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# Some Decompositions of UP-ideals and Proper UP-filters

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**Abstract:** In this article we show some decompositions of UP-ideals and proper UP-filters of UP-algebras.

## 1. Introduction

The idea of the algebraic structure of 'UP-algebra' was introduced and analyzed by A. Iampan in his article [1]. The author has introduced and analyzed the concepts of UP-subalgebra and UP-ideal and their mutual connections. In the article [7] were introduced the concept of UP-filter in UP algebras by J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan. Also, the article [3], written by P. Mosrijai, A. Satirad and A. Iampan, refers to some properties of UP-ideals in UP-algebras as well as the versions of the isomorphism theorems for this type of algebras. In forthcoming article [2], written by Y. B. Jun and A. Iampan, a decomposition of UP-filters was described through some special sub-sets in UP-algebras.

This author introduced in [4] the concept of proper UP-filter in UP-algebras on something different way then it is common in the available literature. In addition, in [4] he established the connection between UP-ideals and proper UP-filters. This author, in his articles [5, 6], also dealt with the properties of the UP-Ideal and the proper UP-filters in UP-algebras.

In this article, the author further develops the idea of a proper UP-filter by identifying some of the fundamental features of this concept. In this article, he develops and expands Jun and Iampan's idea of a decomposition of a UP-filters in UP-algebras to decomposition of UP-ideals and proper UP-filters of UP-algebras.

## 2. Preliminaries

Let us recall the definition of UP-algebra.

**Definition 2.1** ([1], Definition 1.3)**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP- algebra* if it satisfies the following axioms:
(UP - 1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP - 2): $(\forall x \in A)(0 \cdot x = x)$,
(UP - 3): $(\forall x \in A)(x \cdot 0 = 0)$,
(UP - 4): $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \Longrightarrow x = y)$.

In this algebraic structure, the order relation is determined in the following way

$$(\forall x, y \in A)(x \leq y \Longleftrightarrow x \cdot y = 0).$$

The following definition gives the concept of UP-ideals in a UP-algebra.

**Definition 2.2** ([1], Definition 2.1)**.** Let $A$ be a UP-algebra. A subset $B$ of $A$ is called a *UP-ideal* of $A$ if it satisfies the following properties:
(1) $0 \in B$, and
(2) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in B \wedge y \in B \Longrightarrow x \cdot z \in B)$.

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Some of fundamental properties of UP-ideals is given in the following Proposition

**Proposition 2.3** ( [1], Theorem 2.3 and Corollary 2.4)**.** *Let A be a UP-algebra and B a UP-ideal of A. Then*
(3) $(\forall x, y \in A)(((x \cdot y \in B) \wedge x \in B) \Longrightarrow y \in B)$;
(4) $(\forall x, y \in A)(y \in B \Longrightarrow x \cdot y \in B)$;
(5) $(\forall x, y \in A)((x \leqslant y \wedge x \in B) \Longrightarrow y \in B)$.

The property (3) in previous Proposition of UP-ideals $B$ in a UP-algebra $A$ can be viewed as a kind of consistency with respect to the operation $'.'$ in the following sense $(\forall y \in A)(By \subseteq B \Longrightarrow y \in B)$. On the other hand, on the property (4) in the previous Proposition, it can be viewed as a classical property of the right ideal in an algebraic structure in the following sense $AB \subseteq B$. The property (5) in the previous Proposition suggests that the UP-ideals of UP-algebra can be viewed as some kind of upper subsets in UP-algebra $A$.

In the article [6] it has been shown that the concept of UP-ideals is completely determined by the conditions (3) and (4) in the previous Proposition.

The commitment and intention of the author in his short article [4] was to construct a substructure $G$ in UP-algebras that will have the following property

$$(\forall x, y \in A)((y \in G \wedge x \leq y) \Longrightarrow x \in G)$$

and has a standard attitude toward the UP-ideal. This was done by introducing the concept of a proper UP-filter by the following way.

**Definition 2.4** ([4], Definition 3.1)**.** Let $A$ be a UP-algebra. A subset $G$ of $A$ is called a *proper UP-filter* of $A$ if it satisfies the following properties:
(6) $\neg(0 \in G)$, and
(7) $(\forall x, y, z \in A)((\neg(x \cdot (y \cdot z) \in G) \wedge x \cdot z \in G) \Longrightarrow y \in G)$.

In the mentioned article it was shown

**Proposition 2.5** ([4], Theorem 3.4 and Corollary 3.5)**.** *Let A be a UP-algebra and G a proper UP-filter of A. Then*
(8) $(\forall x, y \in A)((\neg(x \cdot y \in G) \wedge y \in G) \Longrightarrow x \in G)$.
(9) $(\forall x, y \in A)(x \cdot y \in G \Longrightarrow y \in G)$.
(10) $(\forall x, y \in A)((x \leqslant y \wedge y \in G) \Longrightarrow x \in G)$.

The property (8) in the previous Proposition claims that a proper UP filter $G$ is not a UP-subalgebra in a UP-algebra $A$. The property (9) can be viewed as right consistency of a proper filter $G$ of an UP-algebra $A$. Finally, the property (10) suggests that proper UP-filters in a UP-algebra can be viewed as some sort of down subsets in that UP-algebra.

In the article [5] it has been shown that a proper UP-filter $G$ is completely determined by the conditions (8) and (9) in the previous Proposition.

The notations and notions appearing in this text are not predefined, the reader can find in the articles [1, 3, 4].

## 3. The main results

Let $A$ be a UP-algebra and let $a, b$ be elements of $A$. In what follows, we will use the notations

$$\langle a] = \{x \in A : x \leqslant a\}, \quad [a\rangle = \{x \in A : a \leqslant x\} \text{ and}$$
$$X_{ab} = \{z \in A : b(az) = 0\}, \quad Y_{ab} = \{y \in A : a(yb) = 0\}.$$

Note that $\langle 0] = A$ and $[0\rangle = \{0\}$

The idea of decomposition of a substructure in a UP-algebra first time is presented in the article [2]. In this text, we develop and expand this idea on any substructure of a UP-algebra.

Our first result relates to a decomposition of UP-ideals.

**Theorem 3.1.** *Let B be a UP-ideals of UP-algebra A. Then*

$$B = \bigcup_{a \in B} [a\rangle.$$

*Proof.* Let $B$ be a UP-idea of $A$. Then $a \in [a\rangle$ because $a \leqslant a$ is valid. Thus, $B \subseteq \bigcup_{a \in B}[a\rangle$.
Suppose that $y \in \bigcup_{a \in B}[a\rangle$. Then there exists an element $x \in B$ such that $y \in [x\rangle$. Thus $x \leqslant y$. Then $y \in B$ by (5). So, $\bigcup_{a \in B}[a\rangle \subseteq B$. Therefore, $B = \bigcup_{a \in B}[a\rangle$. □

**Theorem 3.2.** *Each UP-ideal B of an UP-algebra A can be decomposed in the following way*

$$G = \bigcup_{x,y \in B} X_{xy}.$$

*Proof.* Let $B$ be a UP-ideal of $A$ and let $b \in B$. Since $X_{0y} = [y\rangle$ for any $y \in A$ we have $b \in [b\rangle = X_{0b} \subseteq \bigcup_{x,y \in B} X_{xy}$ by Theorem 3.1.
Opposite, let $z \in \bigcup_{x,y \in B} X_{xy}$. Then there exist elements $x, y \in B$ such that $z \in X_{xy}$. This means $y \cdot (x \cdot z) = 0$, i.e. $y \leqslant xz$. From $y \in B$ and $y \leqslant xz$ follows $xz \in B$ by (5). From this by (3) we got $z \in B$. Then $\bigcup_{x,y \in B} X_{xy} \subseteq G$. □

Our second result in this article relates to a decomposition of the proper UP-filter.

**Theorem 3.3.** *Let G be a proper UP-filter in a UP-algebra A. Then*

$$G = \bigcup_{a \in G} \langle a].$$

*Proof.* Let $G$ be a proper UP-filter of a UP-algebra $A$ and $a \in G$. Then $a \in \langle a]$ because $a \leqslant a$. Thus, $G \subseteq \bigcup_{a \in G} \langle a]$. If $x \in \bigcup_{a \in G} \langle a]$, then there exists $y \in G$ such that $x \in \langle y]$. This means $x \leqslant y$. From this and from $y \in G$ by (10) follows $x \in G$. Therefore, $G = \bigcup_{a \in G} \langle a]$. □

**Theorem 3.4.** *Each proper UP-filter G of an UP-algebra A can be decomposed in the following way*

$$G = \bigcup_{x,z \in A} \{Y_{xz} : xz \in G\}.$$

*Proof.* Let $G$ be a proper UP-filter of the UP-algebra $A$. Since $Y_{0z} = \langle z]$ for any $z \in A$ , we have $G = \bigcup_{z \in G} \langle z] = \bigcup_{z \in G} Y_{0z} \subseteq \bigcup_{x,z \in A} \{Y_{xz} : xz \in G\}$ by Theorem 3.3.
Conversely, let $y \in \bigcup_{x,z \in A} \{Y_{xz} : xz \in G\}$. Then there exist elements $x, z \in A$ such that $xz \in G$ and $y \in Y_{xz}$. Then $x(yz) = 0$. Thus $\neg(x(yz) \in G)$ by (6). From this and from $xz \in G$ follows $y \in G$ by (7). Therefore, $\bigcup_{x,z \in A} \{Y_{xz} : xz \in G\} \subseteq G$. This and prior inclusion give $G = \bigcup_{x,z \in A} \{Y_{xz} : xz \in G\}$, which was to be proven. □

We emphasize that family members of $\mathfrak{G}_G = \{Y_{ab} : a, b \in A\}$ have the following characteristics:
(a) $Y_{ab} = \{y \in A : a \leqslant yb\}$.
(b) $Y_{0b} = \langle b]$.
(c) $Y_{a0} = A$. From the conditions $ab \in G$, it follows that $\neg(A \in \mathfrak{G}_G)$.
(d) For any $a, b \in A$ such that $ab \in G$ the following $Y_{ab} \subseteq G$ holds.

We also have the elements of family $\mathfrak{B}_B = \{X_{ab} : a \in B \wedge b \in B\}$ having the following characteristics
(e) $X_{ab} = \{z \in A : b \leqslant az\}$;
(f) $X_{0b} = [b\rangle$;
(g) $X_{a0} = \{0\}$;
(h) $(\forall a, b \in B)(X_{ab} \subseteq B)$.

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