(n,1)$-invariant.

Proposition 4.1. *Let* $x_0, x_1, \dots, x_m$ *be vectors in* E_1^n. *Then* $R[x_0, x_1, \dots, x_m]^{M(n,1)} = R[x_1 - x_0, x_2 - x_0, \dots, x_m - x_0]^{O(n,1)}$.

Proof. Let $f(x_0, x_1, \dots, x_m) \in R[x_0, x_1, \dots, x_m]^{M(n,1)}$. Clearly, f is $M(n,1)$-invariant. Then,

$$f(x_0, x_1, \dots, x_m) = f(Fx_0, Fx_1, \dots, Fx_m) \tag{4.5}$$

for all $Fx = gx + b$, $g \in O(n,1)$, $b \in E_1^n$.

Here, specially, put $g = I$ and $b = -x_0$, where I is identity matrix. Then we have $Fx = x - x_0$ and so

$$f(Fx_0, Fx_1, \dots, Fx_m) = f(0, x_1 - x_0, x_2 - x_0, \dots, x_m - x_0). \tag{4.6}$$

Using equality (4.6), we have

$$f(0, x_1 - x_0, x_2 - x_0, \dots, x_m - x_0) = f(0, g(x_1 - x_0), g(x_2 - x_0), \dots, g(x_m - x_0)) \tag{4.7}$$

for all $g \in O(n,1)$.

Using equalities (4.5) and (4.7), $f(x_0, x_1, \dots, x_m)$ is $O(n,1)$-invariant. That is $f(x_0, x_1, \dots, x_m) = \varphi(< x_i - x_0, x_j - x_0 >)$. Conversely, it is obvious. □

Proposition 4.2. *Let* $x_0, x_1, \dots, x_m$ *be vectors in* E_1^n. *Then* $R[x_0, x_1, \dots, x_m]^{SM(n,1)} = R[x_1 - x_0, x_2 - x_0, \dots, x_m - x_0]^{SO(n,1)}$.

Proof. It is similar to Proposition4.1. □

Theorem 4.3. *Let* $x_0, x_1, \dots, x_m$ *be vectors in* E_1^n. *Then the system* $\{< x_i - x_0, x_j - x_0 >, 1 \leq i \leq j \leq m\}$ *is a system of generators of* $R[x_0, x_1, \dots, x_m]^{M(n,1)}$.

Proof. It follows from Theorem 4.1 and Proposition 4.1. □

Theorem 4.4. *Let* $x_0, x_1, \dots, x_m$ *be vectors in* E_1^n. *Then the systems* $\{[x_{i_1} - x_{i_0} x_{i_2} - x_{i_0} \dots x_{i_n} - x_{i_0}], 1 \leq i_0 < \dots < i_n \leq m\}$ *and* $\{< x_i - x_0, x_j - x_0 >, 1 \leq i \leq j \leq m,\}$ *are a system of generators of* $R[x_0, x_1, \dots, x_m]^{SM(n,1)}$.

Proof. It follows from Theorem 4.2 and Proposition 4.2. □

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Affiliations

İDRIS ÖREN
ADDRESS: Karadeniz Technical University, Dept. of Mathematics, 61080, Trabzon-Turkey.
E-MAIL: oren@ktu.edu.tr