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EINSTEIN MANIFOLDS AS AFFINE HYPERSURFACES

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Abstract. In this article we study Einstein manifolds which can be realized either as centroaffine hypersurfaces or as graph hypersurfaces in some affine space. We establish general inequalities for such affine hypersurfaces. We also study Einstein centroaffine and graph hypersurfaces which satisfy the equality case of the inequalities. As immediate applications we give some non-existence results. Furthermore, we provide some examples to show that our inequalities are sharp.

1. Introduction.

A hypersurface $\phi\colon M \to \mathbf{R}^{n+1}$ is called *centroaffine* if the position vector is always transversal to $\phi_*(TM)$ in $\mathbf{R}^{n+1}$. In this case, for any vector fields X, Y tangent to M, one can decompose $D_X\phi_*(Y)$ into its tangential and transverse components. This is written as

$$D_X f_*(Y) = \phi_*(\nabla_X Y) + h(X,Y)\xi, \tag{1.1}$$

where h is a symmetric tensor of type $(0,2)$ and the affine normal ξ is given by $-\phi$. In this article, we assume that h is non-degenerate, so it defines a Riemannian metric on M, called the affine metric.

A Riemannian manifold (M,g) is called Einstein if its Ricci tensor Ric is proportional to its metric tensor so that $Ric = cg$ for some constant c. An Einstein manifold M is said to be realized as an affine hypersurface if there exists a codimension one affine immersion from M into some affine space such that the induced affine metric is exactly the Einstein metric on M (cf. [7, page 333]). We simply call such a hypersurface an Einstein affine hypersurface.

An affine hypersurface $\phi : M \to \mathbf{R}^{\mathbf{n+1}}$ is called a *graph hypersurface* if the affine normal ξ is a constant transversal vector field. A result of [13] states that locally M is affine equivalent to the graph immersion of a certain function F. For a graph hypersurface we also have the decomposition (1.1). Again we assume that h is non-degenerate, so it defines a Riemannian metric.

For an immersed hypersurface $\phi : M \to \mathbf{R}^{n+1}$ in an affine $(n+1)$-space $\mathbf{R}^{n+1}$, a transverse vector field ξ is said to be *equiaffine* if $D\xi$ has its image in $\phi_* T_p M$,

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where D is the canonical flat connection on $\mathbf{R}^{n+1}$. With an equiaffine transversal vector field ξ, one has an equiaffine structure (∇, θ) with $\nabla\theta = 0$, where θ is the induced volume element (cf. [14, pp. 31-32]). An equiaffine transversal vector field is sometimes called a *relative normalization.* Obviously, the transversal vector field ξ of a centroaffine or a graph hypersurface is equiaffine.

Let $\hat{\nabla}$ denote the Levi-Civita connection of h and let K be the difference tensor $\nabla - \hat{\nabla}$ on M. Then, for each $X \in T_pM$,

$$K_X : Y \mapsto K(X, Y)$$

is an endomorphism of T_pM. By taking the trace of K, one obtains a so-called *Tchebychev form*

$$T(X) := \frac{1}{n}\,\mathrm{trace}\,\{\, Y \to K(X, Y)\,\}. \tag{1.2}$$

The *Tchebychev vector field* $T^{\#}$ can then be defined by $h(T^{\#}, X) = T(X)$.

The Tchebychev form and Tchebychev vector field play an important role in centroaffine differential geometry, see for instance [3, 4, 15, 17]. In particular, some general optimal inequalities involving Tchebychev vector field for graph and centroaffine hypersurfaces have been discovered in [3, 4, 9]. A survey of global properties of affine hyperspheres can be found in [12]. An interesting local result on affine hyperspheres is given in [18].

For a Riemannian n-manifold (M, g) with Levi-Civita connection ∇, É. Cartan and A. P. Norden studied non-degenerate affine immersions $f : (M, \nabla) \to \mathbf{R}^{n+1}$ with a transversal field ξ and *with ∇ as its induced affine connection.* The well-known Cartan-Norden theorem states that if f is a such affine immersion, then either ∇ is flat and f is a graph immersion or ∇ is not flat and $\mathbf{R}^{n+1}$ admits a parallel Riemannian metric relative to which f is an isometric immersion and ξ is perpendicular to $f(M)$ (cf. for instance, [14, p. 159]).

In this article, we study Riemannian manifolds in affine geometry from a view point different from the view point of Cartan and Norden. More precisely, we study Riemannian manifolds which can be realized either as centroaffine hypersurfaces or as graph hypersurfaces such that *their induced affine metrics h are exactly the original Riemannian metrics.* In section 3, we establish four general optimal inequalities for Einstein manifolds which are realized as centroaffine hypersurfaces or as graph hypersurfaces. We also establish the necessary and sufficient conditions for an Einstein affine hypersurface to satisfy the equality case of one of the inequalities. In section 4, we give some applications of our inequalities. In the last section, we provide some examples of Einstein affine hypersurfaces to show that our inequalities are sharp.

2. Preliminaries.

We recall some basic facts about affine hypersurfaces (cf. [7, 12, 14, 17]). The main idea of this article is based on [2, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16].

Assume that $\phi : M \to \mathbf{R}^{n+1}$ is a centroaffine hypersurface with affine normal given by $\xi = -\phi$. Then the centroaffine structure equations are given by

$$D_X\phi_*(Y) = \phi_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.1}$$

$$D_X\xi = -\phi_*(X). \tag{2.2}$$

Let us assume that h is positive definite so that h defines a Riemannian metric on M. The corresponding equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{2.4}$$

By definition the *cubic form* C is the totally symmetric (0,3)-tensor field

$$C(X,Y,Z) = (\nabla_X h)(Y,Z).$$

The ∇-Ricci tensor is defined by

$$Ric(Y,Z) = \text{trace}\,\{Z \mapsto R(Z,X)Y\}.$$

It follows from (2.3) that ∇ is projective flat:

$$Ric(X,Y) = (n-1)h(X,Y). \tag{2.5}$$

Let $\hat\nabla$, $\hat K$, $\hat R$, $\widehat{Ric}$ and $\hat\kappa$ be the Levi-Civita connection, the sectional curvature, the curvature tensor and the normalized scalar curvature of h, respectively. The difference tensor K is a symmetric $(1,2)$-tensor field which is defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \hat\nabla_X Y. \tag{2.6}$$

The difference tensor K and the cubic form C are related by

$$C(X,Y,Z) = -2h(K_X Y, Z). \tag{2.7}$$

The Tchebychev form T, the Tchebychev vector field $T^\#$ and the *Pick invariant* J are given by

$$T(X) = \left(\frac{1}{n}\right)\text{trace}\, K_X, \tag{2.8}$$

$$h(T^\#, X) = T(X), \tag{2.9}$$

$$h(C,C) = 4h(K,K) = 4n(n-1)J. \tag{2.10}$$

For centroaffine hypersurfaces we have

$$h(K_X Y, Z) = h(Y, K_X Z), \tag{2.11}$$

$$\hat R(X,Y)Z = K_Y K_X Z - K_X K_Y Z + h(Y,Z)X - h(X,Z)Y, \tag{2.12}$$

$$(\hat\nabla K)(X,Y,Z) = (\hat\nabla K)(Y,Z,X) = (\hat\nabla K)(Z,X,Y). \tag{2.13}$$

Taking the trace once we obtain from (2.12) that

$$\widehat{Ric}(X,Y) = \alpha(X,Y) - nT(K_X Y) + (n-1)h(X,Y), \tag{2.14}$$

where $\alpha(X,Y) = \text{trace}\,(K_X K_Y)$. By taking trace twice we find from (2.12) that

$$\hat\kappa = J + 1 - \frac{n}{n-1}h(T^\#, T^\#). \tag{2.15}$$

When $T = 0$ and if we consider the centroaffine hypersurface M as a hypersurface of the equiaffine space, then M is a so-called *proper affine hypersphere* centered at the origin. If the difference tensor K vanishes identically, then M is a hyperquadric centered at the origin. Because M is assumed to be positive definite, M is an hyperellipsoid.

If M is a *graph hypersurface* in $\mathbf{R}^{n+1}$, we also have the decomposition (2.1). Again when h is positive definite, so it gives a Riemannian metric, called the *Calabi metric*. When $T = 0$, if we consider the graph hypersurface M as a hypersurface of the equiaffine space in $\mathbf{R}^{n+1}$, then M is a so-called *improper affine hypersphere*, with Blaschke normal in the direction of ξ and the affine metric homothetic to the Blaschke metric.

For graph hypersurfaces, equations (2.4), (2.6), (2.7), (2.8) and (2.10) hold as well. On the other hand, equations (2.2), (2.3) and (2.9) shall be replaced respectively by

$$D_X\xi = R(X,Y)Z = 0, \tag{2.16}$$

$$\hat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z. \tag{2.17}$$

If we choose $\xi = (0,\dots,0,1)$, then we can assume locally that the graph hypersurface M is given by

$$x_{n+1} = F(x_1,\dots,x_n).$$

It turns out that the $(x_1,\dots,x_n)$ are ∇-flat coordinates on M and that the Calabi metric is given by

$$h\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) = \frac{\partial^2 F}{\partial x_i \partial x_j}. \tag{2.18}$$

Moreover, it is known that M is an improper affine hypersphere if and only if the Hessian determinant $\det\big(\frac{\partial^2 F}{\partial x_i \partial x_j}\big)$ is constant.

Let M_1 and M_2 be two improper affine hyperspheres defined by equations

$$x_{p+1} = F_1(x_1,\dots,x_p), \quad y_{q+1} = F_2(y_1,\dots,y_q).$$

Then one can define a new improper affine hypersphere M in $\mathbf{R}^{p+q+1}$ by

$$z = F_1(x_1,\dots,x_p) + F_2(y_1,\dots,y_q), \tag{2.19}$$

where $(x_1,\dots,x_p,y_1,\dots,y_q,z)$ are the coordinates on $\mathbf{R}^{p+q+1}$. The affine normal of M is given by $(0,\dots,0,1)$. Obviously, the Calabi metric is the Riemannian product metric. This composition is known as the Calabi composition of M_1 and M_2 (see [10]).

Recall that a *partition* of a natural number n is a way of writing n as a sum of positive integers.

We recall the following algebraic lemma from [1] for later use.

Lemma 2.1. *Let $a_1,\dots,a_n$ be real numbers and let k be an integer in $[2,n-1]$. Then, for any partition $(n_1,\dots,n_k)$ of n, we have*

$$\begin{aligned} &\sum_{1\le i_1<j_1\le n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\le i_2<j_2\le n_1+n_2} a_{i_2}a_{j_2} + \cdots \\ &\qquad + \sum_{n_1\cdots+n_{k-1}+1\le i_1<j_1\le n} a_{i_k}a_{j_k} \\ &\ge \frac{1}{2k}\big\{(a_1+\cdots+a_n)^2 - k(a_1^2+\cdots+a_n^2)\big\} \end{aligned} \tag{2.20}$$

with the equality holding if and only if we have

$$a_1+\cdots+a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_n. \tag{2.21}$$

3. Optimal inequalities for Einstein affine hypersurfaces.

Assume $(M,\hat{g})$ is a Riemannian n-manifold. Let $e_1,\dots,e_n$ be an orthonormal basis of M. Then the sectional curvature of the 2-plane spanned by the orthonormal vectors e_i, e_j is denoted by $\hat{K}(e_i\wedge e_j)=\hat{g}(\hat{R}(e_i,e_j)e_j,e_i)$, where $\hat{R}$ is the curvature tensor of $(M,\hat{g})$. The Ricci tensor $\widehat{Ric}$ of $(M,\hat{g})$ is given by

$$\widehat{Ric}(X,Y)=\sum_{j=1}^{n}\hat{g}(\hat{R}(e_j,X)Y,e_j).$$

Let p be a point in M, q a natural number $\leq n/2$, and $\pi_1,\dots,\pi_q$ mutually orthogonal 2-plane sections in T_pM. We define two invariants $\hat{\delta}_q^{Ric}$ and $\check{\delta}_q^{Ric}$ on (M,g) respectively by

$$\hat{\delta}_q^{Ric}(p)=\sup_{u\in T_p^1M}\widehat{Ric}(u,u)-\frac{2}{n}\inf_{\pi_1\perp\cdots\perp\pi_q}\big\{\hat{K}(\pi_1)+\cdots+\hat{K}(\pi_q)\big\},\tag{3.1}$$

$$\check{\delta}_q^{Ric}(p)=\inf_{u\in T_p^1M}\widehat{Ric}(u,u)-\frac{2}{n}\sup_{\pi_1\perp\cdots\perp\pi_q}\big\{\hat{K}(\pi_1)+\cdots+\hat{K}(\pi_q)\big\},\tag{3.2}$$

where u runs over all unit vectors in T_pM and $\pi_1,\dots,\pi_q$ run over all mutually orthogonal 2-plane sections in T_pM. In particular, for an Einstein n-manifold M, we have

$$\hat{\delta}_q^{Ric}(p)=(n-1)\hat{\kappa}-\frac{2}{n}\inf_{\pi_1\perp\cdots\perp\pi_q}\big\{\hat{K}(\pi_1)+\cdots+\hat{K}(\pi_q)\big\},\tag{3.3}$$

$$\check{\delta}_q^{Ric}(p)=(n-1)\hat{\kappa}-\frac{2}{n}\sup_{\pi_1\perp\cdots\perp\pi_q}\big\{\hat{K}(\pi_1)+\cdots+\hat{K}(\pi_q)\big\},\tag{3.4}$$

where $\hat{\kappa}$ is the normalized scalar curvature of $(M,\hat{g})$.

For Einstein centroaffine hypersurfaces we have the following general optimal inequalities.

Theorem 3.1. *If an Einstein n-manifold M can be realized as a centroaffine hypersurface in an affine $(n+1)$-space $\mathbf{R}^{n+1}$, then we have the following inequality:*

$$\check{\delta}_q^{Ric}\geq n-1-\frac{2q}{n}-\frac{n(n-q-1)}{n-q}h(T^{\#},T^{\#})\tag{3.5}$$

for any natural number $q<n/2$.

The equality case of (3.5) holds identically if and only if M is realized as a hyper-quadric centered at the origin. Hence M is realized as a hyperellipsoid.

Proof. Let q be a natural number satisfying $q<n/2$. Assume that M is an Einstein n-manifold which can be realized as a centroaffine hypersurface in an affine $(n+1)$-space $\mathbf{R}^{n+1}$. Consider q mutually orthogonal plane sections $\pi_1,\dots,\pi_q$ of M at p. We choose an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM such that π_j is spanned by e_{2j-1} and e_{2j} for $j=1,\dots,q$.

Let us put

$$\hat{\psi}=n(n-1)(1-\hat{\kappa})+\frac{n^2(q+1-n)}{n-q}h(T^{\#},T^{\#}).\tag{3.6}$$

Then we obtain from (2.10), (2.13) and (3.6) that

$$n^2h(T^{\#},T^{\#})=(n-q)\big(h(K,K)+\hat{\psi}\big).\tag{3.7}$$

Let us choose an orthonormal basis $e_1^*,\dots,e_n^*$ of T_pM so that e_1^* is in the direction of the Tchebychev vector $T^\#$ at p (when $T^\#=0$ at a point $p\in M$, we may choose $e_1^*,\dots,e_n^*$ to be any arbitrary orthonormal basis of T_pM). Then we can express (3.7) as

$$\frac{n^2}{n-q}h(T^\#,T^\#)-\sum_{s=1}^n\big(K_{ss}^{1^*}\big)^2=\sum_{1\le s\ne t\le 2k}\big(K_{st}^{1^*}\big)^2+\sum_{r=2}^n\sum_{s,t=1}^n(K_{st}^{r^*})^2+\hat\psi. \tag{3.8}$$

Thus, by applying Lemma 2.1 to (3.8), we obtain

$$\sum_{i=1}^q K_{2i-1\,2i-1}^{1^*}K_{2i\,2i}^{1^*}\ge\frac12\sum_{j=2}^n\sum_{s=1}^n(K_{ss}^{j^*})^2+\sum_{j=1}^n\sum_{1\le s<t\le n}\big(K_{st}^{j^*}\big)^2+\frac{\hat\psi}{2} \tag{3.9}$$

with equality holding if and only if we have

$$K_{11}^{1^*}+K_{22}^{1^*}=\cdots=K_{2q-1\,2q-1}^{1^*}+K_{2q\,2q}^{1^*}=K_{2q+1\,2q+1}^{1^*}=\cdots=K_{n\,n}^{1^*}. \tag{3.10}$$

On the other hand, from (2.12) we have

$$\hat K(e_s\wedge e_t)=1-h(K(e_s,e_s),K(e_t,e_t))+h(K(e_s,e_t),K(e_s,e_t)). \tag{3.11}$$

Hence by applying (3.9) and (3.11) we find

$$\begin{aligned}\sum_{i=1}^q\hat K(e_{2i-1}\wedge e_{2i})&=q-\sum_{i=1}^q\sum_{j=1}^n\big\{K_{2i-1\,2i-1}^{j^*}K_{2i\,2i}^{j^*}-(K_{2i-1\,2i}^{j^*})^2\big\}\\
&\le q-\frac{\hat\psi}{2}-\frac12\sum_{j=2}^n\sum_{s=1}^n(K_{ss}^{j^*})^2-\sum_{j=1}^n\sum_{1\le s<t\le n}\big(K_{st}^{j^*}\big)^2\\
&\quad-\sum_{i=1}^q\sum_{j=2}^n K_{2i-1\,2i-1}^{j^*}K_{2i\,2i}^{j^*}+\sum_{i=1}^q\sum_{j=1}^n(K_{2i-1\,2i}^{j^*})^2\\
&\le q-\frac{\hat\psi}{2}+\sum_{j=1}^n\sum_{s\ne1,2}\big\{(K_{1s}^{j^*})^2+(K_{2s}^{j^*})^2\big\}\\
&\qquad+\cdots\\
&\quad+\sum_{j=1}^n\sum_{s\ne 2q-1,2q}\big\{(K_{2q-1s}^{j^*})^2+(K_{2qs}^{j^*})^2\big\}\\
&\quad+\frac12\sum_{j=2}^n\big\{(K_{11}^{j^*}+K_{22}^{j^*})^2+\cdots+(K_{2q-1\,2q-1}^{j^*}+K_{2q\,2q}^{j^*})^2\big\}\\
&\le q-\frac{\hat\psi}{2}.\end{aligned} \tag{3.12}$$

Therefore, after applying (3.6) and (3.12), we obtain

$$\begin{aligned}\frac n2(n-1)\hat\kappa-\sum_{i=1}^q\hat K(e_{2i-1}\wedge e_{2i})&\ge\frac n2(n-1)\hat\kappa-q+\frac{\hat\psi}{2}\\
&=\frac{n(n-1)-2q}{2}+\frac{n^2(q+1-n)}{2(n-q)}h(T^\#,T^\#),\end{aligned} \tag{3.13}$$

which implies inequality (3.5).

If the equality sign of (3.5) holds identically, then we have (3.10), Moreover, in this case all of the inequalities in (3.12) become equalities. Hence the difference tensor K satisfies

$$K(e_\alpha, e_\beta) = 0 \tag{3.14}$$

whenever $\{\alpha, \beta\} \neq \{2i-1, 2i\}$ for $i = 1, \ldots, q$ and

$$\begin{aligned} K(e_1, e_1) + K(e_2, e_2) &= \cdots = k(e_{2q-1}, e_{2q-1}) + K(e_{2q}, e_{2q}) \\ &= K(e_{2q+1}, e_{2q+1}) = \cdots = K(e_n, e_n) \\ &= 2\Phi \end{aligned} \tag{3.15}$$

with $\Phi = nT^{\#}/(2(n-q))$.

From (3.11), (3.14) and (3.15) we find

$$\begin{aligned} \widehat{Ric}(e_{2j-1}, e_{2j-1}) = n - 2 + \hat{K}(\pi_j) \\ - 2(n-q-1)h(K(e_{2j-1}, e_{2j-1}), \Phi), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \widehat{Ric}(e_{2j}, e_{2j}) = n - 2 + \hat{K}(\pi_j) \\ - 2(n-q-1)h(K(e_{2j}, e_{2j}), \Phi), \end{aligned} \tag{3.17}$$

$$\widehat{Ric}(e_n, e_n) = n - 1 - 4(n-q-1)h(\Phi, \Phi), \tag{3.18}$$

$$\begin{aligned} 2h(K(e_{2j-1}, e_{2j-1}), K(e_{2j}, e_{2j})) = 4h(\Phi, \Phi) \\ - h(K(e_{2j-1}, e_{2j-1}), K(e_{2j-1}, e_{2j-1})) \\ - h(K(e_{2j}, e_{2j}), K(e_{2j}, e_{2j})), \end{aligned} \tag{3.19}$$

where $\hat{K}(\pi_j)$ denotes the sectional curvature of the plane section spanned by the orthonormal vectorse_{2j-1} and e_{2j}.

Since M is Einsteinian, (3.16) and (3.17) imply that

$$h(K(e_{2j-1}, e_{2j-1}), \Phi) = h(K(e_{2j}, e_{2j}), \Phi). \tag{3.20}$$

From (3.15) we also have

$$h(K(e_{2j-1}, e_{2j-1}), \Phi) + h(K(e_{2j}, e_{2j}), \Phi) = 2h(\Phi, \Phi).$$

Hence we obtain

$$h(K(e_{2j-1}, e_{2j-1}), \Phi) = h(K(e_{2j}, e_{2j}), \Phi) = h(\Phi, \Phi). \tag{3.21}$$

Therefore, by applying (3.17), (3.18) and (3.20), we find

$$\hat{K}(\pi_j) = 1 - 2(n-q-1)h(\Phi, \Phi). \tag{3.22}$$

On the other hand, (3.11) and (3.19) yield

$$\begin{aligned} \hat{K}(\pi_j) = 1 - 2h(\Phi, \Phi) + \frac{1}{2}h(K(e_{2j-1}, e_{2j-1}), K(e_{2j-1}, e_{2j-1})) \\ + \frac{1}{2}h(K(e_{2j}, e_{2j}), K(e_{2j}, e_{2j})) \\ + h(K(e_{2j-1}, e_{2j}), K(e_{2j-1}, e_{2j})). \end{aligned}$$

Since $q < n/2$, we may derive from (3.14), (3.15), (3.22) and (3.23) that $K = 0$. Consequently, M can be realized as a hyper-quadric centered at the origin.

The converse is easy to verify. □

For even-dimensional Einstein manifolds we also have the following.

Theorem 3.2. *Let k be a natural number ≥ 2. If an Einstein $2k$-manifold M can be realized as a centroaffine hypersurface in an affine $(2k+1)$-space $\mathbf{R}^{2k+1}$, then we have the following inequality:*

$$\check{\delta}_k^{Ric} \geq 2(k-1)\{1 - h(T^{\#}, T^{\#})\}. \tag{3.23}$$

The equality case of (3.23) *holds identically if and only if, with respect a suitable h-orthonormal basis $e_1, \ldots, e_{2k}$, the difference tensor K satisfies*

$$K(e_\alpha, e_\beta) = 0, \tag{3.24}$$

$$K(e_1, e_1) + K(e_2, e_2) = \cdots = K(e_{2k-1}, e_{2k-1}) + K(e_{2k}, e_{2k}), \tag{3.25}$$

$$h(K(e_i, e_i), T^{\#}) = h(T^{\#}, T^{\#}), \tag{3.26}$$

where $\{\alpha, \beta\} \neq \{2i-1, 2i\}$ and $i = 1, \ldots, k$.

Proof. Let k be a natural number ≥ 2. Assume that M is an Einstein $2k$-manifold which is realized as a centroaffine hypersurface in an affine $(2k+1)$-space $\mathbf{R}^{2k+1}$. Let $\pi_1, \ldots, \pi_k$ be k mutually orthogonal 2-plane sections at $p \in M$. We choose an orthonormal basis $e_1, \ldots, e_{2k}$ of T_pM with

$$\pi_1 = \text{Span}\{e_1, e_2\}, \ldots, \pi_k = \text{Span}\{e_{2k-1}, e_{2k}\}.$$

Let us put

$$\hat{\eta} = 2k(2k-1)(1-\hat{\kappa}) - 4k(k-1)h(T^{\#}, T^{\#}). \tag{3.27}$$

Then (2.10), (2.13) and (3.27) imply that

$$4k \cdot h(T^{\#}, T^{\#}) = h(K, K) + \hat{\eta}. \tag{3.28}$$

Let us choose an orthonormal basis $e_1^*, \ldots, e_n^*$ of T_pM so that e_1^* is in the direction of the Tchebychev vector $T^{\#}$ at p. If we put $K_{\alpha\beta}^{\gamma^*} = h(K(e_\alpha, e_\beta), e_\gamma^*)$, then (3.28) can be expressed as

$$4kh(T^{\#}, T^{\#}) - \sum_{s=1}^{2k} \left(K_{ss}^{1^*}\right)^2 = \sum_{1 \leq s \neq t \leq 2k} \left(K_{st}^{1^*}\right)^2 + \sum_{r=2}^{2k} \sum_{s,t=1}^{2k} (K_{st}^{r^*})^2 + \hat{\eta}. \tag{3.29}$$

If we apply Lemma 2.1 to the left-hand-side of (3.29), we find

$$2 \sum_{i=1}^{k} K_{2i-1\,2i-1}^{1^*} K_{2i\,2i}^{1^*} \geq \sum_{j=2}^{2k} \sum_{s=1}^{2k} (K_{ss}^{j^*})^2 + \sum_{j=1}^{2k} \sum_{1 \leq s \neq t \leq 2k} \left(K_{st}^{j^*}\right)^2 + \hat{\eta} \tag{3.30}$$

with equality holding if and only if we have

$$K_{11}^{1^*} + K_{22}^{1^*} = \cdots = K_{2k-1\,2k-1}^{1^*} + K_{2k\,2k}^{1^*}. \tag{3.31}$$

By combining (3.11) and (3.30), we find

$$\begin{aligned} \sum_{i=1}^{k} \hat{K}(e_{2i-1} \wedge e_{2i}) \leq k - \frac{\hat{\eta}}{2} - \frac{1}{2} \sum_{j=2}^{2k} \sum_{s=1}^{2k} (K_{ss}^{j^*})^2 - \sum_{j=1}^{2k} \sum_{1 \leq s < t \leq 2k} \left(K_{st}^{j^*}\right)^2 \\ - \sum_{i=1}^{k} \sum_{j=2}^{2k} K_{2i-1\,2i-1}^{j^*} K_{2i\,2i}^{j^*} + \sum_{i=1}^{k} \sum_{j=1}^{2k} (K_{2i-1\,2i}^{j^*})^2 \end{aligned} \tag{3.32}$$

$$\begin{aligned}&\le k-\frac{\hat\eta}{2}+\sum_{j=1}^{2k}\sum_{s\ne 1,2}\left\{(K^{j^*}_{1s})^2+(K^{j^*}_{2s})^2\right\}\\ &\qquad +\cdots\\ &\quad +\sum_{j=1}^{2k}\sum_{s\ne 2k-1,2k}\left\{(K^{j^*}_{2k-1s})^2+(K^{j^*}_{2ks})^2\right\}\\ &\quad +\frac{1}{2}\sum_{j=2}^{2k}\left\{(K^{j^*}_{11}+K^{j^*}_{22})^2+\cdots+(K^{j^*}_{2k-1\,2k-1}+K^{j^*}_{2k\,2k})^2\right\}\\ &\le k-\frac{\hat\eta}{2}.\end{aligned}$$

Therefore, after applying (3.27) and (3.32), we derive that

$$(2k-1)\hat\kappa-\frac{1}{k}\sum_{i=1}^{k}\hat K(e_{2i-1}\wedge e_{2i})\ge 2(k-1)-2(k-1)h(T^\#,T^\#). \tag{3.33}$$

Consequently, we obtain inequality (3.23) from (3.4) and (3.33).

If the equality sign of (3.23) holds identically, then we have (3.31), Moreover, all of the inequalities in (3.32) become equalities. Hence the difference tensor K satisfies (3.24), i.e.,

$$K(e_\alpha,e_\beta)=0 \tag{3.34}$$

for $\{\alpha,\beta\}\ne\{2i-1,2i\}$ with $i=1,\dots,k$. Also we find

$$K(e_1,e_1)+K(e_2,e_2)=\cdots=k(e_{2k-1},e_{2k-1})+K(e_{2k},e_{2k})=2T^\#, \tag{3.35}$$

which gives (3.25).

Furthermore, we have

$$\widehat{Ric}(e_{2j-1},e_{2j-1})=2(k-1)+\hat K(\pi_j) -2(k-1)h(K(e_{2j-1},e_{2j-1}),T^\#), \tag{3.36}$$

$$\widehat{Ric}(e_{2j},e_{2j})=2(k-1)+\hat K(\pi_j) -2(k-1)h(K(e_{2j},e_{2j}),T^\#), \tag{3.37}$$

where $\hat K(\pi_j)$ denotes the sectional curvature of the plane section spanned by the orthonormal vectors e_{2j-1} and e_{2j}.

Since M is Einsteinian, (3.36) and (3.37) imply

$$h(K(e_{2j-1},e_{2j-1}),T^\#)=h(K(e_{2j},e_{2j}),T^\#). \tag{3.38}$$

Now, it follows from (3.35) that

$$h(K(e_{2j-1},e_{2j-1}),T^\#)+h(K(e_{2j},e_{2j}),T^\#)=2h(T^\#,T^\#).$$

Thus after combining (3.38) and (3.39) we find

$$h(K(e_{2j-1},e_{2j-1}),T^\#)=h(K(e_{2j},e_{2j}),T^\#)=h(T^\#,T^\#) \tag{3.39}$$

for $j=1,\dots,k$. Consequently, we also obtain (3.26).

The converse can be verified by direct computation. □

For Einsteinian graph hypersurfaces we have the following.

Theorem 3.3. *If an Einstein n-manifold M can be realized as a graph hypersurface in an affine $(n+1)$-space, then we have*

$$\check{\delta}_q^{Ric} \geq \frac{n(q+1-n)}{n-q} h(T^\#, T^\#) \tag{3.40}$$

for any natural number $q < n/2$.

The equality case of (3.40) holds identically if and only if M is realized as a paraboloid centered at the origin.

Theorem 3.4. *If an Einstein 2k-manifold M can be realized as a centroaffine hypersurface in some affine $(2k+1)$-space, then we have*

$$\check{\delta}_k^{Ric} \geq 2(1-k)h(T^\#, T^\#). \tag{3.41}$$

If the equality sign of (3.41) holds identically, then M is an improper affine hypersphere. Moreover, with respect a suitable h-orthonormal basis $e_1, \ldots, e_{2k}$, the difference tensor K satisfies

$$K(e_\alpha, e_\beta) = 0, \tag{3.42}$$

$$K(e_1, e_1) + K(e_2, e_2) = \cdots = k(e_{2k-1}, e_{2k-1}) + K(e_{2k}, e_{2k}), \tag{3.43}$$

$$h(K(e_j, e_j), T^\#) = h(T^\#, T^\#), \;\; j = 1, \ldots, k, \tag{3.44}$$

whenever $\{\alpha, \beta\} \neq \{2j-1, 2j\}$.

Theorem 3.3 and Theorem 3.4 can be proved in a similar way as that of Theorem 3.1 and Theorem 3.2.

4. Some immediate applications.

The following four corollaries are some immediate consequences of our results obtained in Section 3.

Corollary 4.1. *Let M be an Einstein n-manifold. If we have*

$$\check{\delta}_q^{Ric}(p) < n - 1 - \frac{2q}{n} \tag{4.1}$$

for some natural number $q \leq n/2$ at some point $p \in M$, then M cannot be realized as an elliptic proper affine hypersphere in an affine space.

Corollary 4.2. *Let M be an Einstein n-manifold. If we have*

$$\check{\delta}_q^{Ric}(p) < 0 \tag{4.2}$$

for some natural number $q \leq n/2$ at some point $p \in M$, then M cannot be realized as an improper affine hypersphere in an affine space.

Corollary 4.3. *Let M be a Riemannian n-manifold realized as a graph hypersurface in an affine $(n+1)$-space which satisfies*

$$\check{\delta}_q^{Ric}(p) < \frac{n(q+1-n)}{n-q} h(T^\#, T^\#)(p) \tag{4.3}$$

for some natural number $q \leq n/2$ at some point $p \in M$, the M is not an Einstein manifold.

Corollary 4.4. *Let M be a Riemannian n-manifold realized as a centroaffine hypersurface in an affine $(n+1)$-space which satisfies*

$$\check{\delta}_q^{Ric}(p) < n - 1 - \frac{2q}{n} - \frac{n(n-q-1)}{n-q} h(T^{\#}, T^{\#})(p) \tag{4.4}$$

for some natural number $q \leq n/2$ at some point $p \in M$, the M is not an Einstein manifold.

5. Example and remarks.

In this section we give some remarks and we also provide some simple examples of Einstein manifolds which can be realized either as graph hypersurfaces or as centroaffine hypersurfaces.

Example 5.1. The hyperellipsoid in $\mathbf{R}^{n+1}$ provides an example of an Einstein n-manifold satisfying

$$\check{\delta}_q^{Ric} = n - 1 - \frac{2q}{n}, \quad q \leq \frac{n}{2},$$

which can be realized as a centroaffine hypersurface.

This example shows that the conditions on $\check{\delta}_q^{Ric}$ given in Theorems 3.1 and 3.2 are sharp.

Remark 5.1. The hyperellipsoid also provides the simplest example showing that the condition on $\hat{\delta}_q^{Ric}$ mentioned in Corollary 4.1 is sharp.

The hyperboloids provide the simplest examples which illustrate that the condition on $\hat{\delta}_q^{Ric}$ given in Corollary 4.2 is sharp as well.

Example 5.2. Let (M^4, g) denote the Riemannian product $N_1^2(c) \times N_2^2(c)$ of two surfaces of constant curvature $c > 0$. Then (M^4, g) is an Einstein 4-manifold which satisfies

$$\widehat{Ric}(X, Y) = cg(X, Y), \quad \hat{\delta}_2^{Ric} = c. \tag{5.1}$$

Consider an immersion of M^4 into the affine 5-space $\mathbf{R}^5$ defined by:

$$\phi(x_1, \ldots, x_n) = \left(x_1, x_2, x_3, x_4, \ln \left\{ \left(1 + e^{2x_1^2 + 2x_2}\right)^{\frac{1}{4c}} \left(1 + e^{2x_3^2 + 2x_4}\right)^{\frac{1}{4c}} \right\} \right). \tag{5.2}$$

By applying (2.18), it is direct to show that this immersion is a realization of the Einstein 4-manifold $(N_1^2(c) \times N_2^2(c), g)$ as a Einstein graph hypersurface in $\mathbf{R}^5$ with the Blaschke normal given by $\xi = (0, 0, 0, 0, 1)$.

Remark 5.2. By applying a straight-forward computation we may prove that the Tchebychev vector field $T^{\#}$ of the Einstein graph hypersurface given in Example 5.2 is non-trivial.

Remark 5.3. The author thanks Professor Luc Vrancken for his suggestions to improve the original version of this article.

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