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CURVES OF GENERALIZED $AW(k)$-TYPE IN EUCLIDEAN SPACES

KADRI ARSLAN AND ŞABAN GÜVENÇ

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Abstract. In this study, we consider curves of generalized $AW(k)$-type of Euclidean n-space. We give curvature conditions of these kind of curves.

1. Introduction

In [7], the first author and A. West defined the notion of submanifolds of $AW(k)$-type. Since then, many works have been done related to these type of manifolds (for example, see [15], [5], [6] and [3]). In [15], the first author and B. Kılıç studied curves and surfaces of $AW(k)$-type. Further, in [27], C. Özgür and F. Gezgin carried out the results for where given in [5] to Bertrand curves and new special curves defined in [13] by S. Izumiya and N. Takeuchi. For example, in [5] and [15], the authors gave curvature conditions and characterizations related to these curves in $\mathbb{R}^n$. Also many results are obtained in Lorentzian spaces in [17], [22], [21], [18] and [8]. In [32], D. Yoon investigate curvature conditions of curves of $AW(k)$-type in Lie group G. Recently, C. Özgür and the second author studied some types of slant curves of pseudo-Hermitian $AW(k)$-type in [26].

In the present study, we give a generalization of AW(k)-type curves in Euclidean n-space $\mathbb{E}^n$. We also give curvature conditions of these type of curves.

2. Basic Notation

Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve in $\mathbb{E}^n$. The curve γ is called a Frenet curve of osculating order d if its higher order derivatives $\gamma'(s), \gamma''(s), ..., \gamma^{(d)}(s)$ $(d \leq n)$ are linearly independent and $\gamma'(s), \gamma''(s), ..., \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order d, one can associate an orthonormal d-frame $v_1, v_2, ..., v_d$ along γ (such that $\gamma'(s) = v_1$) called the Frenet d-frame and $(d-1)$ functions $\kappa_1, \kappa_2, ..., \kappa_{d-1} : I \to \mathbb{R}$ called the Frenet curvatures

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such that the Frenet formulas are defined in the usual way:

$$
\left.\begin{array}{l}
D_{v_1}v_1 = \kappa_1 v_2, \\
D_{v_1}v_2 = -\kappa_1 v_1 + \kappa_2 v_3, \\
\qquad ... \\
D_{v_1}v_i = -\kappa_{i-1}v_{i-1} + \kappa_i v_{i+1}, \\
D_{v_1}v_d = -\kappa_{d-1}v_{d-1},
\end{array}\right\} \tag{2.1}
$$

where $3 \leq i \leq d-1$.

3. Curves of Generalized $AW(k)$-type

Let γ be a unit speed curve in n-dimensional Euclidean space $\mathbb{E}^n$. By the use of Frenet formulas (2.1), we obtain the higher order derivatives of γ as follows:

$$
\left.\begin{array}{rl}
\gamma''(s) = & \kappa_1 v_2, \\
\gamma'''(s) = & -\kappa_1^2 v_1 + \kappa_1' v_2 + \kappa_1\kappa_2 v_3, \\
\gamma^{(iv)}(s) = & -3\kappa_1\kappa_1' v_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2\right) v_2 \\
& + \left(2\kappa_1'\kappa_2 + \kappa_1\kappa_2'\right) v_3 + \kappa_1\kappa_2\kappa_3 v_4, \\
\gamma^{(v)}(s) = & \left[-3(\kappa_1')^2 - 4\kappa_1\kappa_1'' + \kappa_1^4 + \kappa_1^2\kappa_2^2\right] v_1 \\
& + \left(\kappa_1''' - 6\kappa_1^2\kappa_1' - 3\kappa_1'\kappa_2^2 - 3\kappa_1\kappa_2\kappa_2'\right) v_2 \\
& + \left(3\kappa_1''\kappa_2 + 3\kappa_1'\kappa_2' - \kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2\right) v_3 \\
& + (3\kappa_1'\kappa_2\kappa_3 + 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3')v_4 + \kappa_1\kappa_2\kappa_3\kappa_4 v_5.
\end{array}\right\} \tag{3.1}
$$

Let us write

$$
\left.\begin{array}{l}
N_1 = \kappa_1 v_2, \\
N_2 = \kappa_1' v_2 + \kappa_1\kappa_2 v_3, \\
N_3 = \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4, \\
N_4 = \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 + \mu_5 v_5,
\end{array}\right\} \tag{3.2}
$$

where

$$
\begin{array}{l}
\lambda_2 = \kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2, \\
\lambda_3 = 2\kappa_1'\kappa_2 + \kappa_1\kappa_2', \\
\lambda_4 = \kappa_1\kappa_2\kappa_3
\end{array} \tag{3.3}
$$

and

$$
\begin{array}{l}
\mu_2 = \kappa_1''' - 6\kappa_1^2\kappa_1' - 3\kappa_1'\kappa_2^2 - 3\kappa_1\kappa_2\kappa_2', \\
\mu_3 = 3\kappa_1''\kappa_2 + 3\kappa_1'\kappa_2' - \kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2, \\
\mu_4 = 3\kappa_1'\kappa_2\kappa_3 + 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3', \\
\mu_5 = \kappa_1\kappa_2\kappa_3\kappa_4
\end{array} \tag{3.4}
$$

are differentiable functions.

We give the following definition:

Definition 3.1. Frenet curves are

i) of generalized $AW(1)$-type if they satisfy $N_4 = 0$,

ii) of generalized $AW(2)$-type if they satisfy

$$\|N_2\|^2 N_4 = \langle N_2, N_4\rangle N_2, \tag{3.5}$$

iii) of generalized $AW(3)$-type if they satify

$$\|N_1\|^2 N_4 = \langle N_1, N_4\rangle N_1, \tag{3.6}$$

iv) of generalized $AW(4)$-type if they satisfy

$$\|N_3\|^2 N_4 = \langle N_3, N_4\rangle N_3, \tag{3.7}$$

$v)$ of generalized $AW(5)$-type if they satisfy

$$N_4 = a_1 N_1 + b_1 N_2, \tag{3.8}$$

$vi)$ of generalized $AW(6)$-type if they satisfy

$$N_4 = a_2 N_1 + b_2 N_3, \tag{3.9}$$

$vii)$ of generalized $AW(7)$-type if they satisfy

$$N_4 = a_3 N_2 + b_3 N_3, \tag{3.10}$$

where a_i, b_i $(1 \leq i \leq 3)$ are non-zero real valued differentiable functions.

Remark 3.1. We use notation $GAW(k)$-type for the curves of generalized $AW(k)$-type.

Geometrically, a curve of $GAW(k)$-type is a curve whose fifth derivative's normal part is either zero or linearly dependent with one or two of its previous derivatives' normal parts.

Firstly, we give the following proposition:

Proposition 3.1. *The osculating order of a Frenet curve of any $GAW(k)$-type can not be bigger than or equal to* 5.

Proof. Let $\gamma : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^n$ be a Frenet curve of osculating order d. If γ is of any $GAW(k)$-type, since none of N_i $(1 \leq i \leq 3)$ contains a component in the direction of v_5, we find $\mu_5 = \kappa_1\kappa_2\kappa_3\kappa_4 = 0$. This concludes $d \leq 4$, which completes the proof. □

Using equations 3.2 and Definition 3.1, we obtain the following main theorem:

Theorem 3.1. *Let γ be a unit speed Frenet curve of osculating order $d \leq 4$ in n-dimensional Euclidean space $\mathbb{E}^n$. Then γ is*

$i)$ of $GAW(1)$-type if and only if

$$\mu_2 = \mu_3 = \mu_4 = 0,$$

$ii)$ of $GAW(2)$-type if and only if

$$\mu_4 = 0,$$
$$\kappa_1\kappa_2\mu_2 - \kappa_1'\mu_3 = 0,$$

$iii)$ of $GAW(3)$-type if and only if

$$\mu_3 = \mu_4 = 0,$$

$iv)$ of $GAW(4)$-type if and only if

$$\lambda_2\mu_3 - \lambda_3\mu_2 = 0,$$
$$\lambda_2\mu_4 - \lambda_4\mu_2 = 0,$$

$v)$ of $GAW(5)$-type if and only if

$$\mu_2 = a_1\kappa_1 + b_1\kappa_1',$$
$$\mu_3 = b_1\kappa_1\kappa_2,$$
$$\mu_4 = 0,$$

$vi)$ of $GAW(6)$-type if and only if

$$\mu_2 = a_2\kappa_1 + b_2\lambda_2,$$

$$\mu_3 = b_2\lambda_3,$$
$$\mu_4 = b_2\lambda_4,$$

vii) of $GAW(7)$-type if and only if

$$\mu_2 = a_3\kappa_1' + b_3\lambda_2,$$
$$\mu_3 = a_3\kappa_1\kappa_2 + b_3\lambda_3,$$
$$\mu_4 = b_3\lambda_4.$$

Proof. $i)$ Let γ be of $GAW(1)$-type. Then, from equations (3.2) and Definition 3.1 , we have $N_4 = \mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = 0$. Since v_2, v_3 and v_4 are linearly independent, we get $\mu_2 = \mu_3 = \mu_4 = 0$. The sufficiency is trivial.

$ii)$ Let γ be of $GAW(2)$-type. If we calculate $\|N_2\|^2$ and $\langle N_2, N_4\rangle$, by the use of equations (3.2) and (3.5), we obtain

$$[(\kappa_1')^2 + \kappa_1^2\kappa_2^2](\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = (\kappa_1'\mu_2 + \kappa_1\kappa_2\mu_3)(\kappa_1' v_2 + \kappa_1\kappa_2 v_3).$$

Since v_2, v_3 and v_4 are linearly independent, we find $\mu_4 = 0$ and $\kappa_1\kappa_2\mu_2 - \kappa_1'\mu_3 = 0$. Conversely, if $\mu_4 = 0$ and $\kappa_1\kappa_2\mu_2 - \kappa_1'\mu_3 = 0$, one can easily show that equation (3.5) is satisfied.

$iii)$ Let γ be of $GAW(3)$-type. We get $\|N_1\|^2 = \kappa_1^2$ and $\langle N_1, N_4\rangle = \kappa_1\mu_2$. So, if we write these equations in (3.6), we have

$$\kappa_1^2(\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = \kappa_1\mu_2(\kappa_1 v_2).$$

Thus, $\mu_3 = \mu_4 = 0$. Converse theorem is clear.

$iv)$ Let γ be of $GAW(4)$-type. We can easily calculate $\|N_3\|^2 = \lambda_2^2 + \lambda_3^2 + \lambda_4^2$ and $\langle N_3, N_4\rangle = \lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4$. So equation (3.7) gives us

$$(\lambda_2^2 + \lambda_3^2 + \lambda_4^2)(\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4) = (\lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4)(\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4).$$

Hence, we can write

$$(\lambda_2^2 + \lambda_3^2 + \lambda_4^2)\mu_2 = (\lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4)\lambda_2, \tag{3.11}$$

$$(\lambda_2^2 + \lambda_3^2 + \lambda_4^2)\mu_3 = (\lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4)\lambda_3, \tag{3.12}$$

$$(\lambda_2^2 + \lambda_3^2 + \lambda_4^2)\mu_4 = (\lambda_2\mu_2 + \lambda_3\mu_3 + \lambda_4\mu_4)\lambda_4. \tag{3.13}$$

If we multiply (3.11) with λ_3 and use equation (3.12), we find $\lambda_2\mu_3 - \lambda_3\mu_2 = 0$. Multiplying (3.11) with λ_4 and using equation (3.13), we have $\lambda_2\mu_4 - \lambda_4\mu_2 = 0$. Conversely, it is easy to show that equation (3.7) is satisfied if $\lambda_2\mu_3 - \lambda_3\mu_2 = 0$ and $\lambda_2\mu_4 - \lambda_4\mu_2 = 0$.

$v)$ Let γ be of $GAW(5)$-type. Then, in view of equations (3.8) and (3.2), we can write

$$\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_1(\kappa_1 v_2) + b_1(\kappa_1' v_2 + \kappa_1\kappa_2 v_3),$$

which gives us $\mu_2 = a_1\kappa_1 + b_1\kappa_1'$, $\mu_3 = b_1\kappa_1\kappa_2$ and $\mu_4 = 0$. Conversely, if these last three equations are satisfied, one can show that $N_4 = a_1 N_1 + b_1 N_2$.

$vi)$ Let γ be of $GAW(6)$-type. By definition, we have $N_4 = a_2 N_1 + b_2 N_3$, that is,

$$\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_2(\kappa_1 v_2) + b_2(\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4).$$

Since v_2, v_3 and v_4 are linearly independent, we can write

$$\mu_2 = a_2\kappa_1 + b_2\lambda_2,$$
$$\mu_3 = b_2\lambda_3,$$

$$\mu_4 = b_2\lambda_4.$$

Conversely, if these last equations are satisfied, then we easily show that $N_4 = a_2N_1 + b_2N_3$.

vii) Let γ be of $GAW(7)$-type. Then using equations (3.10) and (3.2), we obtain

$$\mu_2 v_2 + \mu_3 v_3 + \mu_4 v_4 = a_3(\kappa_1' v_2 + \kappa_1\kappa_2 v_3) + b_3(\lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4).$$

Thus

$$\mu_2 = a_3\kappa_1' + b_3\lambda_2,$$
$$\mu_3 = a_3\kappa_1\kappa_2 + b_3\lambda_3,$$
$$\mu_4 = b_3\lambda_4.$$

Conversely, let γ be a curve satisfying the last three equations. It is easily found that $N_4 = a_3N_2 + b_3N_3$. □

From now on, we consider Frenet curves whose first curvature κ_1 is a constant. We give curvature conditions of such a curve to be of $GAW(k)$-type. We can state following propositions:

Proposition 3.2. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(1)$-type if and only if it is a straight line or a circle.*

Proof. Let γ be of $GAW(1)$-type. Since κ_1 =constant, using (3.4) and Theorem 3.1, we find

$$\mu_2 = -3\kappa_1\kappa_2\kappa_2' = 0, \tag{3.14}$$

$$\mu_3 = -\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2 = 0, \tag{3.15}$$

$$\mu_4 = 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3' = 0. \tag{3.16}$$

If $\kappa_1 = 0$, then γ is a straight line and above three equations are satisfied. Let κ_1 be a non-zero constant. If $\kappa_2 = 0$, then γ is a circle and equations (3.14), (3.15) and (3.16) are satisfied again. Assume that $\kappa_2 \neq 0$. Then (3.14) gives us $\kappa_2' = 0$, that is, κ_2 is a constant. In this case, from equation (3.15), we get $(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) = 0$, which means $\kappa_1 = \kappa_2 = \kappa_3 = 0$. This is a contradiction. So $\kappa_2 = 0$.

Conversely, let γ be a straight line or a circle. Thus $\kappa_1 = 0$; or κ_1 =constant and $\kappa_2 = 0$. So $\mu_2 = \mu_3 = \mu_4 = 0$, which completes the proof. □

Proposition 3.3. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(2)$-type if and only if*

i) it is a straight line; or

ii) it is a circle; or

iii) it is a helix of order 3 or 4.

Proof. Let γ be of $GAW(2)$-type. Since κ_1 =constant, using (3.4) and Theorem 3.1, we obtain

$$\mu_4 = 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3' = 0, \tag{3.17}$$

$$\kappa_1\kappa_2\left(-3\kappa_1\kappa_2\kappa_2'\right) = 0. \tag{3.18}$$

One can easily see that κ_2 and κ_3 must be constants. Thus, γ can be a straight line, a circle or a helix of order 3 or 4. Conversely, if γ is one of these curves, the proof is clear using Theorem 3.1. □

Proposition 3.4. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(3)$-type if and only if*

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 satisfying the second order non-linear ODE

$$\kappa_2'' = \kappa_2(\kappa_1^2 + \kappa_2^2); \text{ or}$$

iv) it is a Frenet curve of osculating order 4 with

$$\kappa_2 = \frac{c}{\sqrt{\kappa_3}}$$

and its third curvature satisfies the second order non-linear ODE

$$\kappa_3'' - \frac{3(\kappa_3')^2}{2\kappa_3} + 2\kappa_3(\kappa_1^2 + \kappa_3^2) + 2c^2 = 0, \tag{3.19}$$

and where $c > 0$ is an arbitrary constant.

Proof. Let γ be of $GAW(3)$-type. Since κ_1 =constant, using (3.4) and Theorem 3.1, we have

$$\mu_3 = -\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2 = 0, \tag{3.20}$$

$$\mu_4 = 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3' = 0. \tag{3.21}$$

If $d = 1$ or $d = 2$, we obtain line and circle cases, both of which do not contradict above two equations. Let $d = 3$. Then κ_1 =constant> 0, $\kappa_2 > 0$ and $\kappa_3 = 0$. (3.21) is satisfied directly and (3.20) gives us

$$\kappa_2'' = \kappa_2(\kappa_1^2 + \kappa_2^2),$$

which is a second order non-linear ODE. Now, let $d = 4$. Thus, κ_1 =constant> 0, $\kappa_2 > 0$ and $\kappa_3 > 0$. If we solve (3.21), we find

$$\kappa_2 = \frac{c}{\sqrt{\kappa_3}}, \tag{3.22}$$

where $c > 0$ is an arbitrary constant. Then

$$\kappa_2' = \frac{-c\kappa_3'}{2\kappa_3^{3/2}},$$

$$\kappa_2'' = c.\left[\frac{3(\kappa_3')^2}{4\kappa_3^{5/2}} - \frac{\kappa_3''}{2\kappa_3^{3/2}}\right]. \tag{3.23}$$

If we multiply equation (3.20) with $\frac{\kappa_2}{\kappa_1}$, using (3.22) and (3.23), we obtain the second order non-linear ODE (3.19). Conversely, if γ is one of these curves, one can show that $\mu_3 = \mu_4 = 0$. □

Proposition 3.5. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(4)$-type if and only if*

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 satisfying the second order non-linear ODE

$$3\kappa_2(\kappa_2')^2 = (\kappa_1^2 + \kappa_2^2)\left[\kappa_2'' - \kappa_2(\kappa_1^2 + \kappa_2^2)\right]; \text{ or} \tag{3.24}$$

iv) it is a Frenet curve of osculating order 4 *with*

$$\kappa_2^2\kappa_3 = c.\left(\kappa_1^2+\kappa_2^2\right)^{3/2} \tag{3.25}$$

and its curvatures satify

$$3\kappa_2(\kappa_2')^2 = (\kappa_1^2+\kappa_2^2)\left[\kappa_2'' - \kappa_2(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right].$$

Here, c is an arbitrary constant.

Proof. Let γ be of $GAW(4)$-type. Since κ_1 =constant, using (3.3), (3.4) and Theorem 3.1, we find

$$(-\kappa_1^3-\kappa_1\kappa_2^2)(-\kappa_1^3\kappa_2-\kappa_1\kappa_2^3+\kappa_1\kappa_2''-\kappa_1\kappa_2\kappa_3^2)-(\kappa_1\kappa_2')(-3\kappa_1\kappa_2\kappa_2')=0, \tag{3.26}$$

$$(-\kappa_1^3-\kappa_1\kappa_2^2)(2\kappa_1\kappa_2'\kappa_3+\kappa_1\kappa_2\kappa_3')-(\kappa_1\kappa_2\kappa_3)(-3\kappa_1\kappa_2\kappa_2')=0. \tag{3.27}$$

(3.26) and (3.27) give us

$$3\kappa_2(\kappa_2')^2 = (\kappa_1^2+\kappa_2^2)\left[\kappa_2'' - \kappa_2(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right], \tag{3.28}$$

$$(2\kappa_1^2-\kappa_2^2)\kappa_3\kappa_2' + \kappa_2(\kappa_1^2+\kappa_2^2)\kappa_3' = 0. \tag{3.29}$$

Now, if $\kappa_1 = 0$, then γ is a straight line and equations (3.26) and (3.27) are satisfied. Let κ_1 be a non-zero constant. If $\kappa_2 = 0$, then γ is a circle. Let $\kappa_2 > 0$ and $\kappa_3 = 0$. Then, from equation (3.28), we obtain (3.24). Now, let $d = 4$. Then, using equation (3.29), we can write

$$\int \frac{(2\kappa_1^2-\kappa_2^2)}{\kappa_2(\kappa_1^2+\kappa_2^2)}d\kappa_2 + \int \frac{1}{\kappa_3}d\kappa_3 = \ln c.$$

Remember that $\kappa_1 > 0$ is a constant. So we find

$$2\ln(\kappa_2) - \frac{3}{2}\ln(\kappa_1^2+\kappa_2^2) + \ln(\kappa_3) = \ln c,$$

which gives us (3.25). Furthermore, γ must also satisfy (3.28). Conversely, if γ is one of the curves above, we can show that (3.26) and (3.27) are satisfied. □

Proposition 3.6. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(5)$-type if and only if*

i) it is a straight line; or

ii) it is a Frenet curve of osculating order 3 *with*

$$\kappa_2 \neq \text{constant}$$

and

$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2+\kappa_2^2\right); \text{ or}$$

iii) it is a Frenet curve of osculating order 4 *with*

$$\kappa_2 \neq \text{constant},\ \kappa_3 \neq \text{constant},$$

$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)$$

and

$$\kappa_2 = \frac{c}{\sqrt{\kappa_3}},$$

where $c > 0$ is an arbitrary constant.

Proof. Let γ be of $GAW(5)$-type. Since κ_1 =constant, by the use of Theorem 3.1 and equations (3.4), we have

$$-3\kappa_1\kappa_2\kappa_2' = a_1\kappa_1, \tag{3.30}$$

$$-\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2 = b_1\kappa_1\kappa_2, \tag{3.31}$$

$$2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3' = 0. \tag{3.32}$$

If $d = 1$, then γ is a straight line and above equations are satisfied. If $d = 2$, then γ is a circle. From (3.30), we find $a_1 = 0$, which contradicts the definition. Now, let $d = 3$. Then, using (3.30) and (3.31), we find

$$a_1 = -3\kappa_2\kappa_2',$$
$$b_1 = \frac{\kappa_2''}{\kappa_2} - \kappa_1^2 - \kappa_2^2.$$

Since a_1 and b_1 are non-zero functions, then $\kappa_2 \neq$constant and $\kappa_2'' \neq \kappa_2\left(\kappa_1^2 + \kappa_2^2\right)$. Finally, let $d = 4$. Then, equation (3.32) gives us

$$\kappa_2 = \frac{c}{\sqrt{\kappa_3}}, \tag{3.33}$$

where $c > 0$ is an arbitrary constant. In this case, from (3.30) and (3.31), we find

$$a_1 = -3\kappa_2\kappa_2', \tag{3.34}$$

$$b_1 = \frac{\kappa_2''}{\kappa_2} - \kappa_1^2 - \kappa_2^2 - \kappa_3^2. \tag{3.35}$$

Thus, (3.33) and (3.34) give us

$$\kappa_2 \neq \text{constant}, \ \kappa_3 \neq \text{constant}. \tag{3.36}$$

Also, from (3.35), we can write

$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right). \tag{3.37}$$

Converse proposition is trivial. □

Proposition 3.7. *Let $\gamma : I \subseteq \mathbb{R} \to \mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d \leq 4$ with κ_1 =constant. Then γ is of $GAW(6)$-type if and only if*

i) it is a straight line; or

ii) it is a circle; or

iii) it is a Frenet curve of osculating order 3 with

$$\kappa_2 \neq \textit{constant},$$
$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2 + \kappa_2^2\right)$$

and

$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2 + \kappa_2^2\right) + \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2 + \kappa_2^2}; \textit{ or}$$

iv) it is a Frenet curve of osculating order 4 with

$$\kappa_2 \neq \textit{constant},$$
$$\kappa_2 \neq \frac{c}{\sqrt{\kappa_3}},$$
$$\frac{2\kappa_2'}{\kappa_2} + \frac{\kappa_3'}{\kappa_3} = \frac{\kappa_2''}{\kappa_2'} - \frac{\kappa_2}{\kappa_2'}(\kappa_1^2 + \kappa_2^2 + \kappa_3^2)$$

and

$$\kappa_2'' \neq \kappa_2\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)+\frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2+\kappa_2^2}. \tag{3.38}$$

Here, $c>0$ is an arbitrary constant.

Proof. Let γ be of $GAW(6)$-type. Since κ_1 =constant, by the use of equations (3.3), (3.4) and Theorem 3.1, we have

$$-3\kappa_1\kappa_2\kappa_2' = a_2\kappa_1 + b_2\left(-\kappa_1^3-\kappa_1\kappa_2^2\right), \tag{3.39}$$

$$-\kappa_1^3\kappa_2-\kappa_1\kappa_2^3+\kappa_1\kappa_2''-\kappa_1\kappa_2\kappa_3^2 = b_2\kappa_1\kappa_2', \tag{3.40}$$

$$2\kappa_1\kappa_2'\kappa_3+\kappa_1\kappa_2\kappa_3' = b_2\kappa_1\kappa_2\kappa_3. \tag{3.41}$$

If $\kappa_1=0$, then γ is a straight line. Let $d=2$. Then γ is a circle and from (3.39), we obtain

$$a_2-b_2\kappa_1^2=0,$$

which is satisfied for some a_2, b_2 non-zero differentiable functions. (3.40) and (3.41) are also satisfied. Now, let $d=3$. Then we have

$$-3\kappa_2\kappa_2' = a_2-b_2\left(\kappa_1^2+\kappa_2^2\right), \tag{3.42}$$

$$\kappa_2''-\kappa_2(\kappa_1^2+\kappa_2^2) = b_2\kappa_2'. \tag{3.43}$$

Thus κ_2 can not be constant. So (3.42) and (3.43) give us

$$b_2=\frac{\kappa_2''}{\kappa_2'}-\frac{\kappa_2}{\kappa_2'}(\kappa_1^2+\kappa_2^2),$$

$$a_2=-3\kappa_2\kappa_2'+\frac{\kappa_2''}{\kappa_2'}(\kappa_1^2+\kappa_2^2)-\frac{\kappa_2}{\kappa_2'}(\kappa_1^2+\kappa_2^2)^2,$$

both of which must be non-zero. Finally, let $d=4$. From (3.40), $\kappa_2\neq$constant. In this case, by the use of (3.39), (3.40) and (3.41), we obtain

$$b_2=\frac{2\kappa_2'}{\kappa_2}+\frac{\kappa_3'}{\kappa_3}=\frac{\kappa_2''}{\kappa_2'}-\frac{\kappa_2}{\kappa_2'}(\kappa_1^2+\kappa_2^2+\kappa_3^2), \tag{3.44}$$

$$a_2=-3\kappa_2\kappa_2'+\frac{\kappa_2''}{\kappa_2'}(\kappa_1^2+\kappa_2^2)-\frac{\kappa_2}{\kappa_2'}(\kappa_1^2+\kappa_2^2)(\kappa_1^2+\kappa_2^2+\kappa_3^2). \tag{3.45}$$

Thus, from equation (3.44), we have

$$\kappa_2\neq\frac{c}{\sqrt{\kappa_3}},$$

where $c>0$ is an arbitrary constant. We also have (3.38) from (3.45).

Converse proposition is done easily. □

Proposition 3.8. *Let $\gamma: I\subseteq\mathbb{R}\to\mathbb{E}^n$ be a unit speed Frenet curve of osculating order $d\leq 4$ with κ_1 =constant. Then γ is of $GAW(7)$-type if and only if*

i) it is a straight line; or

ii) it is a Frenet curve of osculating order 3 satisying

$$\kappa_2\neq \text{constant},$$

$$\kappa_2''\neq\frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2+\kappa_2^2}+\kappa_2(\kappa_1^2+\kappa_2^2);\ \textit{or}$$

iv) it is a Frenet curve of osculating order 4 *satisfying*

$$\kappa_2 \neq \text{constant},$$

$$\kappa_2 \neq \frac{c}{\sqrt{\kappa_3}},$$

$$\frac{3\kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2} = \frac{2\kappa_2'}{\kappa_2} + \frac{\kappa_3'}{\kappa_3},$$

$$\kappa_2'' \neq \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2+\kappa_2^2} + \kappa_2(\kappa_1^2+\kappa_2^2+\kappa_3^2),$$

where c *is an arbitrary constant.*

Proof. Let γ be of $GAW(7)$-type. If we use equations (3.3), (3.4) and Theorem 3.1, we obtain

$$-3\kappa_1\kappa_2\kappa_2' = b_3(-\kappa_1^3 - \kappa_1\kappa_2^2), \tag{3.46}$$

$$-\kappa_1^3\kappa_2 - \kappa_1\kappa_2^3 + \kappa_1\kappa_2'' - \kappa_1\kappa_2\kappa_3^2 = a_3\kappa_1\kappa_2 + b_3\kappa_1\kappa_2', \tag{3.47}$$

$$2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3' = b_3\kappa_1\kappa_2\kappa_3. \tag{3.48}$$

If $d = 1$, γ is a straight line. Let $d = 2$. Then, from (3.46), we find $\kappa_1 = 0$. This is a contradiction. Let $d = 3$. Then, using (3.46), κ_2 can not be constant. By the use of (3.46) and (3.47), we get

$$b_3 = \frac{3\kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2}, \tag{3.49}$$

$$a_3 = \frac{\kappa_2''}{\kappa_2} - \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_2\left(\kappa_1^2+\kappa_2^2\right)} - (\kappa_1^2+\kappa_2^2),$$

both of which are non-zero differentiable functions. Again, equation (3.49) requires κ_2 is not a constant. We also have

$$\kappa_2'' \neq \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2+\kappa_2^2} + \kappa_2(\kappa_1^2+\kappa_2^2).$$

Now, let $d = 4$. Then, using equations (3.46), (3.47) and (3.48), we obtain

$$b_3 = \frac{3\kappa_2\kappa_2'}{\kappa_1^2+\kappa_2^2} = \frac{2\kappa_2'}{\kappa_2} + \frac{\kappa_3'}{\kappa_3}, \tag{3.50}$$

$$a_3 = \frac{\kappa_2''}{\kappa_2} - \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_2\left(\kappa_1^2+\kappa_2^2\right)} - (\kappa_1^2+\kappa_2^2+\kappa_3^2),$$

which give us

$$\kappa_2 \neq \text{constant},$$

$$\kappa_2 \neq \frac{c}{\sqrt{\kappa_3}},$$

$$\kappa_2'' \neq \frac{3\kappa_2\left(\kappa_2'\right)^2}{\kappa_1^2+\kappa_2^2} + \kappa_2(\kappa_1^2+\kappa_2^2+\kappa_3^2).$$

Here, c is an arbitrary constant.

Converse proposition is trivial. □

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Department of Mathematics, Uludag University, Görükle Campus, Bursa-TURKEY
E-mail address: arslan@uludag.edu.tr

Department of Mathematics, Balikesir University, Çağiş Campus, Balikesir-TURKEY
E-mail address: sguvenc@balikesir.edu.tr