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GENERALIZED $(\tilde{\kappa} \neq -1, \tilde{\mu})$-PARACONTACT METRIC MANIFOLDS WITH $\xi(\tilde{\mu}) = 0$

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Abstract. We give a local classification of generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ which satisfies the condition $\xi(\tilde{\mu}) = 0$. An example of such manifolds is presented.

1. Introduction

The study of paracontact geometry was introduced by Kaneyuki and Williams in [11]. A systematic study of paracontact metric manifolds started with the paper [19], were the Levi-Civita connection, the curvature and a canonical connection (analogue to the Tanaka Webster connection of the contact metric case) of a paracontact metric manifold have been described. However such structures were studied before [17], [4], [5]. Note also [3]. These authors called such structures almost paracoHermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [10], [18], [19]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1],[2],[7],[8],[9]). Paracontact metric manifolds have been studied under several different points of view. The case when the Reeb vector field satisfies a nullity condition was studied in [7]. The study of three-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces were obtained in [15].

A remarkable class of paracontact metric manifolds $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is that of paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-spaces, which satisfy the nullity condition

$$\tilde{R}(X,Y)\xi = \tilde{\kappa}\left(\eta\left(Y\right)X - \eta\left(X\right)Y\right) + \tilde{\mu}(\eta\left(Y\right)\tilde{h}X - \eta\left(X\right)\tilde{h}Y), \tag{1.1}$$

for all X, Y vector fields on M, where $\tilde{\kappa}$ and $\tilde{\mu}$ are constants and $\tilde{h} = \frac{1}{2}\mathcal{L}_{\xi}\tilde{\varphi}$.

This new class of pseudo-Riemannian manifolds was introduced in [6]. In [7], the authors showed that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change the form of (1.1) remains

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unchanged under $\mathcal{D}$-homothetic deformations. There are differences between a contact metric (κ,μ)-space $(M^{2n+1},\varphi,\xi,\eta,g)$ and a paracontact metric $(\tilde{\kappa},\tilde{\mu})$-space $(M^{2n+1},\tilde{\varphi},\xi,\eta,\tilde{g})$. Namely, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa},\tilde{\mu})$-manifold such that $\tilde{\kappa}=-1$ in general is not para-Sasakian. In fact, there are paracontact $(\tilde{\kappa},\tilde{\mu})$-manifolds such that $\tilde{h}^2=0$ (which is equivalent to take $\tilde{\kappa}=-1$) but with $\tilde{h}\neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1,2)$-space $(M^{2n+1},\tilde{\varphi},\xi,\eta,\tilde{g})$ with $\tilde{h}^2=0$ but $\tilde{h}\neq 0$ in [6] and then Cappelletti Montano et al. gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [7]. Later, for 3-dimensional, the first numerical example was given in [15]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ,μ)-spaces the constant κ can not be greater than 1, paracontact metric $(\tilde{\kappa},\tilde{\mu})$-space has no restriction for the constants $\tilde{\kappa}$ and $\tilde{\mu}$.

Koufogiorgos and Tsichlias [14] gave a local classification of a non-Sasakian generalized (κ,μ)-contact metric manifold with $\xi(\mu)=0$. This has been our motivation for studying generalized $(\tilde{\kappa}\neq -1,\tilde{\mu})$-paracontact metric manifolds with $\xi(\tilde{\mu})=0$. We would like to emphasize that, as will be shown in this paper, the class of generalized $(\tilde{\kappa}\neq -1,\tilde{\mu})$-paracontact metric manifolds with $\xi(\tilde{\mu})=0$ is much more different than the class of generalized (κ,μ)-contact metric manifolds with $\xi(\mu)=0$.

By a generalized $(\tilde{\kappa},\tilde{\mu})$-paracontact metric manifold we mean a 3-dimensional paracontact metric manifold satisfying (1.1) where $\tilde{\kappa}$ and $\tilde{\mu}$ are non constant smooth functions. In the special case, where $\tilde{\kappa}$ and $\tilde{\mu}$ are constant, then $(M^{2n+1},\tilde{\varphi},\xi,\eta,\tilde{g})$ is called a $(\tilde{\kappa},\tilde{\mu})$-paracontact metric manifold.

In [15], Kupeli Erken and Murathan proved the existence of a new class of paracontact metric manifolds: the so called $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-paracontact metric manifolds. Such a manifold M is defined through the condition

$$\tilde{R}(X,Y)\xi = \tilde{\kappa}\left(\eta\left(Y\right)X-\eta\left(X\right)Y\right)+\tilde{\mu}(\eta\left(Y\right)\tilde{h}X-\eta\left(X\right)\tilde{h}Y) +\tilde{\nu}(\eta\left(Y\right)\tilde{\varphi}\tilde{h}X-\eta\left(X\right)\tilde{\varphi}\tilde{h}Y), \tag{1.2}$$

where $\tilde{\kappa},\tilde{\mu}$ and $\tilde{\nu}$ are smooth functions on M. Furthermore, it is proved that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is a $(\tilde{\kappa},\tilde{\mu})$-paracontact metric manifold.

The paper is organized in the following way. In Section 2, we will report some basic information about paracontact metric manifolds. Some results about generalized $(\tilde{\kappa}\neq -1,\tilde{\mu})$-paracontact metric manifolds will be given in Section 3. In Section 4, we shall locally classify generalized $(\tilde{\kappa}\neq -1,\tilde{\mu})$-paracontact metric manifold with $\xi(\tilde{\mu})=0$ (i.e. the function $\tilde{\mu}$ is constant along the integral curves of the characteristic vector field ξ). We will prove that we can construct in R^3 two families of such manifolds. All manifolds are assumed to be connected.

2. Preliminaries

The aim of this section is to report some basic facts about paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [11], [19] and references therein for more information about paracontact metric geometry.

An $(2n+1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field $\tilde\varphi$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\tilde\varphi^2 = I - \eta \otimes \xi$,

(ii) the tensor field $\tilde\varphi$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the ± 1-eigendistributions, $\mathcal{D}^{\pm} := \mathcal{D}_{\tilde\varphi}(\pm 1)$ of $\tilde\varphi$ have equal dimension n.

From the definition it follows that $\tilde\varphi\xi = 0$, $\eta \circ \tilde\varphi = 0$ and the endomorphism $\tilde\varphi$ has rank $2n$. When the tensor field $N_{\tilde\varphi} := [\tilde\varphi, \tilde\varphi] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde g$ such that

$$\tilde g(\tilde\varphi X, \tilde\varphi Y) = -\tilde g(X,Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde\varphi, \xi, \eta, \tilde g)$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, \xi\}$ such that $\tilde g(X_i, X_j) = \delta_{ij}$, $\tilde g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde\varphi X_i$, for any $i, j \in \{1, \cdots, n\}$. Such basis is called a $\tilde\varphi$-basis.

If in addition $d\eta(X,Y) = \tilde g(X, \tilde\varphi Y)$ for all vector fields X, Y on M, $(M, \tilde\varphi, \xi, \eta, \tilde g)$ is said to be a *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde h := \frac{1}{2}\mathcal{L}_\xi \tilde\varphi$. It is known [19] that $\tilde h$ anti-commutes with $\tilde\varphi$ and satisfies $\tilde h\xi = 0$, tr$\tilde h$ =tr$\tilde h\tilde\varphi = 0$ and

$$\tilde\nabla\xi = -\tilde\varphi + \tilde\varphi\tilde h, \tag{2.2}$$

where $\tilde\nabla$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, \tilde g)$.

Moreover $\tilde h \equiv 0$ if and only if ξ is a Killing vector field and in this case $(M, \tilde\varphi, \xi, \eta, \tilde g)$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$\tilde R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.3}$$

so that it is a $(\tilde\kappa, \tilde\mu)$-space with $\tilde\kappa = -1$. To note that, differently from the contact metric case, condition (2.3) is necessary but not sufficient for a paracontact metric manifold to be para-Sasakian. This fact was already pointed out in [7].

As a natural generalization of the above para-Sasakian condition one can consider contact metric manifolds satisfying (1.1) for some real numbers κ and μ. Paracontact metric manifolds satisfying (1.1) are called $(\tilde\kappa, \tilde\mu)$-paracontact metric manifold. $(\tilde\kappa, \tilde\mu)$-paracontact metric manifolds were introduced and deeply studied in [6] and [7].

By a generalized $(\tilde\kappa, \tilde\mu)$-paracontact metric manifold we mean a 3-dimensional paracontact metric manifold satisfying (1.1) where $\tilde\kappa$ and $\tilde\mu$ are non constant smooth functions. In the special case, where $\tilde\kappa$ and $\tilde\mu$ are constant, then $(M^{2n+1}, \tilde\varphi, \xi, \eta, \tilde g)$ is called a $(\tilde\kappa, \tilde\mu)$-paracontact metric manifold.

Generalized $(\tilde\kappa, \tilde\mu)$-paracontact metric manifolds were studied by Kupeli Erken and Murathan in [15]. A recent generalization of the $(\tilde\kappa, \tilde\mu)$-paracontact metric manifold is given by the following definition.

Definition 2.1 ([15]). A $2n+1$-dimensional paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold is a paracontact metric manifold for which the curvature tensor field satisfies

(2.4)
$$\tilde{R}(X,Y)\xi = \tilde{\kappa}(\eta(Y)X-\eta(X)Y)+\tilde{\mu}(\eta(Y)\tilde{h}X-\eta(X)\tilde{h}Y)+\tilde{\nu}(\eta(Y)\tilde{\varphi}\tilde{h}X-\eta(X)\tilde{\varphi}\tilde{h}Y),$$

for all $X,Y\in\Gamma(TM)$, where $\tilde{\kappa},\tilde{\mu},\tilde{\nu}$ are smooth functions on M.

A paracontact metric manifold whose characteristic vector field ξ is a harmonic vector field is called an H-paracontact manifold. Moreover, Kupeli Erken and Murathan [15] proved that ξ is a harmonic vector field if and only if ξ is an eigenvector of the Ricci operator. In the same study, they characterized the 3-dimensional H-paracontact metric manifolds in terms of $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-paracontact metric manifolds. In particular, they proved the following theorem.

Theorem 2.1 ([15]). *Let $(M,\tilde{\varphi},\xi,\eta,\tilde{g})$ be a 3-dimensional paracontact metric manifold. If the characteristic vector field ξ is harmonic map then the paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold always exists on every open and dense subset of M. Conversely, if M is a paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifold then the characteristic vector field ξ is harmonic map.*

It is shown that condition (2.4) is meaningless for $\tilde{\kappa}\neq -1$ in dimension higher than three, because the functions $\tilde{\kappa},\tilde{\mu}$ are constants and $\tilde{\nu}$ is the zero function.

Given a paracontact metric structure $(\tilde{\varphi},\xi,\eta,\tilde{g})$ and $\alpha>0$, the change of structure tensors

$$\bar{\eta}=\alpha\eta,\quad \bar{\xi}=\frac{1}{\alpha}\xi,\quad \bar{\varphi}=\tilde{\varphi},\quad \bar{g}=\alpha\tilde{g}+\alpha(\alpha-1)\eta\otimes\eta \tag{2.5}$$

is called a $\mathcal{D}_\alpha$*-homothetic deformation.* One can easily check that the new structure $(\bar{\varphi},\bar{\xi},\bar{\eta},\bar{g})$ is still a paracontact metric structure [19]. We now show that while $\mathcal{D}_\alpha$-homothetic deformations destroy conditions like $\tilde{R}_{XY}\xi=0$, they preserve the class of paracontact $(\tilde{\kappa},\tilde{\mu})$-spaces.

Kupeli Erken and Murathan analyzed the different possibilities for the tensor field $\tilde{h}$ in [15]. If $\tilde{h}$ has

$$\begin{pmatrix} \tilde{\lambda} & 0 & 0 \\ 0 & -\tilde{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.6}$$

the form (2.6) respect to local orthonormal $\tilde{\varphi}$-basis $\{X,\tilde{\varphi}X,\xi\}$, the authors called the operator $\tilde{h}$ is of $\mathfrak{h}_1$ *type*.

If the tensor $\tilde{h}$ has the form $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ relative a pseudo orthonormal basis $\{e_1,e_2,e_3\}$. In this case, the authors called $\tilde{h}$ is of $\mathfrak{h}_2$ *type.*

If the matrix form of $\tilde{h}$ is given by

$$\tilde{h}=\begin{pmatrix} 0 & -\tilde{\lambda} & 0 \\ \tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

with respect to local orthonormal basis $\{X,\tilde{\varphi}X,\xi\}$. In this case, the authors said that $\tilde{h}$ is of $\mathfrak{h}_3$ *type.*

3. Generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds

In this section, we will give some basic facts about generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds.

Lemma 3.1. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold. The following identities hold:*

$$\tilde{h}^2 = (1 + \tilde{\kappa})\tilde{\varphi}^2, \tag{3.1}$$

$$\xi(\tilde{\kappa}) = 0, \tag{3.2}$$

$$\tilde{Q}\xi = 2\tilde{\kappa}\xi, \tag{3.3}$$

$$\tilde{Q} = (\frac{\tau}{2} - \tilde{\kappa})I + (-\frac{\tau}{2} + 3\tilde{\kappa})\eta \otimes \xi + \tilde{\mu}\tilde{h}, \quad \tilde{\kappa} \neq -1, \tag{3.4}$$

where $\tilde{Q}$ is the Ricci operator of M, τ denotes scalar curvature of M and $\tilde{l} = \tilde{R}(., \xi)\xi$.

Proof. The proof of (3.1)-(3.3) are similar to that of [15, Lemma 3.2]. The relation (3.4) is an immediate consequence of [15, Lemma 4.4 and Lemma 4.14]. □

Lemma 3.2. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold. Then, for any point $P \in M$, with $\tilde{\kappa}(P) > -1$ there exist a neighborhood U of P and an $\tilde{h}$-frame on U, i.e. orthonormal vector fields ξ, X, $\tilde{\varphi}X$, defined on U, such that*

$$\tilde{h}X = \tilde{\lambda}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X, \quad h\xi = 0, \quad \tilde{\lambda} = \sqrt{1 + \tilde{\kappa}} \tag{3.5}$$

at any point $q \in U$. Moreover, setting $A = X\tilde{\lambda}$ and $B = \tilde{\varphi}X\tilde{\lambda}$ on U the following formulas are true :

$$\tilde{\nabla}_X\xi = (\tilde{\lambda} - 1)\tilde{\varphi}X, \ \tilde{\nabla}_{\tilde{\varphi}X}\xi = -(\tilde{\lambda} + 1)X, \tag{3.6}$$

$$\tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \ \tilde{\nabla}_\xi\tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X, \tag{3.7}$$

$$\tilde{\nabla}_X X = -\frac{B}{2\tilde{\lambda}}\tilde{\varphi}X, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = -\frac{A}{2\tilde{\lambda}}X, \tag{3.8}$$

$$\tilde{\nabla}_{\tilde{\varphi}X}X = -\frac{A}{2\tilde{\lambda}}\tilde{\varphi}X - (\tilde{\lambda} + 1)\xi, \quad \tilde{\nabla}_X\tilde{\varphi}X = -\frac{B}{2\tilde{\lambda}}X + (1 - \tilde{\lambda})\xi, \tag{3.9}$$

$$[\xi, X] = (1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2})\tilde{\varphi}X, \quad [\xi, \tilde{\varphi}X] = (\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2})X, \tag{3.10}$$

$$[X, \tilde{\varphi}X] = -\frac{B}{2\tilde{\lambda}}X + \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X + 2\xi, \tag{3.11}$$

$$\tilde{h}\ grad\tilde{\mu} = grad\tilde{\kappa}, \tag{3.12}$$

$$X\tilde{\mu} = 2A, \tag{3.13}$$

$$\tilde{\varphi}X\tilde{\mu} = -2B, \tag{3.14}$$

$$\xi(A) = (1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2})B, \tag{3.15}$$

$$\xi(B)=(\tilde{\lambda}+1-\frac{\tilde{\mu}}{2})A, \tag{3.16}$$

$$\left[\xi,\tilde{\varphi}grad\tilde{\lambda}\right]=0. \tag{3.17}$$

Proof. The proofs of (3.6)-(3.11) are given in [15]. For the proof of (3.12), we will use well known formula

$$\frac{1}{2}grad\ \tau=\sum_{i=1}^{3}\varepsilon_i(\tilde{\nabla}_{X_i}\tilde{Q})X_i,$$

where $\{X_1=X,\ X_2=\tilde{\varphi}X,\ X_3=\xi\}$. Using the equation (2.2), since $tr\tilde{h}=tr\tilde{h}\tilde{\varphi}=0$, we obtain

$$\begin{aligned}\sum_{i=1}^{3}\varepsilon_i(\tilde{\nabla}_{X_i}\tilde{Q})X_i &= \sum_{i=1}^{3}\varepsilon_iX_i(\frac{\tau}{2}-\tilde{\kappa})X_i+\sum_{i=1}^{3}\varepsilon_i(X_i(\tilde{\mu})\tilde{h}\ X_i)\\ &\quad+\tilde{\mu}\sum_{i=1}^{3}\varepsilon_i(\nabla_{X_i}\tilde{h}\ )X_i\\ &= \frac{1}{2}grad\tau-grad\tilde{\kappa}+\tilde{h}\ grad\tilde{\mu}\\ &\quad+\tilde{\mu}\sum_{i=1}^{3}\varepsilon_i(\tilde{\nabla}_{X_i}\tilde{h}\ )X_i-\frac{1}{2}\xi(\tau)\xi\end{aligned} \tag{3.18}$$

The relations (3.5), (3.8) and (3.9) yield $\sum_{i=1}^{3}(\tilde{\nabla}_{X_i}\tilde{h})X_i=0$. Using the last relation in (3.18), one has

$$\frac{1}{2}grad\ \tau=\frac{1}{2}grad\tau-grad\tilde{\kappa}+\tilde{h}\ grad\tilde{\mu}-\frac{1}{2}\xi(\tau)\xi \tag{3.19}$$

that is

$$-grad\tilde{\kappa}+\tilde{h}\ grad\tilde{\mu}-\frac{1}{2}\xi(\tau)\xi=0. \tag{3.20}$$

Since the vector field $-grad\tilde{\kappa}+\tilde{h}\ grad\tilde{\mu}$ is orthogonal to ξ. So, we get (3.12). The equations (3.13) and (3.14) are immediate consequences of (3.12).

By virtue of (3.2) and (3.10), we have

$$\begin{aligned}\xi(A) &= \xi X\tilde{\lambda}=[\xi,X]\tilde{\lambda}+X\xi\tilde{\lambda}=(1-\tilde{\lambda}-\frac{\tilde{\mu}}{2})\tilde{\varphi}X\tilde{\lambda}\\ &= (1-\tilde{\lambda}-\frac{\tilde{\mu}}{2})B\end{aligned}$$

The relation (3.16) is proved similarly. Using (3.2), we have

$$grad\tilde{\lambda}=-AX+B\tilde{\varphi}X,\quad \tilde{\varphi}grad\tilde{\lambda}=-A\tilde{\varphi}X+BX\ . \tag{3.21}$$

From the relations (3.21), (3.10), (3.15) and (3.16) we obtain

$$\begin{aligned}\left[\xi,\tilde{\varphi}grad\tilde{\lambda}\right] &= [\xi,-A\tilde{\varphi}X+BX]\\ &= -(\xi A)\tilde{\varphi}X-A\left[\xi,\tilde{\varphi}X\right]+(\xi B)X+B\left[\xi,X\right]=0.\end{aligned}$$

□

Lemma 3.3. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold. Then, for any point $P \in M$, with $\tilde{\kappa}(P) < -1$ there exist a neighborhood U of P and an $\tilde{h}$-frame on U, i.e. orthonormal vector fields ξ, X, $\tilde{\varphi}X$, defined on U, such that*

$$\tilde{h}X = \tilde{\lambda}\tilde{\varphi}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X, \quad h\xi = 0, \quad \tilde{\lambda} = \sqrt{-1-\tilde{\kappa}} \tag{3.22}$$

at any point $q \in U$. Moreover, setting $A = X\tilde{\lambda}$ and $B = \tilde{\varphi}X\tilde{\lambda}$ on U the following formulas are true :

$$\tilde{\nabla}_X \xi = -\tilde{\varphi}X + \tilde{\lambda}X, \ \tilde{\nabla}_{\tilde{\varphi}X}\xi = -X - \tilde{\lambda}\tilde{\varphi}X, \tag{3.23}$$

$$\tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2}\tilde{\varphi}X, \ \tilde{\nabla}_\xi \tilde{\varphi}X = -\frac{\tilde{\mu}}{2}X, \tag{3.24}$$

$$\tilde{\nabla}_X X = -\frac{B}{2\tilde{\lambda}}\tilde{\varphi}X + \tilde{\lambda}\xi, \quad \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X = -\frac{A}{2\tilde{\lambda}}X + \tilde{\lambda}\xi, \tag{3.25}$$

$$\tilde{\nabla}_{\tilde{\varphi}X} X = -\frac{A}{2\tilde{\lambda}}\tilde{\varphi}X - \xi, \quad \tilde{\nabla}_X \tilde{\varphi}X = -\frac{B}{2\tilde{\lambda}}X + \xi, \tag{3.26}$$

$$[\xi, X] = -\tilde{\lambda}X + (1 - \frac{\tilde{\mu}}{2})\tilde{\varphi}X, \quad [\xi, \tilde{\varphi}X] = (1 - \frac{\tilde{\mu}}{2})X + \tilde{\lambda}\tilde{\varphi}X, \tag{3.27}$$

$$[X, \tilde{\varphi}X] = -\frac{B}{2\tilde{\lambda}}X + \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X + 2\xi, \tag{3.28}$$

$$\tilde{h} \ grad\tilde{\mu} = grad\tilde{\kappa}, \tag{3.29}$$

$$X\tilde{\mu} = 2B, \tag{3.30}$$

$$\tilde{\varphi}X\tilde{\mu} = -2A, \tag{3.31}$$

$$\xi(A) = -\tilde{\lambda}A + (1 - \frac{\tilde{\mu}}{2})B, \tag{3.32}$$

$$\xi(B) = (1 - \frac{\tilde{\mu}}{2})A + \tilde{\lambda}B, \tag{3.33}$$

$$\left[\xi, \tilde{\varphi}grad\tilde{\lambda}\right] = 0. \tag{3.34}$$

Proof. The proofs of (3.23)-(3.28) are given in [15]. The proof of (3.29) is similar to proof of Lemma (3.2), equation (3.12). The equations (3.30) and (3.31) are immediate consequences of (3.29).

By virtue of (3.2) and (3.27), we have

$$\begin{aligned}\xi(A) &= \xi X\tilde{\lambda} = [\xi, X]\tilde{\lambda} + X\xi\tilde{\lambda} = -\tilde{\lambda}X\tilde{\lambda} + (1 - \frac{\tilde{\mu}}{2})\tilde{\varphi}X\tilde{\lambda} \\ &= -\tilde{\lambda}A + (1 - \frac{\tilde{\mu}}{2})B\end{aligned}$$

The relation (3.33) is proved similarly. Using (3.2), we have

$$grad\tilde{\lambda} = -AX + B\tilde{\varphi}X, \quad \tilde{\varphi}grad\tilde{\lambda} = -A\tilde{\varphi}X + BX \ . \tag{3.35}$$

From the relations (3.35), (3.27), (3.32) and (3.33) we obtain

$$\begin{aligned}\left[\xi, \tilde{\varphi} grad\tilde{\lambda}\right] &= [\xi, -A\tilde{\varphi}X + BX] \\ &= -(\xi A)\tilde{\varphi}X - A[\xi, \tilde{\varphi}X] + (\xi B)X + B[\xi, X] = 0.\end{aligned}$$

□

4. Generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds with $\xi(\tilde{\mu}) = 0$

We shall give a local classification of generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds with $\tilde{\kappa} > -1$ which satisfy the condition $\xi(\tilde{\mu}) = 0$.

Theorem 4.1 (Main Theorem)**.** *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold with $\tilde{\kappa} > -1$ and $\xi(\tilde{\mu}) = 0$. Then*

1) At any point of M, precisely one of the following relations is valid: $\tilde{\mu} = 2(1 + \sqrt{1 + \tilde{\kappa}})$, or $\tilde{\mu} = 2(1 - \sqrt{1 + \tilde{\kappa}})$

2) At any point $P \in M$ there exists a chart $(U, (x, y, z))$ with $P \in U \subseteq M$, such that

i) the functions $\tilde{\kappa}, \tilde{\mu}$ depend only on the variable z.

ii) if $\tilde{\mu} = 2(1 + \sqrt{1 - \tilde{\kappa}})$, (resp. $\tilde{\mu} = 2(1 - \sqrt{1 - \tilde{\kappa}})$), the tensor fields η, ξ, $\tilde{\varphi}$, $\tilde{g}$, $\tilde{h}$ are given by the relations,

$$\xi = \frac{\partial}{\partial x}, \quad \eta = dx - adz$$

$$\tilde{g} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & -1 + a^2 + b^2 \end{pmatrix} \qquad \left(resp. \quad \tilde{g} = \begin{pmatrix} 1 & 0 & -a \\ 0 & -1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix} \right),$$

$$\tilde{\varphi} = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & 1 & -b \end{pmatrix} \qquad \left(resp. \quad \tilde{\varphi} = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & 1 & -b \end{pmatrix} \right),$$

$$\tilde{h} = \begin{pmatrix} 0 & 0 & a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & 2\tilde{\lambda}b \\ 0 & 0 & \tilde{\lambda} \end{pmatrix} \qquad \left(resp. \quad \tilde{h} = \begin{pmatrix} 0 & 0 & -a\tilde{\lambda} \\ 0 & \tilde{\lambda} & -2\tilde{\lambda}b \\ 0 & 0 & -\tilde{\lambda} \end{pmatrix} \right)$$

with respect to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where $a = -2y + f(z)$ (resp. $a = 2y + f(z)$), $b = -\frac{y}{2}\frac{r'(z)}{r(z)} - 2xr(z) + s(z)$, $\tilde{\lambda} = \tilde{\lambda}(z) = r(z)$ and $f(z)$, $r(z)$, $s(z)$ are arbitrary smooth functions of z.

Proof of the Main Theorem: Let $\{\xi, X, \tilde{\varphi}X\}$ be an $\tilde{h}$-frame, such that

$$\tilde{h}X = \tilde{\lambda}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}\tilde{\varphi}X, \quad \tilde{\lambda} = \sqrt{1 + \tilde{\kappa}}$$

in an appropriate neighbourhood of an arbitrary point of M. Using the hypothesis $\xi(\tilde{\mu}) = 0$ and equations (3.13)-(3.17) and (3.21) we have the following relations,

$$(\tilde{\varphi} grad\lambda)\tilde{\mu} = 4AB, \tag{4.1}$$

$$[\xi, \tilde{\varphi} grad\lambda]\, \tilde{\mu} = 0, \tag{4.2}$$

$$\xi(AB) = 0, \tag{4.3}$$

$$A\xi B + B\xi A = 0, \tag{4.4}$$

$$A^2(\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2}) + B^2(1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2}) = 0. \tag{4.5}$$

Differentiating the relation (4.5) with respect to ξ and using the equations (3.2), $\xi(\tilde{\mu}) = 0$, (3.15), (3.16) and (4.5), we obtain

$$(1 + \tilde{\lambda} - \frac{\tilde{\mu}}{2})(-\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2})AB = 0. \tag{4.6}$$

We put $F = (1 + \tilde{\lambda} - \frac{\tilde{\mu}}{2})(-\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2})$ and consider the set $N = \{p \in M \mid (\mathrm{grad}\tilde{\lambda})(p) \neq 0\}$. We will prove that $F = 0$ at any point of N. Let $p \in N$ be such that $F(p) \neq 0$. From (4.6) we get $(AB)(p) = 0$. We consider cases $\{A(p) = B(p) = 0\ \}, \{A(p) \neq 0,\ B(p) = 0\}$ and $\{A(p) = 0,\ B(p) \neq 0\}$. Now we will examine the first case. In this case, by (3.2), we get $(\xi(\tilde{\lambda}))(p) = 0$. As a result we obtain $(\mathrm{grad}\tilde{\lambda})(p) = 0$ which is a contradiction with $(\mathrm{grad}\tilde{\lambda})(p) \neq 0$. So, the first case is impossible. We assume that $\{A(p) \neq 0,\ B(p) = 0\}$. Since the function F is continuous, we find that a neighbourhood $V \subseteq N$ exists, with $p \in V$ such that $F \neq 0$ at any point of V. Similarly, due to the fact that the function A is continuous on its domain, a neighbourhood W of p exists with $p \in W \subset V$, such that $A \neq 0$ at any point of W, and thus $B = 0$ on W. From (4.5), we have $(1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2}) = 0$ at any point of W and thus $F = 0$ on W, which is a contradiction. Since the last case is similar to the second case we omit it. Therefore, $F = 0$ at any point of N. In what follows, we will work on the complement N^C of set N, in order to prove that $F = 0$ on M. If $N^C = \emptyset$, then $F = 0$ on M. Let us suppose that $N^C \neq \emptyset$. Then we have $\mathrm{grad}\tilde{\lambda} = 0$ on N^C and thus the function of $\tilde{\lambda}$ is constant at any connected component of the interior $(N^C)^\circ$. From the constancy of $\tilde{\lambda}$ and the relations (3.13) and (3.14), $\xi(\tilde{\mu}) = 0$, the function $\tilde{\mu}$ is also constant. As a result we find that F is constant on any connected component of $(N^C)^\circ$. Because M is connected and $F = 0$ on N and $F =$ constant on any connected component of $(N^C)^\circ$ we conclude that $F = 0$, or equivalently $(1 + \tilde{\lambda} - \frac{\tilde{\mu}}{2})(-\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2}) = 0$ at any point of M.

Now we consider the open disjoint sets $U_0 = \{p \in M \mid (\tilde{\lambda} + 1 - \frac{\tilde{\mu}}{2})(p) \neq 0\ \}$ and $U_1 = \{p \in M \mid (1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2})(p) \neq 0\ \}$.We have $U_0 \cup U_1 = M$. Due to the fact that M is connected, we conclude that $\{M = U_0$ and $U_1 = \emptyset\}$ or $\{U_0 = \emptyset$ and $U_1 = M\}$. Regarding the set U_0 we have $\tilde{\mu} = 2(1+\tilde{\lambda})$, or equivalently $\tilde{\mu} = 2(1+\sqrt{1 + \tilde{\kappa}})$ at any point M. Similarly, regarding the set U_1 we obtain $\tilde{\mu} = 2(1 - \tilde{\lambda}) = 2(1 - \sqrt{1 + \tilde{\kappa}})$. Therefore, (1) is proved. Now, we will examine the cases $\tilde{\mu} = 2(1 + \sqrt{1 + \tilde{\kappa}})$ and $\tilde{\mu} = 2(1 - \sqrt{1 + \tilde{\kappa}})$.

Case 1. $\tilde{\mu} = 2(1 + \sqrt{1 + \tilde{\kappa}})$.

Let $p \in M$ and $\{\xi, X, \tilde{\varphi}X\}$ be an $\tilde{h}$-frame on a neighborhood U of p. Using the assumption $\tilde{\mu} = 2(1 + \sqrt{1 + \tilde{\kappa}})$ and (4.5) we obtain $B = 0$ and thus the relations (3.10) and (3.11) are reduced to

$$[\xi, X] = -2\tilde{\lambda}\tilde{\varphi}X, \tag{4.7}$$

$$[\xi, \tilde{\varphi}X] = 0, \tag{4.8}$$

$$[X, \tilde{\varphi}X] = -\frac{A}{2\tilde{\lambda}}\tilde{\varphi}X + 2\xi. \tag{4.9}$$

Since $[\xi,\tilde{\varphi}X]=0$, the distribution which is spanned by ξ and $\tilde{\varphi}X$ is integrable and so for any $q\in V$ there exist a chart $(V,(x,y,z)\}$ at $p\in V\subset U$, such that

$$\xi=\frac{\partial}{\partial x},\quad X=a\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z},\quad \tilde{\varphi}X=\frac{\partial}{\partial y}, \tag{4.10}$$

where a, b and c are smooth functions on V. Since ξ, X and $\tilde{\varphi}X$ are linearly independent we have $c\neq 0$ at any point of V. By using (4.10), (3.2) and $B=0$ we obtain

$$\frac{\partial\tilde{\lambda}}{\partial x}=0\quad\text{and}\quad\frac{\partial\tilde{\lambda}}{\partial y}=0\ . \tag{4.11}$$

From (4.11) we find

$$\tilde{\lambda}=r(z), \tag{4.12}$$

where $r(z)$ is smooth function of z defined on V. By using (4.7), (4.9) and (4.10) we have following partial differential equations:

$$\frac{\partial a}{\partial x}=0,\quad\frac{\partial b}{\partial x}=-2\tilde{\lambda},\quad\frac{\partial c}{\partial x}=0, \tag{4.13}$$

$$\frac{\partial a}{\partial y}=-2,\quad\frac{\partial b}{\partial y}=-\frac{A}{2\tilde{\lambda}},\quad\frac{\partial c}{\partial y}=0. \tag{4.14}$$

From $\frac{\partial c}{\partial x}=\frac{\partial c}{\partial y}=0$ it follows that $c=c(z)$ and because of the fact that $c\neq 0$, we can assume that $c=1$ through a reparametrization of the variable z. For the sake of simplicity we will continue to use the same coordinates (x,y,z), taking into account that $c=1$ in the relations that we have occured. From $\frac{\partial a}{\partial x}=0$, $\frac{\partial a}{\partial y}=-2$ we obtain

$$a=a(x,y,z)=-2y+f(z),$$

where $f(z)$ is smooth function of z defined on V. Differentiating $\tilde{\lambda}$ with respect to X and using (4.11) and (4.12) we have

$$A=r'(z), \tag{4.15}$$

where $r'(z)=\frac{dr}{dz}$. By using the relations $\frac{\partial b}{\partial x}=-2\tilde{\lambda}$, $\frac{\partial b}{\partial y}=-\frac{A}{2\tilde{\lambda}}$ and (4.12) we get

$$b=-\frac{y}{2}\frac{r'(z)}{r(z)}-2xr(z)+s(z).$$

where $s(z)$ is arbitrary smooth function of z defined on V. We will calculate the tensor fields η, $\tilde{\varphi}$, $\tilde{g}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$. For the components $\tilde{g}_{ij}$ of the Riemannian metric, using (4.10) we have

$$\tilde{g}_{11}=\tilde{g}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})=1,\ \tilde{g}(\xi,\xi)=1,\ \ \tilde{g}_{22}=\tilde{g}(\frac{\partial}{\partial y},\frac{\partial}{\partial y})=\tilde{g}(\tilde{\varphi}X,\tilde{\varphi}X)=1,$$

$$\tilde{g}_{12}=\tilde{g}_{21}=\tilde{g}(\frac{\partial}{\partial x},\frac{\partial}{\partial y})=0,$$

$$\begin{aligned}\tilde{g}_{13}&=\tilde{g}_{31}=\tilde{g}(\frac{\partial}{\partial x},X-a\frac{\partial}{\partial x}-b\frac{\partial}{\partial y})\\&=\tilde{g}(\xi,X)-a\tilde{g}_{11}=-a,\end{aligned}$$

$$\begin{aligned}\tilde g_{23} &= \tilde g_{32} = \tilde g(\frac{\partial}{\partial y}, X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y})\\ &= \tilde g(\tilde\varphi X, X) - a\tilde g_{12} - b\tilde g_{22} = -b,\end{aligned}$$

$$\begin{aligned}-1 &= \tilde g(X,X) \Rightarrow a^2\tilde g_{11} + 2a\tilde g_{13} + b^2\tilde g_{22} + 2ab\tilde g_{12} + 2b\tilde g_{23} + \tilde g_{33} = -1\\ &= a^2 - 2a^2 + b^2 - 2b^2 + \tilde g_{33} = \tilde g_{33} - a^2 - b^2,\end{aligned}$$

from which we obtain $\tilde g_{33} = -1 + a^2 + b^2$.

The matrix form of $\tilde g$ is given by

$$\tilde g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & -1 + a^2 + b^2 \end{pmatrix}.$$

The components of the tensor field $\tilde\varphi$ are immediate consequences of

$$\tilde\varphi(\xi) = \tilde\varphi(\frac{\partial}{\partial x}) = 0, \quad \tilde\varphi(\frac{\partial}{\partial y}) = X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$\begin{aligned}\tilde\varphi(\frac{\partial}{\partial z}) &= \tilde\varphi(X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = \tilde\varphi X - a\tilde\varphi(\frac{\partial}{\partial x}) - b\tilde\varphi(\frac{\partial}{\partial y})\\ &= \tilde\varphi X - b(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z})\\ &= \frac{\partial}{\partial y} - ab\frac{\partial}{\partial x} - b^2\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}\\ &= -ab\frac{\partial}{\partial x} + (1 - b^2)\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}.\end{aligned}$$

The matrix form of $\tilde\varphi$ is given by

$$\tilde\varphi = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & c & -b \end{pmatrix}.$$

The expression of the 1-form η, immediately follows from $\eta(\xi) = 1$, $\eta(X) = \eta(\tilde\varphi X) = 0$

$$\eta = dx - adz.$$

Now we calculate the components of the tensor field $\tilde h$ with respect to the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$.

$$\tilde h(\xi) = \tilde h(\frac{\partial}{\partial x}) = 0, \quad \tilde h(\frac{\partial}{\partial y}) = -\lambda\frac{\partial}{\partial y},$$

$$\begin{aligned}\tilde h(\frac{\partial}{\partial z}) &= \tilde h(X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y})\\ &= \tilde h X - a\tilde h(\frac{\partial}{\partial x}) - b\tilde h(\frac{\partial}{\partial y})\\ &= \tilde\lambda X + b\tilde\lambda\frac{\partial}{\partial y}\\ &= \tilde\lambda(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}) + b\tilde\lambda\frac{\partial}{\partial y},\end{aligned}$$

$$\tilde h(\frac{\partial}{\partial z}) = \tilde\lambda a\frac{\partial}{\partial x} + 2b\tilde\lambda\frac{\partial}{\partial y} + \tilde\lambda\frac{\partial}{\partial z}.$$

The matrix form of $\tilde{h}$ is given by

$$\tilde{h} = \begin{pmatrix} 0 & 0 & a\tilde{\lambda} \\ 0 & -\tilde{\lambda} & 2\tilde{\lambda}b \\ 0 & 0 & \tilde{\lambda} \end{pmatrix}.$$

Thus the proof of the Case 1 is completed.

Case 2. $\tilde{\mu} = 2(1 - \sqrt{1+\tilde{\kappa}})$.

As in the Case 1, we consider an $\tilde{h}$-frame $\{\xi, X, \tilde{\varphi}X\}$. Using the assumption $\tilde{\mu} = 2(1 - \sqrt{1+\tilde{\kappa}})$ and (4.5) we obtain $A = 0$ and thus the relation (3.10) and (3.11) is written as

$$[\xi, X] = 0, \tag{4.16}$$

$$[\xi, \tilde{\varphi}X] = 2\tilde{\lambda}X, \tag{4.17}$$

$$[X, \tilde{\varphi}X] = -\frac{B}{2\tilde{\lambda}}X + 2\xi. \tag{4.18}$$

Because of (4.16) we find that there is a chart $(V', (x, y, z))$ such that

$$\xi = \frac{\partial}{\partial x}, \qquad X = \frac{\partial}{\partial y}$$

on V'. We put

$$\tilde{\varphi}X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z},$$

where a, b, c are smooth functions defined on V'. As in the Case 1, we can directly calculate the tensor fields η, $\tilde{\varphi}$, $\tilde{g}$ and $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. This completes the proof of the main theorem. □

Now, we give an example of a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold with $\xi(\tilde{\mu}) = 0$ which satisfy the conditions of Main Theorem (Case 1).

Example 4.1. We consider the 3-dimensional manifold

$$M = \{(x, y, z) \in R^3, z \neq 0\}$$

and the vector fields

$$\xi = \frac{\partial}{\partial x}, \quad \tilde{\varphi}X = \frac{\partial}{\partial y}, \quad X = (-2y+1)\frac{\partial}{\partial x} + (-\frac{y}{2z} - 2xz + 2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The 1-form $\eta = dx - (-2y+1)dz$ defines a contact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let $\tilde{g}$, $\tilde{\varphi}$ be the pseudo-Riemannian metric and the $(1,1)$-tensor field given by

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 2y-1 \\ 0 & 1 & \frac{y}{2z} + 2xz - 2 \\ 2y-1 & \frac{y}{2z} + 2xz - 2 & -1 + (-2y+1)^2 + (-\frac{y}{2z} - 2xz + 2)^2 \end{pmatrix},$$

$$\tilde{\varphi} = \begin{pmatrix} 0 & -2y+1 & -(-2y+1)(-\frac{y}{2z} - 2xz + 2) \\ 0 & -\frac{y}{2z} - 2xz + 2 & 1 - (-\frac{y}{2z} - 2xz + 2)^2 \\ 0 & 1 & \frac{y}{2z} + 2xz - 2 \end{pmatrix},$$

$$\tilde{h} = \begin{pmatrix} 0 & 0 & (-2y+1)z \\ 0 & -z & 2z(-\frac{y}{2z} - 2xz + 2) \\ 0 & 0 & z \end{pmatrix}, \quad \tilde{\lambda} = z,$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. Finally, we deduce that M is a generalized $(z^2 - 1, 2(1 + z))$ paracontact metric manifold with $\xi(\tilde\mu) = 0$.

Let $\{\xi, X, \tilde\varphi X\}$ be an $\tilde h$-frame, such that

$$\tilde h X = \tilde\lambda \tilde\varphi X, \quad \tilde h \tilde\varphi X = -\tilde\lambda X, \quad \tilde\lambda = \sqrt{-1 - \tilde\kappa}$$

in an appropriate neighbourhood of an arbitrary point of M. Using the hypothesis $\xi(\tilde\mu) = 0$ and equations (3.30)-(3.34) and (3.35) we have the following relations,

$$(\tilde\varphi grad\lambda)\tilde\mu = 2(A^2 + B^2), \tag{4.19}$$

$$[\xi, \tilde\varphi grad\lambda]\,\tilde\mu = 0, \tag{4.20}$$

$$\xi(A^2 + B^2) = 0, \tag{4.21}$$

$$A\xi A + B\xi B = 0, \tag{4.22}$$

$$-\tilde\lambda A^2 + 2AB(1 - \frac{\tilde\mu}{2}) + \tilde\lambda B^2 = 0. \tag{4.23}$$

Differentiating the relation (4.23) with respect to ξ and using the equations (3.2), $\xi(\tilde\mu) = 0$, (3.32), (3.33) and (4.23), we obtain

$$(A(1 - \frac{\tilde\mu}{2}) + \tilde\lambda B)^2 + (B(1 - \frac{\tilde\mu}{2}) - \tilde\lambda A)^2 = 0. \tag{4.24}$$

From (4.24), precisely following cases occurs.

$$\bullet A = 0 \text{ and } B = 0. \tag{4.25}$$

$$\bullet A \neq 0 \text{ and } \tilde\lambda^2 + (1 - \frac{\tilde\mu}{2})^2 = 0. \tag{4.26}$$

$$\bullet B \neq 0 \text{ and } \tilde\lambda^2 + (1 - \frac{\tilde\mu}{2})^2 = 0. \tag{4.27}$$

$$\bullet A = 0 \text{ and } \tilde\lambda^2 + (1 - \frac{\tilde\mu}{2})^2 \neq 0 \tag{4.28}$$

$$\bullet B = 0 \text{ and } \tilde\lambda^2 + (1 - \frac{\tilde\mu}{2})^2 \neq 0 \tag{4.29}$$

We now check, case by case, whether (4.24) give rise to a local classification of generalized $(\tilde\kappa \neq -1, \tilde\mu)$-paracontact metric manifolds with $\tilde\kappa < -1$. From (4.25) we get $\tilde\kappa$ and $\tilde\mu$ constants. So the manifold returns to a $(\tilde\kappa, \tilde\mu)$-paracontact metric manifold. But we want to give a local classification for generalized $(\tilde\kappa, \tilde\mu)$-paracontact metric manifolds. So we omit this case. (4.26) and (4.27) hold if and only if $\tilde\kappa = -1$ and $\tilde\mu = 2$. But this is a contradiction with $\tilde\kappa < -1$. If we use (4.28) in (4.24) we obtain $B^2((1 - \frac{\tilde\mu}{2})^2 + \tilde\lambda^2) = 0$. But the solution of this equation contradicts with the type of manifold and choosing of $\tilde\kappa$.

So we can give following corollary.

Corollary 4.1. *There is not exist any generalized $(\tilde\kappa < -1, \tilde\mu)$-paracontact metric manifolds which satisfy the condition $\xi(\tilde\mu) = 0$.*

In the following theorem, we will locally construct generalized $(\tilde\kappa, \tilde\mu)$-paracontact metric manifolds with $\tilde\kappa > -1$ and $\xi(\tilde\mu) = 0$.

Theorem 4.2. *Let $\tilde{\kappa} : I \subset \mathbb{R} \to \mathbb{R}$ be a smooth function defined on an open interval I, such that $\tilde{\kappa}(z) > -1$ for any $z \in I$. Then, we can construct two families of generalized $(\tilde{\kappa}_i, \tilde{\mu}_i)$-paracontact metric manifolds $(M, \tilde{\varphi}_i, \xi_i, \eta_i, \tilde{g}_i)$, $i = 1, 2$, in the set $M = \mathbb{R}^2 \times I \subset \mathbb{R}^3$, so that, for any $P(x, y, z) \in M$, the following are valid:*

$$\begin{aligned} \tilde{\kappa}_1(P) &= \tilde{\kappa}_2(P) = \tilde{\kappa}(z), \\ \tilde{\mu}_1(P) &= 2(1 + \sqrt{1 + \tilde{\kappa}(z)}) \quad \textit{and} \quad \tilde{\mu}_2(P) = 2(1 - \sqrt{1 + \tilde{\kappa}(z)}) \end{aligned}$$

Each family is determined by two arbitrary smooth functions of one variable.

Proof. We put $\tilde{\lambda}(z) = \sqrt{1 + \tilde{\kappa}(z)} > 0$ and consider on M the linearly independent vector fields

$$\xi_1 = \frac{\partial}{\partial x}, \quad X_1 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \text{ and } Y_1 = \frac{\partial}{\partial y}, \tag{4.30}$$

where $a(x, y, z) = -2y + f(z)$, $b(x, y, z) = -\frac{y}{2}\frac{\tilde{\lambda}'(z)}{\tilde{\lambda}(z)} - 2x\tilde{\lambda}(z) + s(z)$, $f(z)$, $s(z)$ are arbitrary smooth functions of z and $-\tilde{g}_1(X_1, X_1) = \tilde{g}_1(Y_1, Y_1) = \tilde{g}_1(\xi_1, \xi_1) = 1$. The structure tensor fields $\eta_1, \tilde{g}_1, \tilde{\varphi}_1$ are defined by $\eta_1 = dx - (-2y + f(z))dz$, $\tilde{g}_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & -1 + a^2 + b^2 \end{pmatrix}$ and $\tilde{\varphi}_1 = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & 1 & -b \end{pmatrix}$, respectively. From (4.30), we can easily obtain

$$[\xi_1, X_1] = -2\tilde{\lambda}(z)Y_1, \quad [\xi_1, Y_1] = 0, \tag{4.31}$$

$$[X_1, Y_1] = \frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}Y_1 + 2\xi_1. \tag{4.32}$$

Since $\eta_1 \wedge d\eta_1 = 2dx \wedge dy \wedge dz \neq 0$ everywhere on M, we conclude that η_1 is a contact form. By using just defined $\tilde{g}_1$ and $\tilde{\varphi}_1$, we find $\eta_1 = \tilde{g}(., \xi_1)$, $\tilde{\varphi}_1 X_1 = Y_1$, $\tilde{\varphi}_1 Y_1 = X_1$, $\tilde{\varphi}_1 \xi_1 = 0$ and $d\eta_1(Z, W) = \tilde{g}_1(Z, \tilde{\varphi}_1 W)$, $\tilde{g}_1(\tilde{\varphi}_1 Z, \tilde{\varphi}_1 W) = -\tilde{g}_1(Z, W) + \eta_1(Z)\eta_1(W)$ for any Z, $W \in \Gamma(M)$. Hence $M(\eta_1, \xi_1, \tilde{\varphi}_1, \tilde{g}_1)$ is a paracontact metric manifold. From the well known Koszul's formula $2\tilde{g}_1(\tilde{\nabla}_Z W, T) = Z\tilde{g}_1(W, T) + W\tilde{g}_1(T, Z) - T\tilde{g}_1(Z, W) - \tilde{g}_1(Z, [W, T]) + \tilde{g}_1(W, [T, Z]) + \tilde{g}_1(T, [Z, W])$ and (2.2), we have the following equations

$$\tilde{\nabla}_{X_1}\xi_1 = (\tilde{\lambda}(z) - 1)Y_1, \quad \tilde{\nabla}_{Y_1}\xi_1 = -(1 + \tilde{\lambda}(z))X_1, \tag{4.33}$$

$$\tilde{\nabla}_{\xi_1}\xi_1 = 0, \quad \tilde{\nabla}_{\xi_1}X_1 = -(\tilde{\lambda}(z) + 1)Y_1, \quad \tilde{\nabla}_{\xi_1}Y_1 = -(1 + \tilde{\lambda}(z))X_1, \tag{4.34}$$

$$\tilde{\nabla}_{X_1}X_1 = 0, \quad \tilde{\nabla}_{Y_1}Y_1 = \frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}X_1, \tag{4.35}$$

$$\tilde{\nabla}_{Y_1}X_1 = -\frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}Y_1 - (\tilde{\lambda}(z) + 1)\xi_1, \tag{4.36}$$

$$\tilde{\nabla}_{X_1}Y_1 = (1 - \tilde{\lambda}(z))\xi_1, \tag{4.37}$$

$\tilde{h}_1\tilde{\varphi}_1 X_1 = -\tilde{\lambda}(z)\tilde{\varphi}_1 X_1$ and $\tilde{h}_1 X_1 = \tilde{\lambda}(z)X_1$, where $\tilde{\nabla}$ is Levi-Civita connection of $\tilde{g}_1$. By using the relations (4.33)-(4.37) we obtain

$$\begin{aligned}
\tilde{R}(\xi_1,\xi_1)\xi_1 &= 0,\\
\tilde{R}(X_1,\xi_1)\xi_1 &= \tilde{\kappa}_1 X_1 + \tilde{\mu}_1\tilde{h}_1 X_1,\\
\tilde{R}(Y_1,\xi_1)\xi_1 &= \tilde{\kappa}_1 Y_1 + \tilde{\mu}_1\tilde{h}_1 Y_1,\\
\tilde{R}(X_1,X_1)\xi_1 &= 0,\ \tilde{R}(Y_1,Y_1)\xi_1 = 0\\
\tilde{R}(X_1,Y_1)\xi_1 &= 0.
\end{aligned}$$

From the above relations and by virtue of the linearity of the curvature tensor $\tilde{R}$, we conclude that

$$\tilde{R}(Z,W)\xi_1 = (\tilde{\kappa}_1 I + \tilde{\mu}_1\tilde{h}_1)(\eta_1(Z)W - \eta_1(W)Z)$$

for any $Z, W \in \Gamma(M)$, i.e. $(M, \tilde{\varphi}_1, \xi_1, \eta_1, \tilde{g}_1)$ is a generalized $(\tilde{\kappa}_1, \tilde{\mu}_1)$-paracontact metric manifold with $\xi(\tilde{\mu}_1) = 0$ and thus the construction of the first family is completed. For the second construction, we consider the vector fields

$$\xi_2 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}\ , \tag{4.38}$$

$$Y_2 = (2y + f(z))\frac{\partial}{\partial x} + \Big(-\frac{y}{2}\frac{\tilde{\lambda}'(z)}{\tilde{\lambda}(z)} - 2x\tilde{\lambda}(z) + s(z)\Big)\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \tag{4.39}$$

and define the tensor fields $\eta_2, \tilde{g}_2, \tilde{\varphi}_2, \tilde{h}_2$ as follows:

$$\eta_2 = dx - (2y + f(z))dz$$

$$\tilde{g}_2 = \begin{pmatrix} 1 & 0 & -a \\ 0 & -1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix},\ \tilde{\varphi}_2 = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & 1 & -b \end{pmatrix},$$

$$\tilde{h}_2 = \begin{pmatrix} 0 & 0 & -a\tilde{\lambda}_2 \\ 0 & \tilde{\lambda}_2 & -2\tilde{\lambda}_2 b \\ 0 & 0 & -\tilde{\lambda}_2 \end{pmatrix}$$

with respect to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where $a = 2y + f(z)$, $b = (-\frac{y}{2}\frac{\tilde{\lambda}'(z)}{\tilde{\lambda}r(z)} - 2x\tilde{\lambda}(z) + s(z))$ and $-\tilde{g}_2(X_2, X_2) = \tilde{g}_2(Y_2, Y_2) = \tilde{g}_2(\xi_2, \xi_2) = 1$. As in first construction, we say that $(M, \tilde{\varphi}_2, \xi_2, \eta_2, \tilde{g}_2)$ is a generalized $(\tilde{\kappa}_2, \tilde{\mu}_2)$-paracontact metric manifold with $\xi(\tilde{\mu}_2) = 0$, where $\tilde{\kappa}_2(z) = \tilde{\lambda}(z)^2 - 1$ and $\tilde{\mu}_2(x, y, z) = 2(1 - \sqrt{1 + \kappa_2(z)})$. This completes the proof of the theorem. □

In the following theorem, we give an analytic expression of the scalar curvature τ of generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds. It is interesting that the same formula holds both for the case $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$.

Theorem 4.3. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold. Then,*

$$\triangle\tilde{\lambda} = -X(A) + \tilde{\varphi}X(B) + \frac{1}{2\tilde{\lambda}}(A^2 - B^2) \tag{4.40}$$

and

$$\tau = \frac{1}{\tilde{\lambda}}(\triangle\tilde{\lambda}) - \frac{1}{\tilde{\lambda}^2} \parallel grad\tilde{\lambda} \parallel^2 + 2(\tilde{\kappa} + \tilde{\mu}), \tag{4.41}$$

where $\triangle\tilde{\lambda}$ *is Laplacian of* $\tilde{\lambda}$.

Proof. We will give the proof for $\tilde{\kappa} > -1$. The proof for $\tilde{\kappa} < -1$ is similar to $\tilde{\kappa} > -1$. Using the definition of the Laplacian and equations (3.2) and (3.8) we obtain

$$\begin{aligned}\triangle\tilde{\lambda} &= -XX(\tilde{\lambda}) + \tilde{\varphi}X\tilde{\varphi}X(\tilde{\lambda}) + \xi\xi(\tilde{\lambda}) \\ &\quad +(\tilde{\nabla}_X X)\tilde{\lambda} - (\tilde{\nabla}_{\tilde{\varphi}X}\tilde{\varphi}X)\tilde{\lambda} - (\tilde{\nabla}_\xi\xi)\tilde{\lambda} \\ &= -X(A) + \tilde{\varphi}X(B) + \frac{1}{2\tilde{\lambda}}(A^2 - B^2).\end{aligned}$$

In order to compute scalar curvature τ of M, we will use (3.6)-(3.9). Defining the curvature tensor $\tilde{R}$, after some calculations we get

$$\begin{aligned}
&\tilde{R}(X,\tilde{\varphi}X)\tilde{\varphi}X \\
= \ & \tilde{\nabla}_X\tilde{\nabla}_{\tilde{\varphi}X}\ \tilde{\varphi}X - \tilde{\nabla}_{\tilde{\varphi}X}\tilde{\nabla}_X\ \tilde{\varphi}X - \tilde{\nabla}_{[X,\tilde{\varphi}X]}\tilde{\varphi}X \\
= \ & \tilde{\nabla}_X\left(-\frac{A}{2\tilde{\lambda}}X\right) - \tilde{\nabla}_{\tilde{\varphi}X}\left(-\frac{B}{2\tilde{\lambda}}X + (1-\tilde{\lambda})\xi\right) - \tilde{\nabla}_{-\frac{B}{2\tilde{\lambda}}X + \frac{A}{2\tilde{\lambda}}\tilde{\varphi}X + 2\xi}\tilde{\varphi}X \\
= \ & -X\left(\frac{A}{2\tilde{\lambda}}\right)X - \frac{A}{2\tilde{\lambda}}\tilde{\nabla}_X X + \tilde{\varphi}X\left(\frac{B}{2\tilde{\lambda}}\right)X + \frac{B}{2\tilde{\lambda}}\tilde{\nabla}_{\tilde{\varphi}X}X \\
& +\tilde{\varphi}X(\tilde{\lambda})\xi - (1-\tilde{\lambda})\tilde{\nabla}_{\tilde{\varphi}X}\xi + \frac{B}{2\tilde{\lambda}}\tilde{\nabla}_X\tilde{\varphi}X - \frac{A}{2\tilde{\lambda}}\tilde{\nabla}_{\tilde{\varphi}X}\ \tilde{\varphi}X - 2\tilde{\nabla}_\xi\ \tilde{\varphi}X \\
= \ & -X\left(\frac{A}{2\tilde{\lambda}}\right)X + \frac{A}{2\tilde{\lambda}}\frac{B}{2\tilde{\lambda}}\tilde{\varphi}X + \tilde{\varphi}X\left(\frac{B}{2\tilde{\lambda}}\right)X + \frac{B}{2\tilde{\lambda}}\left(-\frac{A}{2\tilde{\lambda}}\tilde{\varphi}X - (\tilde{\lambda}+1)\xi\right) \\
& +\tilde{\varphi}X(\tilde{\lambda})\xi + (1+\tilde{\lambda})(1-\tilde{\lambda})X + \frac{B}{2\tilde{\lambda}}\left(-\frac{B}{2\tilde{\lambda}}X + (1-\tilde{\lambda})\xi\right) + \frac{A}{2\tilde{\lambda}}\left(\frac{A}{2\tilde{\lambda}}X\right) + 2\left(\frac{\tilde{\mu}}{2}X\right) \\
= \ & \left[-X\left(\frac{A}{2\tilde{\lambda}}\right) + \tilde{\varphi}X\left(\frac{B}{2\tilde{\lambda}}\right) - \frac{B^2}{4\tilde{\lambda}^2} + \frac{A^2}{4\tilde{\lambda}^2} + (-\tilde{\lambda}^2+1) + \tilde{\mu}\right]X \\
= \ & \left[-\frac{1}{2}\left(\frac{X(A)\tilde{\lambda} - A^2}{\tilde{\lambda}^2} + \frac{\tilde{\varphi}X(B)\tilde{\lambda} - B^2}{\tilde{\lambda}^2}\right) + \frac{1}{4\tilde{\lambda}^2}(A^2 - B^2) + (-\tilde{\lambda}^2+1) + \tilde{\mu}\right]X \\
= \ & \left[\frac{1}{2}\frac{-X(A) + \phi X(B)}{\tilde{\lambda}} + \frac{1}{2\tilde{\lambda}^2}(A^2 - B^2) + \frac{1}{4\tilde{\lambda}^2}(A^2 - B^2) + (-\tilde{\lambda}^2+1) + \tilde{\mu}\right]X \\
= \ & \left[\frac{1}{2\tilde{\lambda}}\left(-X(A) + \tilde{\varphi}X(B) + \frac{1}{2\tilde{\lambda}}(A^2 - B^2)\right) + \frac{1}{2\tilde{\lambda}^2}(A^2 - B^2) + (-\tilde{\lambda}^2+1) + \tilde{\mu}\right]X \\
= \ & \left[\frac{1}{2\tilde{\lambda}}\triangle\tilde{\lambda} - \frac{1}{2\tilde{\lambda}^2}\parallel grad\tilde{\lambda}\parallel^2 - \tilde{\kappa} + \tilde{\mu}\right]X
\end{aligned}$$

and namely

$$\tilde{g}(\tilde{R}(X,\tilde{\varphi}X)\tilde{\varphi}X, X) = -\frac{1}{2\tilde{\lambda}}\triangle\tilde{\lambda} + \frac{1}{2\tilde{\lambda}^2}\parallel grad\tilde{\lambda}\parallel^2 + \tilde{\kappa} - \tilde{\mu}.$$

By using definition of scalar curvature, i.e. $\tau = TrQ = -\tilde{g}(QX,X)+\tilde{g}(Q\tilde{\varphi}X,\tilde{\varphi}X)+\tilde{g}(Q\xi,\xi)$, and using (3.3), we have

$$\begin{aligned}\tau &= -2\tilde{g}(\tilde{R}(X,\tilde{\varphi}X)\tilde{\varphi}X,X) + 2\tilde{g}(Q\xi,\xi)\\ &= \frac{1}{\tilde{\lambda}} \triangle \tilde{\lambda} - \frac{1}{\tilde{\lambda}^2} \parallel grad\tilde{\lambda} \parallel^2 -2(\tilde{\kappa}-\tilde{\mu}) + 4\tilde{\kappa}\\ &= \frac{1}{\tilde{\lambda}} \triangle \tilde{\lambda} - \frac{1}{\tilde{\lambda}^2} \parallel grad\tilde{\lambda} \parallel^2 +2(\tilde{\kappa}+\tilde{\mu}).\end{aligned}$$

The last equation gives (4.41). □

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