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Lightlike Submanifolds with Planar Normal Section in Semi Riemannian Product Manifolds

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ABSTRACT

In the present paper we give conditions for screen semi invariant lightlike submanifolds of a semi-Riemannian product manifold to have degenerate planar normal sections. Also we give sufficient and efficient conditions for screen invariant and screen anti invariant lightlike submanifold of a semi-Riemanniann product manifold to have non-degenerate planar normal sections.

Keywords: *Lightlike submanifolds; semi-Riemannian product manifolds; Planar normal section; Screen transversal lightlike submanifolds.*
AMS Subject Classification (2010): *53C40; 53C15; 53C42; 53C50.*

1. Introduction

Surfaces with planar normal sections in submanifolds of Riemannian manifolds were first studied by Bang-Yen Chen [5]. In [6],[8], Y. H. Kim initiated the study of semi-Riemannian setting of such surfaces. Both authors obtained similar results in these spaces. We investigated lightlike and half lightlike submanifolds with planar normal sections in [3] and [4]. We first showed that every lightlike surfaces of Minkowski 3- space has degenerate planar normal sections. Then we studied lightlike surfaces with non- degenerate planar normal sections and obtained a characterization for such lightlike surfaces.

In [4] we investigated half-lightlike submanifolds with planar normal sections of four dimensional semi-Riemanniann space. We obtained necessary and sufficient conditions for a half lightlike submanifold of IR_2^4 such that it had degenerate or non-degenerate planar normal sections.

In [7],[8] Kiliç, Sahin, Atceken and Keles introduced a new class of lightlike submanifold called screen semi-invariant (SSI) lightlike submanifolds of a semi-Riemanniann product manifold. They gave examples of such submanifolds and studied the geometry of leaves of distributions which are involved in the definition of SSI- lightlike submanifolds. They obtained necessary and sufficient conditions for the SSI- lightlike submanifold to be locally product manifold and gave some characterizations for totally umbilical SSI-lightlike and screen anti-invariant lightlike submanifolds of semi-Riemannian product manifold. In this paper, we give conditions for a screen semi invariant or anti-invariant lightlike submanifold of a semi-Riemannian product manifold to have degenerate or non-degenerate planar normal sections.

2. Some fundamental concepts and definitions

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+k}, \bar{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution which is a semi-Riemannian complementary distribution of $RadTM$ is of rank r, where$1 \leq r \leq m$. $RadTM = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \cup_{x \in M} \left\{ u \in T_x\bar{M} \mid \bar{g}(u, v) = 0, \forall v \in T_xM \right\}. \tag{2.1}$$

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Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $RadTM$ in TM. i.e., $TM = RadTM \perp S(TM)$.

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $RadTM$, there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [2]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}^{\perp}\mid_M$. Then

$$\begin{aligned} tr(TM) &= ltrTM \perp S(TM^{\perp}), \qquad (2.2)\\ T\bar{M}\mid_M &= S(TM)\perp[RadTM \oplus ltrTM] \perp S(TM^{\perp}). \end{aligned}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$ [1]. The following result is important to this paper.

Definition 2.1. [3]. The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is
Case1: r-lightlike if $r < \min\{m, k\}$;
Case2: Co-isotropic if $r = k < m; S(TM^{\perp}) = \{0\}$;
Case3: Isotropic if $r = m = k$; $S(TM) = \{0\}$;
Case4: Totally lightlike if $r = k = m$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \forall X, Y \in \Gamma(TM), \qquad (2.3)$$
$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \qquad (2.4)$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X,Y) + h^{s}(X,Y), \forall X, Y \in \Gamma(TM), \qquad (2.5)$$
$$\bar{\nabla}_X N = -A_N X + \nabla^{\ell}_X N + D^{s}(X,N), N \in \Gamma(ltr(TM)), \qquad (2.6)$$
$$\bar{\nabla}_X W = -A_W X + \nabla^{s}_X W + D^{\ell}(X,W), W \in \Gamma\left(S(TM^{\perp})\right). \qquad (2.7)$$

Denote the projection of TM on $S(TM)$ by $\bar{P}$.Then by using (2.3), (2.5)-(2.7) and a metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^{s}(X,Y), W) + \bar{g}\left(Y, D^{\ell}(X,W)\right) = g(A_W X, Y), \qquad (2.8)$$
$$\bar{g}(D^{s}(X,N), W) = \bar{g}(N, A_W X). \qquad (2.9)$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^{*}_X \bar{P}Y + h^{*}\left(X, \bar{P}Y\right), \qquad (2.10)$$
$$\nabla_X \xi = -A^{*}_{\xi} X + \nabla^{*t}_X \xi, \qquad (2.11)$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. By using above equations, we obtain

$$\bar{g}\left(h^{\ell}\left(X, \bar{P}Y\right), \xi\right) = g\left(A^{*}_{\xi} X, \bar{P}Y\right), \qquad (2.12)$$
$$\bar{g}\left(h^{s}\left(X, \bar{P}Y\right), N\right) = g\left(A_N X, \bar{P}Y\right), \qquad (2.13)$$
$$\bar{g}\left(h^{\ell}(X,\xi), \xi\right) = 0, A^{*}_{\xi}\xi = 0. \qquad (2.14)$$

In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.5) we get

$$(\nabla_X g)(Y,Z) = \bar{g}\left(h^{\ell}(X,Y), Z\right) + \bar{g}\left(h^{\ell}(X,Z), Y\right). \qquad (2.15)$$

A lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold is called totally umbilical if there is a smooth function ϱ, such that

$$h(X,Y) = \varrho g(X,Y), \forall X, Y \in \Gamma(TM) \qquad (2.16)$$

where ϱ is non-vanishing smooth function on a neighborhood U in M.

A lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold is called screen locally conformal if the shape operators A_N and A^*_ξ of M and $S(TM)$, respectively, are related by

$$A_N = \varphi A^*_\xi \tag{2.17}$$

where φ is non-vanishing smooth function on a neighborhood U in M. Therefore, it follows that $\forall X, Y \in \Gamma(S(TM)), \xi \in RadTM$

$$h^*(X, \xi) = 0 \tag{2.18}$$

[2]

3. Screen Semi İnvariant lightlike submanifolds of semi-Riemannian Product Manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2 -dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ the projections which are given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$, respectively.

We denote the product manifold by $\bar{M} = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $\forall X, Y \in \Gamma(T\bar{M})$. Then we have

$$\pi_*^2 = \pi_*, \pi_*\sigma_* = \sigma_*\pi_* = 0,$$

$$\sigma_*^2 = \sigma_*, \pi_* + \sigma_* = I,$$

where I is identity transformation. Thus $(\bar{M}, \bar{g})$ is an $(m_1 + m_2)$- dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\bar{M} = M_1 \times M_2$ is characterized by M_1 and M_2 are totally geodesic submanifolds of $\bar{M}$.

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that

$$\begin{aligned} F.F &= (\pi_* - \sigma_*)(\pi_* - \sigma_*) \\ F^2 &= \pi_*^2 - \pi_*\sigma_* - \sigma_*\pi_* + \sigma_*^2 = I \end{aligned}$$

$$F^2 = I, \bar{g}(FX, Y) = \bar{g}(X, FY)$$

for any $X, Y \in \Gamma(T\bar{M})$. If we denote the levi-civita connection on $\bar{M}$ by $\bar{\nabla}$, then it can be seen that $(\bar{\nabla}_X F) Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$, that is, F is parallel with respect to $\bar{\nabla}$ [10].

Let M be a submanifold of a Riemannian (or semi-Riemannian) product manifold $\bar{M} = M_1 \times M_2$. If $F(TM) = TM$, then M is called invariant submanifold, if $F(TM) \subset TM^\perp$, then M is called anti-invariant submanifold.[10].

Definition 3.1. Let $(\bar{M}, \bar{g})$ be a semi Riemannian product manifold and M be a lightlike submanifold of $\bar{M}$. We say that M is SSI-lightlike submanifold of $\bar{M}$ if the following statements are satisfied :

1- There exists a non-null distribution $D \subseteq S(TM)$ such that

$$\begin{aligned} S(TM) &= D \perp D^\perp, \\ FD &= D, \\ FD^\perp &\subseteq S(TM^\perp), \\ D \cap D^\perp &= \{0\}, \end{aligned}$$

where $D^\perp$ is orthogonal complementary to D in $S(TM)$.

2- $RadTM$ is invariant with respect to F,that is $FRadTM = RadTM$. Then we have

$$\begin{aligned} FltrTM &= ltrTM \\ TM &= D' \perp D^\perp, D' = D \perp RadTM. \end{aligned}$$

Hence it follows that D' is also invariant with respect to F. We denote the orthogonal complement to $FD^{\perp}$ in $S(TM^{\perp})$ by D_0. Then, we have

$$tr(TM) = ltrTM \perp FD^{\perp} \perp D_0.$$

If $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$,then we say that M is a proper SSI-lightlike submanifold of $\bar{M}$. Hence, for on proper M, we have $\dim(D) \geq 1$, $\dim(D^{\perp}) \geq 1$, $\dim M \geq 3$ and $\dim \bar{M} \geq 5$. Furthermore, there exists no proper SSI-lightlike hypersurface of a semi-Riemannian product manifold. If $D = \{0\}$, that is $FS(TM) \subseteq S(TM^{\perp})$, then we say that M is screen anti invariant lightlike submanifold.

4. Lightlike Submanifolds with Planar Normal Section in Semi Riemannian Product Manifolds

Let M be a screen semi invariant lightlike submanifold of a $\bar{M}$ semi-Riemannian product manifold. Since M be a screen semi lightlike sunmanifolds of $\bar{M}$, $FRadTM = RadTM$. For a point p in M and lightlike vector $\{\xi_1.\xi_2\}$ which spans the radical distribution of a lightlike submanifold, the vector $F\xi = aF\xi_1 + bF\xi_2$ and transversal distribution $tr(TM)$ to M at p determine a subspace $E(p, F\xi)$ through p in $\bar{M}$. The intersection of M and $E(p, F\xi)$ gives a lightlike curve γ in a neighborhood of p, which is called the normal section of M at the point p in the direction of $F\xi$. From this we have

$$\begin{aligned} F\xi &= aF\xi_1 + bF\xi_2 = a\xi_2 + b\xi_1 && (4.1)\\ \gamma'(s) &= F\xi = a\xi_2 + b\xi_1 && (4.2)\\ \gamma''(s) &= \bar{\nabla}_{F\xi}F\xi = -b\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_1 - a\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_2 && (4.3)\\ \gamma'''(s) &= \bar{\nabla}_{F\xi}\bar{\nabla}_{F\xi}F\xi = -b^2\xi_1\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_1 && (4.4)\\ &\quad +b^2\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\tau(\xi_1)\xi_1 \\ &\quad -ab\xi_1\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_1 - ab\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\tau(\xi_1)\xi_2 \\ &\quad -ba\xi_2\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_1 - ba\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\tau(\xi_2)\xi_1 \\ &\quad -a^2\xi_2\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\xi_2 - a^2\left(b\tau(\xi_1) + a\tau(\xi_2)\right)\tau(\xi_2)\xi_2 \end{aligned}$$

Then, $\gamma'''(s)$ is a linear combination of $\gamma'(s)$ and $\gamma''(s)$. Thus (4.1), (4.3) and (4.4) give $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$. We have a SSI lightlike submanifold of a semi-Riemannian product manifold always has degenerate planar normal sections.

Corollary 4.1. *Every screen semi invariant lightlike submanifolds of semi-Riemannian product manifold has degenerate planar normal sections.*

Now,Let M be a screen anti invariant lightlike submanifold of a $\bar{M}$ semi-Riemannian product manifold. For M be a SSI lightlike submanifolds of $\bar{M}$ we can get $D = \{0\}$ and $F(S(TM)) \subseteq S(TM^{\perp})$. Then $S(TM) = D^{\perp}$, $FD = D$ that is $FTM \subseteq TM^{\perp}$.Now, we will check screen anti lightlike submanifolds with non-degenerate planar normal sections. For a point p in M and a non-degenere vector $w \in S(TM)$ tangent to M at p , the vector w and transversal space $tr(TM)$ to M at p determine a subspace $E(p, w)$ in $\bar{M}$ through p. The intersection of M and $E(p, w)$ give a non-degenerate curve γ in a neighborhood of p, which is called the normal section of M at p in the direction of w. Now, we research the conditions for a screen anti invariant lightlike submanifold of $\bar{M}$ to have non-degenerate planar normal sections.

Let $(M, g, S(TM))$ be a screen conformal lightlike submanifold of $\left(\bar{g}, \bar{M}\right)$. In this case $S(TM)$ is integrable. We denote integral submanifold of $S(TM)$ by M'. Then, using (2.5), (2.10) and (2.17) we obtain

$$\begin{aligned} \gamma'(s) &= w, S(TM) = Sp\{w\}, Fw \in S(TM^{\perp}) && (4.5)\\ \gamma''(s) &= \bar{\nabla}_w w = \nabla^*_w w + h^*(w,w) + h(w,w) && (4.6)\\ \gamma'''(s) &= \nabla^*_w\nabla^*_w w + h^*\left(w, \nabla^*_w w\right) + h\left(w, \nabla^*_w w\right) - A^*_{h^*(w,w)}w && (4.7)\\ &\quad +\nabla^{\perp}_w h^*(w,w) - A_{h(w,w)}w + \nabla^{\perp}_w h(w,w) \end{aligned}$$

Then, using (2.16), (2.17) and (2.18) we find

$$A_{h(w,w)}w = \varphi A^*_{h^*(w,w)}w, \qquad (4.8)$$

where φ is non-vanishing smooth function on a neighborhood U in M.

Where ∇^* and ∇ are linear connections on $S(TM)$ and $\Gamma(TM)$respectively and $\gamma'(s) = w$. From the definition of normal section and that $S(TM) = Sp\{w\}$, we have

$$w \wedge \nabla^*_w w = 0 \tag{4.9}$$

and

$$w \wedge \nabla^*_w \nabla^*_w w = 0. \tag{4.10}$$

Then, from (4.8), (4.5), (4.6) and (4.7), we obtain that if $(h^*(w,w) + h(w,w)) \wedge \left(\nabla^{\perp}_w h^*(w,w) + \nabla^{\perp}_w h(w,w)\right) = 0$, we can say $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$.Thus, M has non-degenerate planar normal sections.Proof of conversely is trivial.

If M is totally geodesic screen anti invariant lightlike submanifold of $\bar{M}$, we have $h(w,w) = 0$, $A^*_{h^*(w,w)} = 0$. Hence (4.5)-(4.6) become

$$\begin{aligned} \gamma'(s) &= w \\ \gamma''(s) &= \nabla^*_w w + h^*(w,w) \\ \gamma'''(s) &= \nabla^*_w \nabla^*_w w + h^*(w, \nabla^*_w w) \\ &\quad - A^*_{h^*(w,w)} w + \nabla^{\perp}_w h^*(w,w). \end{aligned}$$

,In this case, if $h^*(w,w) \wedge \nabla^{\perp}_w h^*(w,w) = 0$, then ,we have $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$.

Proof of conversely is trivial.

Consequently, we have the following,

Theorem 4.1. *Let M be a screen anti invariant lightlike submanifold of a $\bar{M}$ semi Riemannian product manifold. Then*

1. *M is screen conformal lightlike submanifold with non-degenerate planar normal sections if and only if*

$$(h^*(w,w) + h(w,w)) \wedge \left(\nabla^{\perp}_w h^*(w,w) + \nabla^{\perp}_w h(w,w)\right) = 0. \tag{4.11}$$

Where $w \in S(TM)$

2. *M is totally geodesic lightlike submanifold with non-degenerate planar normal sections if and only if*

$$h^*(w,w) \wedge \nabla^{\perp}_w h^*(w,w) = 0. \tag{4.12}$$

Now,let M be a screen invariant lightlike submanifold of a $\bar{M}$ semi-Riemannian product manifold. Since M be a screen invariant lightlike submanifolds of $\bar{M}$, we can get $D^{\perp} = \{0\}$ and $F(S(TM)) \subseteq S(TM)$. Then $S(TM) = D$, $FD = D$ that is $FTM = TM$.Now, we will check screen invariant lightlike submanifold with non-degenerate planar normal sections. For a point p in M and a non-degenerate vector $Fw_1 \in S(TM)$ tangent to M at p and $S(TM) = sp\{w_1, w_2\}$, $Fw_1 = w_2$, $Fw_2 = w_1$, the vector w and transversal space $tr(TM)$ to M at p determine a subspace $E(p, Fw_1)$ in $\bar{M}$ through p. The intersection of M and $E(p, Fw_1)$ give a non-degenerate curve γ in a neighborhood of p, which is called the normal section of M at p in the direction of Fw_1. Now, we research the conditions for a screen invariant lightlike submanifold of $\bar{M}$ to have non-degenerate planar normal sections.

Now, we assume that $S(TM)$ is integrable, then, we find

$$\begin{aligned} \gamma'(s) &= Fw_1 = w_2 \\ \gamma''(s) &= \bar{\nabla}_{Fw_1} Fw_1 = \bar{\nabla}_{w_2} w_2 = \nabla^*_{w_2} w_2 + h^*(w_2, w_2) + h(w_2, w_2) \\ \gamma'''(s) &= \bar{\nabla}_{w_2} \nabla^*_{w_2} w_2 + \bar{\nabla}_{w_2} h^*(w_2, w_2) + \bar{\nabla}_{w_2} h(w_2, w_2) \\ &= \nabla^*_{w_2} \nabla^*_{w_2} w_2 + h^*\left(w_2, \nabla^*_{w_2} w_2\right) + h\left(w_2, \nabla^*_{w_2} w_2\right) \\ &\quad - A^*_{h^*(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h^*(w_2, w_2) - A_{h(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h(w_2, w_2). \end{aligned}$$

If M is totally geodesic, $h^*(w_2, w_2) = h(w_2, w_2) = 0$, that is

$$\begin{aligned} \gamma'(s) &= w_2 \\ \gamma''(s) &= \nabla^*_{w_2} w_2 \\ \gamma'''(s) &= \nabla^*_{w_2} \nabla^*_{w_2} w_2. \end{aligned}$$

Because of $\dim(S(TM)) = 2$, we find $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$.

Now ,we assume that M is totally umbilical and $S(TM)$ is parallel distribution, then $h^* = 0$ and $A_{h(w_2,w_2)} \in \Gamma(Rad(TM))$ in M, we can find

$$\begin{aligned}
\gamma'(s) &= w_2 \\
\gamma''(s) &= \nabla^*_{w_2} w_2 + h(w_2, w_2) \\
\gamma'''(s) &= \nabla^*_{w_2}\nabla^*_{w_2} w_2 + h\left(w_2, \nabla^*_{w_2} w_2\right) - A_{h(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h(w_2, w_2).
\end{aligned}$$

If M has non-degenerate planar normal section, then we use above equations

$$\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = w_2 \wedge (h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2} h(w_2, w_2)\right) = 0,$$

namely, $h(w_2, w_2) \wedge \bar{\nabla}_{w_2} h(w_2, w_2) = 0$.

If M is screen conformal submanifold,

$$A_N = \varphi A^*_{\xi} \tag{4.13}$$

where φ is non-vanishing smooth function on a neighborhood U in M.

$$\begin{aligned}
\gamma'(s) &= w_2 \\
\gamma''(s) &= \nabla^*_{w_2} w_2 + h^*(w_2, w_2) + h(w_2, w_2) \\
\gamma'''(s) &= \nabla^*_{w_2}\nabla^*_{w_2} w_2 + h^*\left(w_2, \nabla^*_{w_2} w_2\right) + h\left(w_2, \nabla^*_{w_2} w_2\right) \\
&\quad - A^*_{h^*(w_2,w_2)} w_2 - A_{h(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h(w_2, w_2),
\end{aligned}$$

since $A^*_{h^*(w_2,w_2)} w_2, A_{h(w_2,w_2)} w_2 \in S(TM)$ and $\gamma'''(s) \wedge \gamma''(s) \wedge \gamma'(s) = 0$, we find

$$w_2 \wedge \begin{pmatrix} \nabla^*_{w_2} w_2 \\ +h^*(w_2, w_2) + h(w_2, w_2) \end{pmatrix} \wedge \begin{pmatrix} \nabla^*_{w_2}\nabla^*_{w_2} w_2 + h^*\left(w_2, \nabla^*_{w_2} w_2\right) \\ +h\left(w_2, \nabla^*_{w_2} w_2\right) - A^*_{h^*(w_2,w_2)} w_2 \\ -A_{h(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h(w_2, w_2) \end{pmatrix} = 0,$$

namely, we obtain

$$(h^*(w_2, w_2) + h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2}(h^*(w_2, w_2) + h(w_2, w_2))\right) = 0.$$

Conversely, we assume that M has planar non-degenerate normal sections. Then,

$$w_2 \wedge (h^*(w_2, w_2) + h(w_2, w_2)) \wedge \begin{pmatrix} \nabla^*_{w_2}\nabla^*_{w_2} w_2 + h^*\left(w_2, \nabla^*_{w_2} w_2\right) \\ +h\left(w_2, \nabla^*_{w_2} w_2\right) - A^*_{h^*(w_2,w_2)} w_2 \\ -A_{h(w_2,w_2)} w_2 + \nabla^{\perp}_{w_2} h(w_2, w_2) \end{pmatrix} = 0,$$

namely

$$(h^*(w_2, w_2) + h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2}(h^*(w_2, w_2) + h(w_2, w_2))\right) = 0,$$

or

$$w_2 \wedge (h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2} h(w_2, w_2)\right) = 0.$$

If $h^*(w_2, w_2) + h(w_2, w_2) = 0$, M is totally geodesic, if $A^*_{h^*(w_2,w_2)} w_2, A_{h(w_2,w_2)} w_2 \in S(TM)$, M is screen conformal,$(h^*(w_2, w_2) + h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2}(h^*(w_2, w_2) + h(w_2, w_2))\right) = 0$, if $h^* = 0$ and $A_{h(w_2,w_2)} \in \Gamma(Rad(TM))$ in M, namely M is totally umbilical , then $(h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2} h(w_2, w_2)\right) = 0$

Consequently, we have the following,

Theorem 4.2. *Let M be a screen invariant lightlike submanifold of $\bar{M}$ semi-Riemannian product manifold.. Then*

1. *M is screen invariant submanifold with non degenerate planar normal sections if and only if M is totally geodesic submanifold,*
2. *M is totally umbilical screen invariant submanifold with non degenerate planar normal sections if and only if*

$$(h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2} h(w_2, w_2)\right) = 0, \tag{4.14}$$

3. *M is screen conformal screen invariant submanifold with non degenerate planar normal sections if and only if*

$$(h^*(w_2, w_2) + h(w_2, w_2)) \wedge \left(\bar{\nabla}_{w_2}(h^*(w_2, w_2) + h(w_2, w_2))\right) = 0. \tag{4.15}$$

Example 4.1. Let M_1 and M_2 be R_1^3 and R^2, respectively. Then $\bar{M} = M_1 \times M_2$ is a semi-Riemannian product manifold with metric tensor $\bar{g} = \pi^* g_1 + \sigma^* g_2$, where g_1 and g_2 are standard metric tensors R_1^3 and R^2 with (-,+,+) and (+,+), π^* and σ^* are projections of $\Gamma(T\bar{M})$ to $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. Let M be a submanifold of $\bar{M}$ given by equations

$$\begin{aligned} x_1 &= \sqrt{2}u_1 + u_3, x_2 = u_1 + u_3 \\ x_3 &= u_1 + \left(\sqrt{2} - 1\right) u_3, x_4 = u_2 + \frac{\sqrt{2}-1}{\sqrt{2}} u_3 \\ x_5 &= u_2 - \frac{\sqrt{2}-1}{\sqrt{2}} u_3 \end{aligned}$$

M is SSI-lightlike submanifold of $\bar{M}$[9]. Now, we choose degenerate normal section curve γ through $\xi = \left(\sqrt{2}u_1, u_1, u_1, 0, 0\right)$. In this case, we find

$$\begin{aligned} \gamma'(s) &= \xi = \left(\sqrt{2}u_1, u_1, u_1, 0, 0\right) \\ \gamma''(s) &= \nabla_\xi \xi = \left(2u_1, \sqrt{2}u_1, \sqrt{2}u_1, 0, 0\right) \\ \gamma'''(s) &= \nabla_\xi \nabla_\xi \xi = \left(2\sqrt{2}u_1, 2u_1, 2u_1, 0, 0\right). \end{aligned}$$

From this,we can say that $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linear dependent, Namely $\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s) = 0$. Therefore M has degenerate planar normal sections.

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