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On Deterministic and Random Rolling of Polyhedra

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ABSTRACT

For a convex polyhedron standing with one of its face on a fixed plane we mean rolling when it is rotated into another similar position around any of its edge lying on the plane. A set is said to be the trace of the polyhedron $\mathcal{P}$ if some point of it coincides of some vertex of $\mathcal{P}$ in some position. In this note we investigate the trace of deterministic and random rolling of polyhedra.

Keywords: *pseudo-Finsler manifold; Finsler connection; almost hypercomplex structure.*

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1. Introduction

Let us take a convex polyhedron $\mathcal{P}$ standing with one of its face on a fixed plane Σ. We mean *rolling* of $\mathcal{P}$ when it is rotated into another similar position around any of its edge lying on Σ. So after rolling another face of $\mathcal{P}$ will lean on the plane. We take $\mathcal{P}$ in an arbitrary initial position and we denote by R rolling and $R(\mathcal{P})$ the position of $\mathcal{P}$ after R, and a word $R_n \dots R_2 R_1$ means that after rotation R_1 we act $R_2, \dots, R_n$ respectively, and let $R_n \dots R_2 R_1(\mathcal{P}) = R_n \dots R_2(R_1(\mathcal{P}))$.

Set of all points

$$\mathfrak{M}_{\mathcal{P}} := \{X \in \Sigma : X \text{ coincides of some vertex of } \mathcal{P} \text{ in some position of } \mathcal{P}\}.$$

We say that $\mathfrak{M}_{\mathcal{P}}$ is a *trace* of $\mathcal{P}$ after all rolling and we say that $\mathcal{P}$ is D-polyhedron if the set $\mathfrak{M}_{\mathcal{P}}$ is everywhere dense on Σ. E.g. when $\mathcal{P}$ is a cube then $\mathfrak{M}_{\mathcal{P}}$ is a lattice (so a cube is not D-polyhedron), and when $\mathcal{P}$ is a rectangular parallelepiped with at least two edges having irrational ratio is D-polyhedron (see in [3]).

In [3] I gave a sufficient condition for a general polyhedron to be D-polyhedron, and I characterized all regular D-polyhedra. In [4] we also charaterized all semi-regular (Archimedean) D-polyhedra. See related problems in [5] and [1].

2. Deterministic rolling of a polyhedron

We say that a subset Y of Σ is locally-dense if there is a point $P \in \Sigma$ and a neighborhood $U(P)$ of P such that $Y \cap U(P)$ is dense in $U(P)$.

A question may arise whether there is a polyhedron for which the set $\mathfrak{M}_{\mathcal{P}}$ is "spotted", i.e. $\mathfrak{M}_{\mathcal{P}}$ is locally-dense but $\mathcal{P}$ is not D-polyhedron. Our first result is that the answer is no:

Theorem 2.1. *If a polyhedron $\mathcal{P}$ is locally-dense then it is a D-polyhedron.*

Definition 2.1. Let $\mathcal{S} \subseteq \mathbb{R}^2$ be a subset of the plane and $\varepsilon > 0$. A set $\mathcal{X}$ is said to be $(\mathcal{S}, \varepsilon)$-dense if for every $P \in \mathcal{S}$ there is a point $X \in \mathcal{X} \cap \mathcal{S}$ for which $d(P, X) < \varepsilon$. $\mathcal{X}$ is said to be $\mathcal{S}$-dense if it is $(\mathcal{S}, \varepsilon)$-dense for every $\varepsilon > 0$.

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In the rest of this paper the set $\mathcal{S}$ will be special; it will be a boundary of a circle.
We use the usual notations $\mathbb{N}, \mathbb{R}, \mathbb{R}^+, \mathbb{Q}, \mathbb{Q}^* = \mathbb{R} \setminus \mathbb{Q}$.

3. Random rolling of a polyhedron

We start again with a convex polyhedron $\mathcal{P}$ standing in some initial position of Σ. Assume that the touching face $\mathcal{F}$ is a $k-$gon. Now roll $\mathcal{P}$ around one of the edges of $\mathcal{F}$– or keep the position of $\mathcal{P}$ – with probability $\frac{1}{k+1}$ uniformly at random. (For a technical reason we include the identical rotation as a rolling too). Define the trace in a similar way as it is in the first paragraph and denote it by $RAN(\mathfrak{M})$. Now one can ask what the structure of $RAN(\mathfrak{M})$ is. Is it true that with probability 1 after finitely many rolling $\mathcal{P}$ will be "close" to the initial position of it?

In this paragraph we are going to investigate the simplest case. We say that a tetrahedron is general if the four vertices are selected at random. Then we act a random roll on a general tetrahedron. We will show

Theorem 3.1. *Let $\varepsilon > 0$. For almost all tetrahedron $\mathcal{P}$ with probability 1 there are infinitely many circle $\mathcal{S}$ for which $RAN(\mathfrak{M}_{\mathcal{P}}) \cap \mathcal{S}$ is an ε-dense set on $\mathcal{S}$*

4. Proofs

Proof of Theorem 2.1: Let $P \in \Sigma$ be a point in the plane for which $U(P) \cap \mathfrak{M}_{\mathcal{P}}$ is a dense set, where $U(P) := U_\Delta(P) = \{Q \in \Sigma : d(P, Q) < \Delta\}$ for some radius $\Delta \in \mathbb{R}^+$, and $d(\cdot, \cdot)$ is the usual metric. Denote by $\{Y_1, Y_2, \ldots, Y_s\}$ the set of all vertices and by $\{F_1, F_2, \ldots, F_r\}$ the set of all faces of $\mathcal{P}$, and write

$$\mathfrak{M}_{\mathcal{P}}(Y_i, F_j) :=$$

$$= \{X \in \Sigma : X \text{ coincides of vertex } Y_i \text{ in some position of } \mathcal{P} \text{ standing on face } F_j\}.$$

We claim that there exist a pair i, j $(1 \le i \le s;\ 1 \le j \le r)$ and an $U(P') \subseteq U(P)$ for which $U(P') \cap \mathfrak{M}_{\mathcal{P}}(Y_i, F_j)$ is a dense set. To see this, let us suppose the opposite: assume that there is a neighborhood, $U(P_1) \subseteq U(P)$ for which $U(P_1) \cap \mathfrak{M}_{\mathcal{P}}(Y_1, F_1) = \{\emptyset\}$. Presume now that the sets $U(P_z) \subseteq \cdots \subseteq U(P_1) \subseteq U(P)$ $(z \ge 1)$ have been defined for which for every pair (t, p); $t + p = k,\ (1 \le k \le z)$ we get that

$$U(P_k) \cap \mathfrak{M}_{\mathcal{P}}(Y_t, F_p) = \{\emptyset\}.$$

This process is terminated since $z \le s + r$ and since $\cup_{t,p} \mathfrak{M}_{\mathcal{P}}(Y_t, F_p) = \mathfrak{M}_{\mathcal{P}}$ we obtain that there is a subset $U(P") \subseteq U(P)$ such that

$$U(P") \cap \mathfrak{M}_{\mathcal{P}} = \{\emptyset\}.$$

This contradicts the fact that $U(P) \cap \mathfrak{M}_{\mathcal{P}}$ is a dense set.
So there are $i, j;\ 1 \le i \le s;\ 1 \le j \le r$ and $U(P')$ for which $U(P') \cap \mathfrak{M}_{\mathcal{P}}(Y_i, F_j)$ is a dense set. Without loss of generality we can assume that $P' \in \mathfrak{M}_{\mathcal{P}}(Y_i, F_j)$. Let the radius of the disc $U(P')$ be ϱ. Our task is to show that there is a radius $\delta \in \mathbb{R}^+$, such that for every point $Q \in \Sigma$, $U_\delta(Q) \cap \mathfrak{M}_{\mathcal{P}}$ is a dense set.
We are going to prove that $\delta = \frac{\varrho}{3}$ is admissible. Write briefly $\mathfrak{M}^0_{\mathcal{P}} := \mathfrak{M}_{\mathcal{P}}(Y_i, F_j)$
We follow an iteration process. Let $X_1 = P'$. For $i = 2, 3, \ldots$ we proceed as follows:
If $d(X_{i-1}, Q) < \frac{\varrho}{2}$ then we are done.
If not, pick a point $X_i \in U(X_{i-1})_\varrho \cap \mathfrak{M}^0_{\mathcal{P}}$ for which the following two conditions are valid:

$$(i)\ d(X_i, X_{i-1}) \ge \frac{9\varrho}{10},$$

and

$$(ii)\ X_i X_{i-1} Q\angle < \frac{\pi}{10}.$$

An easy calculation shows that $d(X_i, Q) < d(X_{i-1}, Q) - \varrho/2$. Then increase i by 1 and repeat the previous steps. This process is terminated since $i \le \frac{d(P', Q)}{\varrho/2}$. Therefore there is an $i \in \mathbb{N}$ for which $Q \in U(X_i)_\varrho$ and $d(X_i, Q) < \varrho/2$. It implies that

$$X_i \in U(Q)_{\varrho/3} \cap \mathfrak{M}^0_{\mathcal{P}} \subseteq X_i \in U(X_i)_\varrho \cap \mathfrak{M}^0_{\mathcal{P}}$$

as required.

Proof of Theorem 3.1:

Let $\mathcal{P}$ be a general tetrahedron and denote the vertices by A_1, A_2, A_3, A_4 its vertices respectively. Denote by $\sigma(A_i)$; $(i = 1, 2, 3, 4)$ the sum of all angles which occur at A_i on the faces.

Firstly note that for almost all $\mathcal{P}$, there exists an $i \in \{1, 2, 3, 4\}$ for which $\sigma(A_i)/\pi \in \mathbb{Q}^*$. Indeed seven data determine uniquely a tetrahedron; let them be

$$(d(A_1, A_2); A_3A_1A_2\angle, A_1A_2A_3\angle, A_3A_4A_1\angle, A_1A_3A_4\angle, A_1A_4A_3\angle, \sigma(A_1)).$$

The set of tetrahedra $\mathcal{P}$ for which $\sigma(A_1)/\pi \in \mathbb{Q}$ has Lebesgue measure 0 in $\mathbb{R}^7$. Hence for almost all tetrahedron we have $\sigma(A_1)/\pi \in \mathbb{Q}^*$.

Denote F_1, F_2, F_3, F_4 the faces $A_1A_2A_3$, $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$ of $\mathcal{P}$ respectively. Assume that $\mathcal{P}$ stands at the initial position on F_1.

For a random rolling of the tetrahedron $\mathcal{P}$ corresponds a random sequence of

$$\widehat{\mathcal{F}} := \{F_{i_1}, F_{i_2}, \dots, F_{i_j}, \dots\}$$

$i_j \in \{1, 2, 3, 4\}$. It is easy to see, that it is a bijection between the sequence of rolling and the sequence of $\widehat{\mathcal{F}}$.

We need to look at the Erdős-Rényi type result as follows.

Lemma 4.1. *Let $k \in \mathbb{N}$. Then with probability 1 in a random sequence of $F_{i_1}, F_{i_2}, \dots, F_{i_j}, \dots$ the longest run of the pattern $F_1F_3F_4$ is bigger than k.*

It is a very special case of [1, Theorem 2].

Take two rollings that correspond to $F_1F_3F_4$. At F_1 the vertices A_1, A_2 and A_3 touch the plane. After $F_1F_3F_4$ denote the new position of A_2 by A'_2. Then $A'_2A_1A_2\angle = \sigma(A_1)$.

Now consider a random sequence of rolling. Let $\varepsilon > 0$ real number be given. Since $\sigma(A_1) \in \mathbb{Q}^*$, by the Dirichlet approximation theory we get that there is $k_0(\varepsilon)$ such that for every $k > k_0$ the set $\{j\sigma(A_1)/2\pi\}_{j=1}^{k} mod(1)$ is ε-dense. By Lemma 4.1 we obtain that there are k many consecutive repetition of the pattern $F_1F_3F_4$ in $\widehat{\mathcal{F}}$. Hence there is a circle with radius $d(A_1, A_2)$ for which $\mathfrak{M}_{\mathcal{P}} \cap \mathcal{S}$ is ε-dense in $\mathcal{S}$.

Since infinitely many times there are k many consecutive repetitions of the pattern $F_1F_3F_4$ in $\widehat{\mathcal{F}}$, we obtain the theorem.

5. Concluding remarks

Considering Theorem 2.1 it is reasonable to ask the following

Problem 1: Let $\mathcal{P}$ be a polyhedron, $\mathcal{S}$ is a smooth curve, say a boundary of a circle. Assume that $\mathfrak{M}_{\mathcal{P}}$ is $\mathcal{S}$-dense. Is it true that $\mathfrak{M}_{\mathcal{P}}$ is locally-dense and a fortiori is a D-polyhedron?

Furthermore we mention two questions on random rolling of a polyhedron.

Problem 2: Assume that $\mathcal{P}$ is a $D-$polyhedron. Is it true that with probability of 1, $RAN(\mathfrak{M})$ is an everywhere dense subset of Σ ?

Problem 3: Is it true that for almost all tetrahedron $\mathcal{P}$ with probability 1 $RAN(\mathfrak{M}_{\mathcal{P}})$ is a dense set?

I conjecture that the answers for these questions will be yes.

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