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Eigenvalue Rigidity in Nonnegative Ricci Curvature

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ABSTRACT

Assume that M is a compact orientable Lorentz manifold with timelike boundary $\partial_t M$. If the Ricci curvature of M is bounded below by a positive constant k, then $\lambda_1 < (n-1)\max_M |H| - k$ where λ_1 is the first eigenvalue of the Laplacian of M.

Keywords: *First Eigenvalue; Lorentz manifold; Nondegenerate timelike boundary.*

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1. Introduction

From the differential geometric point of view, the study of boundaries of Riemannian and Lorentz manifolds has its own interest. Many interesting results on Riemannian and Lorentz manifolds have been obtained by many mathematicians (see [2-5]).

In [4], Choi and Wang proved that if M is a compact orientable hypersurface minimally embedded in N, then $\lambda_1 \geq k/2$ where λ_1 is the first eigenvalue of the Laplacian of M. Ho showed that if M is a compact orientable hypersurface embedded in N, then $2\lambda_1 > k - (n-1)\max_M |H|$ in [5].

In this paper, we studied Lorentz manifold with timelike boundary $\partial_t M$ and we obtained rigidity theorem under the assumption on nonnegative Ricci curvature and we proved the rigidity theorem by considering Stokes theorem. The Stokes theorem is one of the most beautiful topics in mathematics. This beauty comes from bringing together a variety of topics: integration, differentiation, manifolds and boundaries. Furthermore, it is widely used in other sciences such as engineering and physics.

The main purpose of this paper is to carry out some results which were given in [4] and [5] to Lorentz manifold with timelike boundary $\partial_t M$.

2. Preliminaries

Let R^n be the real n-dimensional vector space with standart basis $\{e_1, e_2, ..., e_n\}$. A inner product on R^n is defined by

$$\langle X, Y \rangle = \sum_{i=1}^{n} \varepsilon_i x_i y_i$$

for each vectors $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$, where $\varepsilon_1 = -1$ and $\varepsilon_i = 1$ for $2 \leq i \leq n$ [3]. This inner product is called the Lorentz metric on R^n. Then the pairs $(R^n, \langle , \rangle)$ is called the n-dimensional Lorentz space and denoted by L^n. A vector X in L^n is respectively called spacelike, timelike or lightlike(null) if $\langle X, X \rangle > 0$, $\langle X, X \rangle < 0$ or $\langle X, X \rangle = 0$, $X \neq 0$ [6].

Consider a Lorentz manifold $(M, \langle , \rangle)$ with boundary ∂M. A normal vector to ∂M at a point may have one of the three Lorentzian causal characters with respect to the induced metric on ∂M. Denote $\partial_s M$, $\partial_t M$, $\partial_o M$ the sets of the points where normal vectors are spacelike, timelike, lightlike(null) respectively. The subsets $\partial_s M$ and $\partial_t M$ are open in ∂M and the subset $\partial_o M$ is closed in ∂M. Clearly, ∂M consists of $\partial_s M$, $\partial_t M$, $\partial_o M$ and

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those subsets are pairwise disjoint. Consequently, $\partial M^{'}$consists of $\partial_s M$ and $\partial_t M$. $\partial M^{'}$ is an open submanifold of ∂M and may be considered as the nondegenerate boundary of M and $\partial_o M$ is referred to as the degenerate boundary of M.

The main difficulty in stating a boundary for a Lorentz manifold is that the boundary may become degenerate at some of its points, and hence there exists no well-defined unit outward normal at such points.

Let M be an n-dimensional Lorentz manifold with nondegenerate timelike boundary $\partial_t M$. Let u be a function defined on M smooth up to $\partial_t M$. Then Δu and $gradu$ denotes the Laplacian and the gradient of u with respect to the Lorentz metric on M respectively, similarly Lu and $Gradu$ denotes the Laplacian and the gradient of u with respect to the induced Lorentz metric on $\partial_t M$ respectively. For $x \in M$ and $X, Y \in T_x M$, we define the Hessian tensor $Hessu(X, Y) = X(Yu) - (D_X Y)u$ where $D_X Y$ is the covariant derivative of the Lorentz connection of M. We denote the covariant derivative of the Lorentz connection of $\partial_t M$ by $\overline{D}_X Y$.

Let $\{e_1, e_2, ..., e_{n-1}, e_n\}$ be a local orthonormal frame such that at $x \in \partial_t M$, $e_1, e_2, ..., e_{n-1}$ are tangent to $\partial_t M$ and e_n is the outward normal vector. We define the second fundemantal form II as, $II(v, w) = \langle D_v e_n, w\rangle$, v and w are vectors tangent to $\partial_t M$, and H to be the mean curvature, that is, $H = \frac{1}{n-1}\sum_{i=1}^{n-1} \varepsilon_i II(e_i, e_i)$.

In differential geometry, the Stokes Theorem can be stated as follows [1]:

Let M be a compact orientable k-dimensional manifold with boundary. If w is $(k-1)-$ form on M, then

$$\int_M dw = \int_{\partial M} w$$

where ∂M is the oriented boundary of M.

3. Main Theorem

Theorem 3.1. *Suppose that M is a compact orientable Lorentz manifold with nondegenerate timelike boundary $\partial_t M$. If the Ricci curvature of M is bounded below by a positive constant k, then $\lambda_1 < (n-1)\max_M |H| - k$ where λ_1 is the first eigenvalue of the Laplacian of M.*

Proof. Let f be the first eigenvalue of M, that is,

$$\overline{\Delta} f + \lambda_1 f = 0.$$

(3.1)

Let u be the solution of the Dirichlet problem such that:

$$\begin{cases} \Delta u = 0, \text{ in } M; \\ u = f, \text{ in } \partial_t M. \end{cases} \tag{3.2}$$

Then u is a function defined on M smooth up to $\partial_t M$. Then

$$\Delta u = \sum_{i=1}^{n} Hessu(e_i, e_i) = \sum_{i=1}^{n+1} u_{ii}, \tag{3.3}$$

where $u_{ij} = Hessu(e_i, e_j)$ for $i, j = 1, ..., n$. When $x \in \partial_t M$, when $i \neq n$, we have $grad(e_i, e_i) = Grad(e_i, e_i) - II_{ii} e_n$, where $II_{ij} = II(e_i, e_j)$. Hence, by (3.2) and (3.3), when $x \in \partial_t M$,

$$\begin{aligned} \Delta u &= u_{nn} + \overline{\Delta} f + \sum_{i=1}^{n-1} II_{ii} e_n(u) \\ &= u_{nn} + \overline{\Delta} f + (n-1)Hu_n, \end{aligned} \tag{3.4}$$

where H is the mean curvature of M. Then, by (3.1) and (3.4), for $x \in \partial_t M$,

$$u_{nn} = \lambda_1 f - (n-1)Hu_n. \tag{3.5}$$

For $x \in M$, by the fact that $\Delta u = 0$, we have $\frac{1}{2}\Delta \left|gradu\right|^2 = \sum\limits_{i,j=1}^{n} u_{ij}^2 + \sum\limits_{i,j=1}^{n} R_{ij} u_i u_j$, where $R_{ij} = Ric(e_i, e_j)$. By our assumption that Ricci curvature of $\partial_t M$ is bounded below by k, $\frac{1}{2}\Delta \left|gradu\right|^2 \geq \left|Hessu\right|^2 + k\left|gradu\right|^2$, which implies

$$\int\limits_M \frac{1}{2}\Delta \left|gradu\right|^2 \geq \int\limits_M \left|Hessu\right|^2 + k\int\limits_M \left|gradu\right|^2. \tag{3.6}$$

When $i \neq n$, $u_{in} = Hessu(e_i, e_n) = e_i(e_n u) - grad(e_i, e_n)u = e_i(u_n) - \sum\limits_{j=1}^{n-1} II_{ij} u_j$. On the other hand, using the Stokes theorem,

$$\begin{aligned}
\int\limits_M \frac{1}{2}\Delta \left|gradu\right|^2 &= \int\limits_{\partial_t M} \sum_{i=1}^{n-1} u_i u_{in} + \int\limits_{\partial_t M} u_n u_{nn} \\
&= \int\limits_{\partial_t M} Gradf.Gradu_n - \int\limits_{\partial_t M} \sum_{i,j=1}^{n-1} II_{ij} u_i u_j + \int\limits_{\partial_t M} u_n u_{nn} \\
&= \int\limits_{\partial_t M} u_n \overline{\Delta} f - \int\limits_{\partial_t M} II(Gradu, Gradu) + \int\limits_{\partial_t M} u_n u_{nn} \\
&= -\lambda_1 \int\limits_{\partial_t M} u_n f - \int\limits_{\partial_t M} II(Gradu, Gradu) - (n-1)\int\limits_{\partial_t M} H u_n^2.
\end{aligned} \tag{3.7}$$

Here we have used (3.1) and (3.5). By Stokes theorem and (3.2),

$$\int\limits_M \left|gradu\right|^2 = -\int\limits_M u\Delta u + \int\limits_{\partial_t M} u u_n = \int\limits_{\partial_t M} u_n f. \tag{3.8}$$

Hence, by (3.7) and (3.8), we have

$$\begin{aligned}
\int\limits_M \frac{1}{2}\Delta \left|gradu\right|^2 &= -\lambda_1 \int\limits_M \left|gradu\right|^2 - \int\limits_{\partial_t M} II(Gradu, Gradu) \\
&\quad -(n-1)\int\limits_{\partial_t M} H u_n^2.
\end{aligned} \tag{3.9}$$

Combining (3.6) and (3.9), we obtain

$$\begin{aligned}
(-\lambda_1 - k)\int\limits_M \left|gradu\right|^2 &\geq \int\limits_{\partial_t M} II(Gradu, Gradu) + (n-1)\int\limits_{\partial_t M} H u_n^2 \\
&\quad + \int\limits_{\partial_t M} \left|Hessu\right|^2.
\end{aligned} \tag{3.10}$$

Note that $\int\limits_{\partial_t M} II(Gradu, Gradu) = \int\limits_M II(gradf, gradf)$ and we can assume that $\int\limits_{\partial_t M} II(Gradu, Gradu) \geq 0$. On the other hand, by Stokes theorem and Holder's inequality

$$\begin{aligned}
\int\limits_{\partial_t M} H u_n^2 &\geq -\max_M |H| \int\limits_{\partial_t M} u_n^2 \\
&= -\max_M |H| \left(\int\limits_M u_n \Delta u + \int\limits_M gradu_n.gradu \right) \\
&\geq -\max_M |H| \left(\int\limits_M \left|gradu_n\right|^2 \right)^{1/2} . \left(\int\limits_M \left|gradu\right|^2 \right)^{1/2} .
\end{aligned} \tag{3.11}$$

Here we have used (2). Now we consider two cases:

Case I: If $\int\limits_M |gradu_n|^2 \leq \int\limits_M |gradu|^2$, then from (3.11), we have

$$\int\limits_{\partial_t M} Hu_n^2 \geq -\max_M |H| \int\limits_M |gradu|^2 .$$

Since $\int\limits_M |Hessu|^2 \geq 0$ and $\int\limits_{\partial_t M} II(Gradu, Gradu) \geq 0$, by (3.10), we have

$$(-\lambda_1 - k)\int\limits_M |gradu|^2 \geq -(n-1)\max_M |H| \int\limits_M |gradu|^2 .$$

Since $\int\limits_M |gradu|^2 > 0$, we get $\lambda_1 \leq (n-1)\max_M |H| - k$.

Case II: If $\int\limits_M |gradu_n|^2 \geq \int\limits_M |gradu|^2$, then from (3.11), we have

$$\int\limits_{\partial_t M} Hu_n^2 \geq -\max_M |H| \int\limits_M |gradu_n|^2 .$$

Therefore, (3.10) implies that

$$(-\lambda_1 - k)\int\limits_M |gradu|^2 + (n-1)\max_M |H| \int\limits_M |gradu_n|^2 \geq$$
$$\geq \int\limits_M |Hessu|^2 + \int\limits_{\partial_t M} II(Gradu, Gradu) \geq 0.$$

Since $\int\limits_M |gradu_n|^2 \geq \int\limits_M |gradu|^2$, we have

$$(-\lambda_1 - k + (n-1)\max_M |H|)\int\limits_M |gradu_n|^2 \geq$$
$$\geq (-\lambda_1 - k)\int\limits_M |gradu|^2 + (n-1)\max_M |H| \int\limits_M |gradu_n|^2 \geq 0.$$

Since $\int\limits_M |gradu|^2 > 0$, $\int\limits_M |gradu_n|^2 \geq \int\limits_M |gradu|^2 > 0$ by our assumption. Hence, $\lambda_1 \leq (n-1)\max_M |H| - k$.

Therefore, in both cases, we have $\lambda_1 \leq (n-1)\max_M |H| - k$. We claim that it is impossible for the equality holds. Suppose not, then by above argument, we must have $\int\limits_M |Hessu|^2 = 0$. From this, we get $u_{ij} = 0$ on M for all $1 \leq i, j \leq n$. Since u is smooth up to $\partial_t M$, we get $f_{ij} = 0$ on $\partial_t M$ for $1 \leq i, j \leq n-1$, which implies that $Lf = 0$ which is impossible since f is the first eigenfunction of M. This proves our claim. We have $\lambda_1 < (n-1)\max_M |H| - k$ as required. □

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