Konuralp Journal of Mathematics, 8 (2) (2020) 234-243

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Higher Order Splitting Method for Numerical Solution Of Nonlinear Coupled System Viscous Burger's Equation

Nuri Murat Yağmurlu[1*], Yusuf Uçar[1] and İhsan Çelikkaya[2]

[1]*Department of Mathematics, Faculty of Science and Arts, İnönü University, Malatya, Turkey*
[2]*Batman University, Batman, Turkey*
[*]*Corresponding author*

Abstract

In this study, the numerical solutions of nonlinear coupled system viscous Burgers equation with appropriate initial and boundary conditions are going to be obtained by Strang splitting method and also Ext4 and Ext6 methods obtained by extrapolation technique. To apply splitting methods, coupled system viscous Burgers equation split up into two subequation, one is linear and the other is nonlinear equation. Cubic B-spline functions and derivatives are used for the dependent variables u(x,t) and v(x,t) in each sub-equation obtained. Numerical schemas were obtained by applying each sub-equation of the collocation finite element method and the stability analyzes were investigated by the von-Neumann method. The effectiveness of the method was tested on three commonly used test problems in the literature. It was observed that the calculated numerical results were in agreement with the exact solution and compared with the previous studies.

Keywords: *Operator Splitting method; Coupled viscous Burgers equation; Cubic B-spline functions; Collocation method; Strang splitting.*
2010 Mathematics Subject Classification: *65L60; 65N22; 68W30; 65N12.*

1. Introduction

This work is concerned with the numerical solutions of the coupled viscous Burgers equation derived by Episov [1], a simple model of sedimentation or evolution in liquid suspension or colloids under the influence of gravity, of two types of particle-sized volume concentrations. Episov said that if the weights of the particles are heavier than the fluid that envelops them, the motion of the particles will result in sedimentation, but creaming if they are lighter. The coupled system of viscous Burgers equation is given

$$u_t - u_{xx} + \eta u u_x + \alpha (uv)_x = 0, x \in [a,b], t \in [0,T], \tag{1.1}$$

$$v_t - v_{xx} + \eta v v_x + \beta (uv)_x = 0, x \in [a,b], t \in [0,T], \tag{1.2}$$

with the following initial

$$u(x,0) = \phi_1(x), \quad v(x,0) = \phi_2(x),$$

and boundary conditions

$$\begin{aligned} u(a,t) &= f_1(t), \quad u(b,t) = f_2(t) \\ v(a,t) &= g_1(t), \quad v(b,t) = g_2(t) \end{aligned}$$

Here η is a real constant, $\phi_1(x)$, $\phi_2(x)$, $f_1(t)$, $f_2(t)$, $g_1(t)$ $g_2(t)$ are given functions, α and β are Brownian diffusivity, system parameters such as Stokes velocity or Peclet number originating from particle gravity [2].
In recent years, active research efforts focused on nonlinear dynamical systems that emerge in various fields, such as fluid mechanics, plasma physics, biology, hydrodynamics, solid-state physics and optical fibers. These nonlinear phenomena are often referred to as nonlinear wave equations [3]. Thus, the analytical or numerical solution of these nonlinear wave equations is becoming very important in the approach

Email addresses: murat.yagmurlu@inonu.edu.tr (Nuri Murat Yağmurlu), yusuf.ucar@inonu.edu.tr (Yusuf Uçar), ihsancelikkaya85@gmail.com (İhsan Çelikkaya)

theory of fluids. The solutions of the coupled Burgers equation were found by different researchers in various ways. Jain and Kadalbajoo [4] combined technique based on linear approximation and invariant embedding is proposed for solving coupled Burgers' equations over irregular regions. Dehghan et al. [5] have solved coupled Burgers equation using a combination of Adomian decomposition method (modified Adomian decomposition method) and Pade approximation. Mittal and Arora [6] proposed a numerical method for the numerical solution of a coupled system of viscous Burgers' equation with appropriate initial and boundary conditions, by using the cubic B-spline collocation scheme on the uniform mesh points. Mittal and Tripathi [7] have used modified cubic B-spline functions to obtain approximate solutions of coupled Burgers' equations by collocation method. İslam et al. [8] have obtained the numerical solution of the transient nonlinear coupled Burgers' equations by a Local Radial Basis Functions Collocation Method (LRBFCM) for large values of Reynolds number. Rashid et al. [9] have considered Chebyshev–Legendre Pseudo-Spectral (CLPS) method for solving coupled viscous Burgers (VB) equations. Rashid and İsmail [10] have used The Fourier pseudo-spectral method for numerical solutions of one-dimensional coupled system of viscous Burgers equations. Khater et al. [11] have obtained solutions of Burgers'-type equations using a spectral collocation method based on differentiated Chebyshev polynomials. Kutluay and Uçar [12] have used Galerkin quadratic B-spline finite element method to numerical solution of coupled Burgers' equation. Uçar [13] have solved coupled Burgers equations via quintic B-spline collocation finite element method in his Ph.D. thesis. Srivastava et al. [14] have implemented an implicit logarithmic finite difference method (I-LFDM) for the numerical solution of one dimensional coupled nonlinear Burgers equation. Li et al. [15] have applied a new lattice Boltzmann model for coupled Burgers' equations is proposed through selecting proper distribution functions. Lai and Ma [16] have proposed lattice Boltzmann model for the coupled nonlinear system of viscous Burgers' equation using the double evolutionary equations. Mokhtari et al. [17] have applied the generalized differential quadrature method (GDQM) to obtain numerical solution coupled Burgers' equations. Mittal and Jiwari [18] have solved the coupled viscous Burgers' equations by using the differential quadrature method. The exact solution of the equation has been obtained by Kaya [19] using Adomian Decomposition method and Soliman [20] presented modified extended tanh-function method to obtain its exact solution. Abazari and Borhanifar [21] obtained the numerical/analytical solutions of the Burgers and coupled Burgers equations by Differential Transformation Method (DTM). Başhan [22] has considered a numerical treatment of the coupled viscous Burgers' equation in the presence of very large Reynolds number Uçar et al. [23] have investigated numerical solutions and stability analysis of modified Burgers equation via modified cubic B-spline differential quadrature method. Başhan et al. [24] have applied B-spline differential quadrature method for the modified Burgers' equation. Karakoç et al. [25] have applied two different methods for numerical solution of the modified Burgers' equation and so on.
The main purpose of this work is to obtain numerical solutions of the nonlinear coupled system viscous Burger's equation using the operator splitting methods together with the cubic B-spline collocation finite element method. Numerical solutions of several nonlinear partial differantial equations of different kind have been obtained by using operator splitting methods with various numerical methods. For example, they examined the effect of operator splitting methods on the solution of the Bahar and Gurarslan [26] advection-diffusion equations. Holden et al. [27] have provided a new analytical approach to operator splitting for equations of the type $u_t = Au + uu_x$. Holden et al. [28] have applied the method of operator splitting on the generalized Korteweg–de Vries (KdV) equation. Wang [29] have used the split-step finite difference method to solve various nonlinear Schrödinger equations including coupled ones and so on.

2. Splitting methods

Operator splitting is a successful approach in numerical investigation of splitting complex and high-dimensional equations. The basic idea of the operator splitting methods is divide and conquer: split the complex problem into a sequence of simpler subproblems and solve these subproblems [30]. The situation including two linear operators is going to be considered. Now let us consider the following Cauchy problem

$$\frac{du(t)}{dt} = Cu(t), \quad t \in [0,T], \quad u(0) = u_0 \tag{2.1}$$

and assume that it is split as $C = A + B$. Eq. (2.1) can be seen as a semi-discretization of a linear PDE with a homogeneous periodic boundary condition. Here, an initial function $u_0 \in X$ is assumed to be a finite linear operator in X Banach space, with $C = A + B$, $A, B : X \to X$. There is also a norm associated with the X-space, and if A and B are matrices, then this norm is the Euclid norm [31]. The formal solution of Eq.(2.1) is in the form of $u(t_{n+1}) = e^{tC}u(t_n)$. Then, $\Delta t = t_{n+1} - t_n$ is being the simplest splitting method, one can get

$$u(t_{n+1}) \simeq e^{\Delta tB} e^{\Delta tA} u(t_n) \tag{2.2}$$

If the operators A and B are commute, then the method is exact. (2.2) is the simplest splitting technique and refers to the solution of two sub-problems as follows

$$\frac{du(t)}{dt} = Au^*(t),\ u^*(0) = u_0 \text{ on } [0,\Delta t],$$
$$\frac{du(t)}{dt} = Bu^{**}(t),\ u^{**}(0) = u^*(\Delta t) \text{ on } [0,\Delta t]$$

Thus, the solutions at the desired time step are calculated by $u^{**}(\Delta t)$. This technique is called the $A-B$ splitting scheme. One can easily obtain the $B-A$ splitting scheme by replacing the locations of operators A and B [32].

2.1. Symmetric Strang splitting

For better accuracy, Strang [33] first handles the following scheme

$$u(\Delta t) = \frac{1}{2}\left[u_{AB}(\Delta t) + u_{BA}(\Delta t)\right] \tag{2.3}$$

Where u_{AB} and u_{BA} are solutions calculated by AB and BA splitting schemes, respectively. Since each operator needs to be calculated twice in this scheme, the calculation cost is high. In place of Eq. (2.3), the symmetric schme $u(t_{n+1}) \simeq \left(e^{\frac{\Delta t}{2}A}e^{\frac{\Delta t}{2}B}\right)\left(e^{\frac{\Delta t}{2}B}e^{\frac{\Delta t}{2}A}\right)u(t_n) = e^{\frac{\Delta t}{2}A}e^{\Delta tB}e^{\frac{\Delta t}{2}A}u(t_n)$ is proposed due to its low computational cost [34]. This scheme can be explicitly stated as follows

$$\begin{aligned}
\frac{du^*(t)}{dt} &= Au^*(t),\ u^*(0) = u_0 \text{ on } [0,\Delta t/2],\\
\frac{du^{**}(t)}{dt} &= Bu^{**}(t),\ u^{**}(0) = u^*(\Delta t/2) \text{ on } [0,\Delta t],\\
\frac{du^{***}(t)}{dt} &= Au^{***}(t),\ u^{***}(0) = u^{**}(\Delta t) \text{ on } [0,\Delta t/2]
\end{aligned} \tag{2.4}$$

Finally, the numerical schemes are solved with the term $u^{***}(\Delta t/2)$. As it is seen in the Eq. (2.4), the term $u^*(0)$ is calculated from the original initial condition of the problem, the other two initial conditions are taken from the previous calculated ones. If the scheme in Eq. (2.4) is called "$A-B-A$", one can obtain the scheme "$B-A-B$" in a similar way. Now, in order to increase and improve the convergence, the extrapolation techniques $\frac{4}{3}\varphi_{\frac{\Delta t}{2}} * \varphi_{\frac{\Delta t}{2}} - \frac{1}{3}\varphi_{\Delta t}$ and $\frac{81}{40}\varphi_{\frac{\Delta t}{3}} * \varphi_{\frac{\Delta t}{3}} * \varphi_{\frac{\Delta t}{3}} - \frac{16}{15}\varphi_{\frac{\Delta t}{2}} * \varphi_{\frac{\Delta t}{2}} + \frac{1}{24}\varphi_{\Delta t}$ given in [35] are going to be used. If these techniques are applied to Strang splitting technique, the fourth and sixth order methods as follows are obtained, respectively

$$\begin{aligned}
\text{Ext4} &= \frac{4}{3}\left(S_{\frac{\Delta t}{2}}\right)^2 - \frac{1}{3}S_{\Delta t}\\
&= \frac{4}{3}\varphi^{[A]}_{\frac{\Delta t}{4}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{4}} - \frac{1}{3}\varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\Delta t} \circ \varphi^{[A]}_{\frac{\Delta t}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\text{Ext6} &= \frac{81}{40}\left(S_{\frac{\Delta t}{3}}\right)^3 - \frac{16}{15}\left(S_{\frac{\Delta t}{2}}\right)^2 + \frac{1}{24}S_{\Delta t}\\
&= \frac{81}{40}\varphi^{[A]}_{\frac{\Delta t}{6}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \circ \varphi^{[A]}_{\frac{\Delta t}{3}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \circ \varphi^{[A]}_{\frac{\Delta t}{3}} \circ \varphi^{[B]}_{\frac{\Delta t}{3}} \varphi^{[A]}_{\frac{\Delta t}{6}}\\
&- \frac{16}{15}\varphi^{[A]}_{\frac{\Delta t}{4}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\frac{\Delta t}{2}} \circ \varphi^{[A]}_{\frac{\Delta t}{4}} + \frac{1}{24}\varphi^{[A]}_{\frac{\Delta t}{2}} \circ \varphi^{[B]}_{\Delta t} \circ \varphi^{[A]}_{\frac{\Delta t}{2}}.
\end{aligned}$$

3. Cubic B-spline functions and its derivatives

Let us assume the solution domain $[a,b]$ is closed interval, and a smooth finite uniform fragment x_m, $m = 0,1,...,N$ of this region is $a = x_0 < x_1 < ... < x_N = b$. Cubic B-spline functions can be stated as follows, with $\Phi_m(x)$, $m = -1(1)N+1$, if the distance between two consecutive node points is expressed as $h = x_{m+1} - x_m$ on the basis of x_m node points on $[a,b]$ as shown by the [36]

$$\Phi_m(x) = \frac{1}{h^3}\begin{cases}
(x-x_{m-2})^3, & x \in [x_{m-2},x_{m-1}]\\
h^3 + 3h^2(x-x_{m-1}) + 3h(x-x_{m-1})^2 - 3(x-x_{m-1})^3, & x \in [x_{m-1},x_m]\\
h^3 + 3h^2(x_{m+1}-x) + 3h(x_{m+1}-x)^2 - 3(x_{m+1}-x)^3, & x \in [x_m,x_{m+1}]\\
(x_{m+2}-x)^3, & x \in [x_{m+1},x_{m+2}]\\
0, & otherwise
\end{cases} \tag{3.1}$$

An approximation to the exact solution $u(x,t)$ and $v(x,t)$ can be expressed as follows in terms of cubic B-spline functions (3.1) and time dependent parameters $\delta_m(t)$ and $\gamma_m(t)$

$$U(x,t) \simeq \sum_{m=-1}^{N+1}\delta_m(t)\Phi_m(x),\ V(x,t) \simeq \sum_{m=-1}^{N+1}\gamma_m(t)\Phi_m(x) \tag{3.2}$$

Using the Eqs.(3.1) and (3.2), the nodal values of functions $u(x,t)$ and $v(x,t)$ and their first and second order derivatives are found as follows

$$\begin{aligned}
U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}\\
V_m &= V(x_m) = \gamma_{m-1} + 4\gamma_m + \gamma_{m+1}\\
U'_m &= U'(x_m) = \frac{3}{h}(\delta_{m+1} - \delta_{m-1})\\
V'_m &= V'(x_m) = \frac{3}{h}(\gamma_{m+1} - \gamma_{m-1})\\
U''_m &= U''(x_m) = \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1})\\
V''_m &= V''(x_m) = \frac{6}{h^2}(\gamma_{m-1} - 2\gamma_m + \gamma_{m+1})
\end{aligned} \tag{3.3}$$

Here $'$ and $''$ denote the first order and second order derivatives with respect to space variable x, respectively.

4. Application of the method

In the present study, the coupled viscous Burgers equation is split as follows

$$u_t = u_{xx} \tag{4.1}$$

$$u_t = -\eta u u_x - \alpha (uv)_x \tag{4.2}$$

$$v_t = v_{xx} \tag{4.3}$$

$$v_t = -\eta v v_x - \beta (uv)_x \tag{4.4}$$

In order to apply the method to the Eqs. (4.1), (4.2), (4.3) and (4.4) if forward finite difference for time derivative and Crank-Nicolson finite difference approximation for space derivative are used, the following equalities are obtained

$$\left[\frac{u^{n+1}-u^n}{\Delta t}\right] - \left[\frac{u_{xx}^{n+1}+u_{xx}^n}{2}\right] = 0, \tag{4.5}$$

$$\left[\frac{u^{n+1}-u^n}{\Delta t}\right] + \eta\left[\frac{(uu_x)^{n+1}+(uu_x)^n}{2}\right] + \alpha\left[\frac{(vu_x)^{n+1}+(vu_x)^n}{2}\right] + \alpha\left[\frac{(uv_x)^{n+1}+(uv_x)^n}{2}\right] = 0 \tag{4.6}$$

and

$$\left[\frac{v^{n+1}-v^n}{\Delta t}\right] - \left[\frac{v_{xx}^{n+1}+v_{xx}^n}{2}\right] = 0, \tag{4.7}$$

$$\left[\frac{v^{n+1}-v^n}{\Delta t}\right] + \eta\left[\frac{(vv_x)^{n+1}+(vv_x)^n}{2}\right] + \beta\left[\frac{(vu_x)^{n+1}+(vu_x)^n}{2}\right] + \beta\left[\frac{(uv_x)^{n+1}+(uv_x)^n}{2}\right] = 0. \tag{4.8}$$

In Eq. (4.6) in place of nonlinear term, the following approximation proposed by Rubin and Graves is used [37]

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u^n u_x^{n+1} - (uu_x)^n,$$
$$(vu_x)^{n+1} = v^{n+1}u_x^n + v^n u_x^{n+1} - (vu_x)^n,$$
$$(uv_x)^{n+1} = u^{n+1}v_x^n + u^n v_x^{n+1} - (uv_x)^n.$$

In the approximations given in Eq. (3.3) are written in their places in Eqs. (4.5)-(4.8) the following difference equations in terms of parameters $\delta_m(t)$ and $\gamma_m(t)$ are obtained

$$a_1\delta_{m-1}^{n+1} + a_2\delta_m^{n+1} + a_1\delta_{m+1}^{n+1} = a_3\delta_{m-1}^n + a_4\delta_m^n + a_3\delta_{m+1}^n, \tag{4.9}$$

$$a_5\delta_{m-1}^{n+1} + a_6\delta_m^{n+1} + a_7\delta_{m+1}^{n+1} + a_8\gamma_{m-1}^{n+1} + a_9\gamma_m^{n+1} + a_{10}\gamma_{m+1}^{n+1} = \delta_{m-1}^n + \delta_m^n + \delta_{m+1}^n, \tag{4.10}$$

and

$$a_1\gamma_{m-1}^{n+1} + a_2\gamma_m^{n+1} + a_1\gamma_{m+1}^{n+1} = a_3\gamma_{m-1}^n + a_4\gamma_m^n + a_3\gamma_{m+1}^n, \tag{4.11}$$

$$b_1\gamma_{m-1}^{n+1} + b_2\gamma_m^{n+1} + b_3\gamma_{m+1}^{n+1} + b_4\delta_{m-1}^{n+1} + b_5\delta_m^{n+1} + b_6\delta_{m+1}^{n+1} = \gamma_{m-1}^n + \gamma_m^n + \gamma_{m+1}^n, \tag{4.12}$$

where

$$a_1 = 1 - \frac{3\Delta t}{h^2},\ a_2 = 4 + \frac{6\Delta t}{h^2},\ a_3 = 1 + \frac{3\Delta t}{h^2},\ a_4 = 4 - \frac{6\Delta t}{h^2},$$
$$a_5 = 1 + \eta\frac{\Delta t u_x^n}{2} - \eta\frac{3\Delta t u^n}{2h} - \alpha\frac{3\Delta t v^n}{2h} + \frac{\alpha\Delta t v_x^n}{2},\ a_6 = 4 + 2\eta\Delta t u_x^n + 2\alpha\Delta t v_x^n$$
$$a_7 = 1 + \eta\frac{\Delta t u_x^n}{2} + \eta\frac{3\Delta t u^n}{2h} + \alpha\frac{3\Delta t v^n}{2h} + \frac{\alpha\Delta t v_x^n}{2},\ a_8 = \frac{\alpha\Delta t u_x^n}{2} - \alpha\frac{3\Delta t u^n}{2h},$$
$$a_9 = 2\alpha\Delta t u_x^n,\ a_{10} = \frac{\alpha\Delta t u_x^n}{2} + \alpha\frac{3\Delta t u^n}{2h},\ b_1 = 1 + \eta\frac{\Delta t v_x^n}{2} - \eta\frac{3\Delta t v^n}{2h} - \beta\frac{3\Delta t u^n}{2h} + \frac{\beta\Delta t u_x^n}{2},$$
$$b_2 = 4 + 2\eta\Delta t v_x^n + 2\beta\Delta t u_x^n,\ b_3 = 1 + \eta\frac{\Delta t v_x^n}{2} + \eta\frac{3\Delta t v^n}{2h} + \beta\frac{3\Delta t u^n}{2h} + \frac{\beta\Delta t u_x^n}{2},$$
$$b_4 = \frac{\beta\Delta t v_x^n}{2} - \beta\frac{3\Delta t v^n}{2h},\ b_5 = 2\beta\Delta t v_x^n,\ b_6 = \frac{\beta\Delta t v_x^n}{2} + \frac{3\beta\Delta t v^n}{2h}.$$

Here, the terms on the left side contains $(n+1)^{th}$ the time step, and the terms on the right side contain the n^{th} time step, with $m=0(1)N$. The systems in Eqs. (4.9) and (4.10) consist of $(2N+2)$ linear equations and $(2N+6)$ unknowns namely $(\delta_{-1},\delta_0,\delta_1,...,\delta_N)$, $(\gamma_{-1},\gamma_0,\gamma_1,...,\gamma_N)$ For this system to be solvable, the parameters $\delta_{-1},\gamma_{-1},\delta_{N+1},\gamma_{N+1}$ should be eliminated using the boundary conditions. By applying the boundary conditions, a system which can be obtained in $(2N+2)\times(2N+2)$ dimensional. By doing similar operations equations are obtained for Eqs. in (4.11) and (4.12).
In order to solve these systems, first of all, the initial vectors δ^0 and γ^0 are needed and these vectors are obtained from the initial conditions $u(x,0)=\phi_1(x)$ and $v(x,0)=\phi_2(x)$ as follows

$$
\begin{aligned}
&u(x_m,0)=\phi_1(x_m)=U(x_m,0), \quad v(x_m,0)=\phi_2(x_m)=V(x_m,0),\ m=0(1)N\\
&\quad u_m=\delta^0_{m-1}+4\delta^0_m+\delta^0_{m+1}, \qquad v_m=\gamma^0_{m-1}+4\gamma^0_m+\gamma^0_{m+1}\\
&\quad u_0=\delta^0_{-1}+4\delta^0_0+\delta^0_1, \qquad v_0=\gamma^0_{-1}+4\gamma^0_0+\gamma^0_1\\
&\quad u_1=\delta^0_0+4\delta^0_1+\delta^0_2, \qquad v_1=\gamma^0_0+4\gamma^0_1+\gamma^0_2\\
&\quad \vdots \qquad\qquad \vdots\\
&\quad u_N=\delta^0_{N-1}+4\delta^0_N+\delta^0_{N+1}, \qquad v_N=\gamma^0_{N-1}+4\gamma^0_N+\gamma^0_{N+1}
\end{aligned}
$$

For these systems to be solvable the parameters $\delta^0_{-1},\gamma^0_{-1}$ and $\delta^0_{N+1},\gamma^0_{N+1}$ are eliminated using the boundary conditions $U''(a,0)=U''(b,0)=V''(a,0)=V''(b,0)=0$.Thus a band matrix of type $(N+1)\times(N+1)$ which can be easily solved by Thomas algorith is obtained as follows

$$
\begin{bmatrix} 6 & 0 & 0 & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & & & \\ & & 1 & 4 & 1 \\ & & 0 & 0 & 6 \end{bmatrix}
\begin{bmatrix} \delta^0_0 \\ \delta^0_1 \\ \vdots \\ \delta^0_{N-1} \\ \delta^0_N \end{bmatrix}
=
\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix},
\begin{bmatrix} 6 & 0 & 0 & & & \\ 1 & 4 & 1 & & & \\ & & \ddots & & & \\ & & 1 & 4 & 1 \\ & & 0 & 0 & 6 \end{bmatrix}
\begin{bmatrix} \gamma^0_0 \\ \gamma^0_1 \\ \vdots \\ \gamma^0_{N-1} \\ \gamma^0_N \end{bmatrix}
=
\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix}.
$$

5. von-Neumann stability analysis

Here, while applying Strang splitting scheme, the systems (4.9) and (4.11) are solved by $\Delta t/2$, and at time steps Δt for the systems (4.10) and (4.12) the stability analysis of the systems is considered together. If the equations $\delta^n_m=A\xi^n e^{i\beta mh}$ and $\gamma^n_m=B\xi^n e^{i\beta mh}$ are written in their places in Eqs. (4.9) and (4.11) with $i=\sqrt{-1}$, A and B are harmonic amplitutes, β is mod number, h is element size, the following equations are obtained

$$
\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)_{1,2}=\frac{P-Q}{P+Q}
$$

$$
P=2\cos\beta h+4,\ \ Q=\frac{6\Delta t}{h^2}(1-\cos\beta h)
$$

From here, the condition $\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)_{1,2}\right|\le 1$ is satisfied since $Q=\frac{6\Delta t}{h^2}(1-\cos\beta h)\ge 0$ is valid. Similarly, since Eqs. (4.2) and (4.4) are solved together, their stability analyses are considered after discretizing them using standard finite difference method. For this, if in places of u and v in nonlinear terms uu_x, vv_x and $(uv)_x=uv_x+vu_x$ in Eqs. (4.2) and (4.4) the constants z_1 and z_2 are taken, for $\sigma_1=\max\{(\eta z_1+\alpha z_2),(\eta z_2+\beta z_1)\}$, $\sigma_2=\max\{\alpha z_1,\beta z_2\}$, $\lambda=3\Delta t\sigma_1/2h$ and $\mu=3\Delta t\sigma_2/2h$, the following equations are obtained

$$
\begin{aligned}
(1-\lambda)\delta^{n+1}_{m-1}+4\delta^{n+1}_m+(1+\lambda)\delta^{n+1}_{m+1}+(-\mu)\gamma^{n+1}_{m-1}+\mu\gamma^{n+1}_{m+1}&=(1+\lambda)\delta^n_{m-1}+4\delta^n_m+(1-\lambda)\delta^n_{m+1}\\
&+\mu\gamma^n_{m-1}+(-\mu)\gamma^n_{m+1}
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
(1-\lambda)\gamma^{n+1}_{m-1}+4\gamma^{n+1}_m+(1+\lambda)\gamma^{n+1}_{m+1}+(-\mu)\delta^{n+1}_{m-1}+\mu\delta^{n+1}_{m+1}&=(1+\lambda)\gamma^n_{m-1}+4\gamma^n_m+(1-\lambda)\gamma^n_{m+1}\\
&+\mu\delta^n_{m-1}+(-\mu)\delta^n_{m+1}.
\end{aligned} \tag{5.2}
$$

Now, if in systems (5.1) and (5.2) the terms $\delta^n_m=A\xi^n e^{i\beta mh}$ and $\gamma^n_m=B\xi^n e^{i\beta mh}$ are written in their places and Euler formula is used

$$
x_1=\frac{\lambda_1-\lambda_2 i}{\lambda_1+\lambda_2 i} \quad \text{and} \quad x_2=\frac{\lambda_1-\lambda_3 i}{\lambda_1+\lambda_3 i}
$$

is obtained where

$$
\lambda_1=2\cos\beta h+4,\ \lambda_2=(2\lambda+2\mu)\sin\beta h,\ \lambda_3=(2\lambda-2\mu)\sin\beta h
$$

Since the conditions $|x_1|\le 1$ and $|x_2|\le 1$ are satisfied, it is obvious that $\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)_{1,2}\right|\le 1$. Thus the Strang splitting scheme obtained for coupled viscous Burgers' eqution is unconditionally stable, since the condition

$$
|\rho(\xi)|\le\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)_{1,2}\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)_{1,2}\right|\left|\rho_A\left(\frac{\xi^{n+1/2}}{\xi^n}\right)_{1,2}\right|\le 1
$$

is always valid.

6. Numerical Examples and Results

The equations in Eqs. (4.1) and (4.3) are first solved by the $\Delta t/2$ time step while the $A-B-A$ Strang splitting scheme is applied to the split coupled Burgers equation. The solution vectors obtained here are used as starting vectors in Eqs. (4.2) and (4.4) and are solved at time step Δt. Finally, to obtain the desired solutions, the solution vectors obtained from the time-stepped solutions are taken as initial conditions for the equations (4.1) and (4.3) and it was solved at time step $\Delta t/2$. In order to observe the effectiveness of the proposed method, three test problems were addressed and tested with the error norms L_2 and L_∞ given below.

$$L_2 = \sqrt{\sum_{i=0}^{N} \left|U_i^{exact} - U_i^{approx}\right|^2} \Big/ \sqrt{\sum_{i=0}^{N} \left|U_i^{exact}\right|^2}$$

$$L_\infty = \max_i \left|U_i^{exact} - U_i^{approx}\right|.$$

Problem 1

In this problem, the nonlinear coupled viscous Burgers equation system given in (1.1) and (1.2) is considered for $\alpha = \beta = 1$, $\eta = -2$ following initial conditions

$$u(x,0) = v(x,0) = \sin x, \qquad -\pi \le x \le \pi$$

and boundary conditions

$$u(-\pi,t) = u(\pi,t) = 0, \qquad 0 \le t \le T$$
$$v(-\pi,t) = v(\pi,t) = 0, \qquad 0 \le t \le T.$$

The exact solution of this problem is given as $u(x,t) = v(x,t) = e^{-t}\sin x$ by Kaya [19].

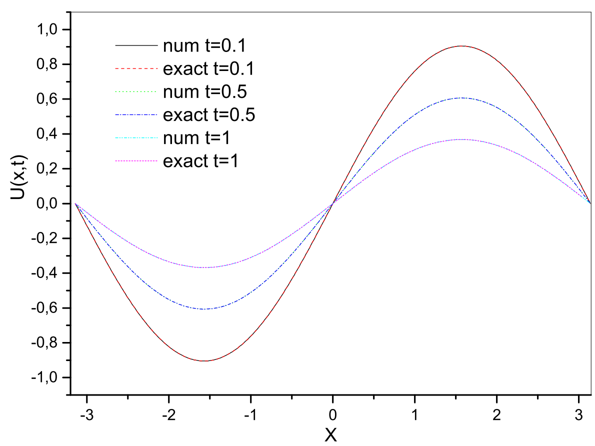

Figure 6.1: The graphs of the calculated numerical values of $U(x,t)$ for Problem 1 at times $t = 0.1, 0.5$ and 1.

Table 1: A comparison of the error norms L_2 and L_∞ of $u(x,t)$ for Problem 1 for values of $\Delta t = 0.001$ and $N = 200, 400$ at various times with those in Ref. [6], Ref. [13] and Ref. [12]

t	$S_{\Delta t}$		Ext4		Ext6	
$N = 200$	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.1	$8.226E-6$	$7.443E-6$	$8.195E-6$	$7.415E-6$	$8.189E-6$	$7.409E-6$
0.5	$4.113E-5$	$2.495E-5$	$4.110E-5$	$2.493E-5$	$4.111E-5$	$2.494E-5$
1	$8.227E-5$	$3.026E-5$	$8.223E-5$	$3.025E-5$	$8.227E-5$	$3.026E-5$
$N = 400$						
0.1	$2.057E-6$	$1.861E-6$	$2.026E-6$	$1.833E-6$	$2.020E-6$	$1.828E-6$
0.5	$1.029E-5$	$6.242E-6$	$1.026E-5$	$6.221E-6$	$1.027E-5$	$6.229E-6$
1	$2.058E-5$	$7.572E-6$	$2.054E-5$	$7.557E-6$	$2.058E-5$	$7.572E-6$
$N = 200$	[6]		[13]		[12]	
0.1	$8.21E-06$	$7.45E-06$	$1.47E-06$	$4.06E-06$	$0.17E-06$	$0.52E-06$
0.5	$2.49E-05$	$4.10E-05$	$2.46E-06$	$2.78E-06$	$0.27E-06$	$0.36E-06$
1	$3.00E-05$	$8.21E-05$	$3.45E-06$	$1.70E-06$	$0.36E-06$	$0.22E-06$
$N = 400$						
0.1	$2.05E-06$	$1.86E-06$	$0.69E-06$	$1.99E-06$	$0.07E-06$	$0.14E-06$
0.5	$1.02E-05$	$6.22E-06$	$1.17E-06$	$1.35E-06$	$0.16E-06$	$0.14E-06$
1	$2.04E-05$	$7.56E-06$	$1.66E-06$	$0.82E-06$	$0.15E-06$	$0.10E-06$

Table 3: A comparison of the error norms L_2 and L_∞ of $u(x,t)$ for Problem 2 for values of $\Delta t = 0.01$, $N = 100$ and various α and β at times $t = 0.1$ and 0.5 with those in Ref.[6, 13, 12, 11, 10, 17]

			$S_{\Delta t}$		Ext4		Ext6	
t	α	β	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	0.1	0.3	$6.737E-4$	$4.187E-5$	$6.729E-4$	$4.186E-5$	$6.724E-4$	$4.185E-5$
	0.3	0.03	$7.411E-4$	$4.591E-5$	$7.387E-4$	$4.591E-5$	$7.375E-4$	$4.591E-5$
1	0.1	0.3	$1.325E-3$	$8.280E-5$	$1.324E-3$	$8.277E-5$	$1.323E-3$	$8.275E-5$
	0.3	0.03	$1.463E-3$	$9.182E-5$	$1.457E-3$	$9.182E-5$	$1.456E-3$	$9.182E-5$
			[6]		[13]		[12]	
0.5	0.1	0.3	$6.736E-4$	$4.167E-5$	$6.732E-4$	$4.187E-5$	$6.783E-4$	$4.208E-5$
	0.3	0.03	$7.326E-4$	$4.590E-5$	$7.430E-4$	$4.591E-5$	$7.609E-4$	$4.703E-5$
1	0.1	0.3	$1.325E-3$	$8.258E-5$	$1.323E-3$	$8.277E-5$	$1.334E-3$	$8.320E-5$
	0.3	0.03	$1.452E-3$	$9.182E-5$	$1.464E-3$	$9.183E-5$	$1.500E-3$	$9.409E-5$
			[11]		[10]		[17]	
0.5	0.1	0.3	$1.44E-3$	$4.38E-5$	$3.2453E-5$	$9.6185E-4$	$2.02E-3$	$1.00E-4$
	0.3	0.03	$6.68E-4$	$4.58E-5$	$2.7326E-5$	$4.3102E-4$	$5.07E-3$	$2.52E-4$
1	0.1	0.3	$1.27E-3$	$8.66E-5$	$2.4054E-5$	$1.1529E-3$	$4.03E-3$	$2.01E-4$
	0.3	0.03	$1.30E-3$	$9.16E-5$	$2.8316E-5$	$1.2684E-3$	$1.00E-2$	$5.04E-4$

Table 2: A comparison of the error norms L_2 and L_∞ of $u(x,t)$ for Problem 1 for $\Delta t = 0.001$ various values of N at times $t = 0.1, 0.5$ with those in Ref. [6] and Ref. [7].

N	$S_{\Delta t}$		Ext4		Ext6	
$t=0.1$	L_2	L_∞	L_2	L_∞	L_2	L_∞
32	$3.2164E-4$	$2.9103E-4$	$3.2161E-4$	$2.9100E-4$	$3.2160E-4$	$2.9100E-4$
64	$8.0342E-5$	$7.2697E-5$	$8.0311E-5$	$7.2669E-5$	$8.0305E-5$	$7.2663E-5$
128	$2.0082E-5$	$1.8171E-5$	$2.0051E-5$	$1.8143E-5$	$2.0045E-5$	$1.8137E-5$
256	$0.5021E-5$	$0.4543E-5$	$0.4990E-5$	$0.4515E-5$	$0.4984E-5$	$0.4509E-5$
512	$1.2556E-6$	$1.1361E-6$	$0.1225E-6$	$1.1082E-6$	$0.1219E-6$	$0.1103E-6$
$t=0.5$						
32	$16.0715E-4$	$9.7479E-4$	$16.0712E-4$	$9.7477E-4$	$16.0713E-4$	$9.7478E-4$
64	$4.0165E-4$	$2.4362E-4$	$4.0162E-4$	$2.4359E-4$	$4.0163E-4$	$2.4360E-4$
128	$10.0412E-5$	$6.0903E-5$	$10.0377E-5$	$6.0882E-5$	$10.0392E-5$	$6.0891E-5$
256	$2.5110E-5$	$1.5230E-5$	$2.5075E-5$	$1.5209E-5$	$2.5090E-5$	$1.5218E-5$
512	$0.6285E-5$	$0.3812E-5$	$0.6250E-5$	$0.3791E-5$	$0.6265E-5$	$0.3800E-5$
		[6]			[7]	
	$t=0.1$		$t=0.5$		$t=0.1$	$t=0.5$
	L_2	L_∞	L_2	L_∞	L_∞	L_∞
32	–	$2.9104E-04$	–	$9.7478E-04$	$3.0973141E-04$	$1.0373845E-04$
64	–	$7.2704E-05$	–	$2.4361E-04$	$7.4999576E-05$	$2.5132688E-04$
128	–	$1.8178E-05$	–	$6.0896E-05$	$1.8456076E-05$	$6.1854865E-05$
256	–	$4.5497E-05$	–	$1.5223E-05$	–	–
512	–	$1.1430E-06$	–	$3.8052E-05$	–	–

In Problem 1, numerical results and graphs are given for $u(x,t)$ function since $u(x,t)$ and $v(x,t)$ have the same initial, boundary conditions and exact solution. Numerical solution and complete solution graph at time $t = 0.1$,0.5 and 1 for $u(x,t)$ are given in Fig. 6.1. As you can see from Fig. 6.1 , the numerical solution and the exact solution are very close to each other. L_2 and L_∞ error norms calculated with Strang, Ext4 and Ext6 methods at time $t = 0.1$,0.5 and 1 for $N = 200$ and 400 values were given in Table 1 and the references [6, 13, 12]. As it is seen in Table 1 , the newly obtained error norms L_2 and L_∞ are in agreement with those in Ref. [6] and are larger than those calculated in [13, 12]. In the Table 2, error norms L_2 and L_∞ calculated at $t = 0.1$ and 0.5 times for $\Delta t = 0.001$ are given. When looked at this table, it is clear that the calculated error norms are better than those calculated in the Refs. [6, 7].

Problem 2

As a second problem, the solutions of the nonlinear coupled viscous Burgers equation system given in Eqs. (1.1) and (1.2) were investigated for $\eta = 2$ for different α and β values at time $t = 0.5$ and 1

Soliman [20] has been given the exact solution to this problem

$$u(x,t) = a_0\left(1-\tanh(A(x-2At)\right)$$

$$v(x,t) = a_0\left(\left(\frac{2\beta-1}{2\alpha-1}\right)-\tanh(A(x-2At))\right)$$

where

$$a_0 = 0.05 \quad \text{and} \quad A = \frac{1}{2}a_0\left(\frac{4\alpha\beta-1}{2\alpha-1}\right)$$

The initial and boundary conditions of the problem are taken from the exact solution for $u(x,t)$ and $v(x,t)$. The solution region for this problem was taken as the range $[-10,10]$ and all the results were calculated with $N = 100$, $\Delta t = 0.01$.

Table 4: A comparison of the error norms L_2 and L_∞ of $v(x,t)$ for Problem 2 for values of $\Delta t = 0.01$, $N = 100$ at times $t = 0.1$ and 0.5 for different values of α ve β with those in Refs.[6, 13, 12, 11, 10, 17]

		β	$S_{\Delta t}$		Ext4		Ext6	
t	α	β	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	0.1	0.3	$5.014E-4$	$0.218E-4$	$5.000E-4$	$0.218E-4$	$4.993E-4$	$0.217E-4$
	0.3	0.03	$1.319E-3$	$1.809E-4$	$1.318E-3$	$1.809E-4$	$1.318E-3$	$1.809E-4$
1	0.1	0.3	$0.979E-3$	$4.205E-5$	$0.976E-3$	$4.197E-5$	$0.975E-3$	$4.193E-5$
	0.3	0.03	$2.603E-3$	$3.618E-4$	$2.601E-3$	$3.618E-4$	$2.600E-3$	$3.618E-4$
			[6]		[13]		[12]	
0.5	0.1	0.3	$9.057E-4$	$1.480E-4$	$5.015E-4$	$0.218E-4$	$5.101E-4$	$0.221E-4$
	0.3	0.03	$1.591E-3$	$5.729E-4$	$1.319E-3$	$1.809E-4$	$1.327E-3$	$1.818E-4$
1	0.1	0.3	$1.251E-3$	$4.770E-5$	$0.977E-3$	$4.205E-5$	$0.995E-3$	$4.255E-5$
	0.3	0.03	$2.250E-3$	$3.617E-4$	$2.600E-3$	$3.618E-4$	$2.617E-3$	$3.636E-4$
			[11]		[10]		[17]	
0.5	0.1	0.3	$5.42E-4$	$4.99E-5$	$2.746E-5$	$3.332E-4$	$1.56E-3$	$3.80E-5$
	0.3	0.03	$1.20E-3$	$1.81E-4$	$2.454E-4$	$1.148E-3$	$1.59E-3$	$1.85E-4$
1	0.1	0.3	$1.29E-3$	$9.92E-5$	$3.745E-5$	$1.162E-3$	$3.10E-3$	$7.58E-5$
	0.3	0.03	$2.35E-3$	$3.62E-4$	$4.525E-4$	$1.638E-3$	$3.15E-2$	$3.67E-4$

The error norms of L_2 and L_∞ calculated at times $t = 0.1$ and 0.5 for this problem are given by Table 3 and Table 4 for $u(x,t)$ and $v(x,t)$, respectively. It is seen that the results obtained with Strang, Ext4 and Ext6 are better than those obtained from [13, 12, 17] studies and are in agreement with [6] in Table 3. In addition, although our L_∞ norm is worse than in the [11, 10] studies, our L_2 norm is better

Problem 3

As a final problem, the nonlinear coupled viscous Burgers equation system in Eqs. (1.1) and (1.2)has been studied on $[-20,20]$ for $\eta = 2$ and $\alpha = \beta = 5/2$ parameters .The exact solution of the problem given with the initial condition

$$u(x,0) = K\left(1-\tanh(\frac{3Kx}{2})\right), \quad x \in [-20,20]$$
$$v(x,0) = K\left(1-\tanh(\frac{3Kx}{2})\right), \quad x \in [-20,20]$$

has been given byAbazari [21] as follows

$$u(x,t) = K\left(1-\tanh(\frac{3K}{2}(x-3Kt))\right), \quad x \in [-20,20]$$
$$v(x,t) = K\left(1-\tanh(\frac{3K}{2}(x-3Kt))\right), \quad x \in [-20,20]$$

The boundary conditions of the problem are taken from the exact solution. Here, various anti-kink wave solutions are obtained for different K values since the initial condition depends on the K parameter [16]. The results obtained with $S_{\Delta t}$, Ext4 and Ext6 methods for this problem are given in Table 5. The L_2 and L_∞ error norms for $K = 0.1$ and 0.5 were compared with those of [15, 16] in Table6. As it is seen from Table 6,the results obtained by the methods $S_{\Delta t}$, Ext4 and Ext6 are much better in general. The results calculated for $K = 1$ were compared with the [15] study and Table 7. Also, the graphs of the solutions calculated for $K = 0.1, 0.5, 1$ and 5 at different times are given in Fig. 6.2. As it is seen in Fig. 6.2, as the K value increases, the solution curves are also staggered and the numerical solution appears to be in agreement with the exact solution.

Table 5: The error norms L_2 and L_∞ of $u(x,t)$ for Problem 3 using $S_{\Delta t}$,Ext4 and Ext6 for values of $\Delta t = 0.001$ and $K = 0.1, 0.5, 1$ at various times.

			$S_{\Delta t}$		Ext4		Ext6	
K	N	t	L_2	L_∞	L_2	L_∞	L_2	L_∞
		1	$1.0283E-6$	$0.6902E-6$	$1.1765E-6$	$0.7641E-6$	$0.9540E-6$	$0.4694E-6$
		2	$1.7781E-6$	$0.7829E-6$	$1.8199E-6$	$0.7648E-6$	$1.5650E-6$	$0.5264E-6$
0.1	320	3	$2.3353E-6$	$0.7425E-6$	$2.3347E-6$	$0.7140E-6$	$2.1065E-6$	$0.6131E-6$
		4	$2.7935E-6$	$0.7536E-6$	$2.7634E-6$	$0.7557E-6$	$2.5753E-6$	$0.7556E-6$
		5	$3.1816E-6$	$0.8758E-6$	$3.1268E-6$	$0.8780E-6$	$2.9831E-6$	$0.8777E-6$
		1	$1.5432E-4$	$5.4661E-4$	$1.5430E-4$	$5.4664E-4$	$1.5431E-4$	$5.4661E-4$
		2	$1.6042E-4$	$5.7424E-4$	$1.6041E-4$	$5.7403E-4$	$1.6041E-4$	$5.7416E-4$
0.5	200	3	$1.5517E-4$	$5.7845E-4$	$1.5516E-4$	$5.7857E-4$	$1.5517E-4$	$5.7844E-4$
		4	$1.4936E-4$	$5.6625E-4$	$1.4935E-4$	$5.6600E-4$	$1.4936E-4$	$5.6624E-4$
		5	$1.4442E-4$	$5.6815E-4$	$1.4428E-4$	$5.6832E-4$	$1.4428E-4$	$5.6810E-4$
		1	$0.1736E-3$	$1.7595E-3$	$0.1736E-3$	$1.7615E-3$	$0.1736E-3$	$1.7611E-3$
		2	$0.1611E-3$	$1.7281E-3$	$0.1610E-3$	$1.7306E-3$	$0.1610E-3$	$1.7298E-3$
1	320	3	$0.1522E-3$	$1.7247E-3$	$0.1521E-3$	$1.7273E-3$	$0.1521E-3$	$1.7261E-3$
		4	$0.1448E-3$	$1.7243E-3$	$0.1447E-3$	$1.7271E-3$	$0.1447E-3$	$1.7255E-3$
		5	$0.1384E-3$	$1.7243E-3$	$0.1383E-3$	$1.7273E-3$	$0.1383E-3$	$1.7252E-3$

Table 6: A comparison of the error norms L_2 and L_∞ for Problem 3 for values of $\Delta t = 0.001$ and $K = 0.1, 0.5$ with those in Ref. [16] and [15].

			$S_{\Delta t}$		Ext4		Ext6	
K	N	t	L_2	L_∞	L_2	L_∞	L_2	L_∞
		1	$1.0283E-6$	$0.6902E-6$	$1.1765E-6$	$0.7641E-6$	$0.9540E-6$	$0.4694E-6$
		2	$1.7781E-6$	$0.7829E-6$	$1.8199E-6$	$0.7648E-6$	$1.5650E-6$	$0.5264E-6$
0.1	320	3	$2.3353E-6$	$0.7425E-6$	$2.3347E-6$	$0.7140E-6$	$2.1065E-6$	$0.6131E-6$
		4	$2.7935E-6$	$0.7536E-6$	$2.7634E-6$	$0.7557E-6$	$2.5753E-6$	$0.7556E-6$
		5	$3.1816E-6$	$0.8758E-6$	$3.1268E-6$	$0.8780E-6$	$2.9831E-6$	$0.8777E-6$
		1	$1.5432E-4$	$5.4661E-4$	$1.5430E-4$	$5.4664E-4$	$1.5431E-4$	$5.4661E-4$
		2	$1.6042E-4$	$5.7424E-4$	$1.6041E-4$	$5.7403E-4$	$1.6041E-4$	$5.7416E-4$
0.5	200	3	$1.5517E-4$	$5.7845E-4$	$1.5516E-4$	$5.7857E-4$	$1.5517E-4$	$5.7844E-4$
		4	$1.4936E-4$	$5.6625E-4$	$1.4935E-4$	$5.6600E-4$	$1.4936E-4$	$5.6624E-4$
		5	$1.4442E-4$	$5.6815E-4$	$1.4428E-4$	$5.6832E-4$	$1.4428E-4$	$5.6810E-4$
			[16]		[15]			
		1	$1.4829E-6$	$5.7788E-7$	$2.7344E-5$	–		
		2	$2.7955E-6$	$1.0754E-6$	$6.4798E-5$	–		
0.1	320	3	$3.9298E-6$	$1.4861E-6$	$1.0832E-4$	–		
		4	$4.9434E-6$	$1.8800E-6$	$1.5709E-4$	–		
		5	$5.8615E-6$	$2.2034E-6$	$2.1113E-4$	–		
		1	$1.6362E-4$	$6.7505E-4$	$6.6534E-5$	–		
		2	$1.9746E-4$	$8.1705E-4$	$6.3686E-5$	–		
0.5	320	3	$2.0557E-4$	$8.6375E-4$	$6.0667E-5$	–		
		4	$2.0543E-4$	$8.8160E-4$	$5.8210E-5$	–		
		5	$2.0231E-4$	$8.9060E-4$	$5.6210E-5$	–		

Table 7: A comparison of the error norms L_2 and L_∞ of $u(x,t)$ for Problem 3 for values of $K = 1, \Delta t = 0.001$ and $N = 320$ with those in Ref. [15].

	$S_{\Delta t}$		Ext4		Ext6		Li at al.[15]	
t	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	$0.1736E-3$	$1.7595E-3$	$0.1736E-3$	$1.7615E-3$	$0.1736E-3$	$1.7611E-3$	$1.7416E-4$	–
2	$0.1611E-3$	$1.7281E-3$	$0.1610E-3$	$1.7306E-3$	$0.1610E-3$	$1.7298E-3$	$1.6157E-4$	–
3	$0.1522E-3$	$1.7247E-3$	$0.1521E-3$	$1.7273E-3$	$0.1521E-3$	$1.7261E-3$	$1.5268E-4$	–
4	$0.1448E-3$	$1.7243E-3$	$0.1447E-3$	$1.7271E-3$	$0.1447E-3$	$1.7255E-3$	$1.4525E-4$	–
5	$0.1384E-3$	$1.7243E-3$	$0.1383E-3$	$1.7273E-3$	$0.1383E-3$	$1.7252E-3$	$1.3883E-4$	–

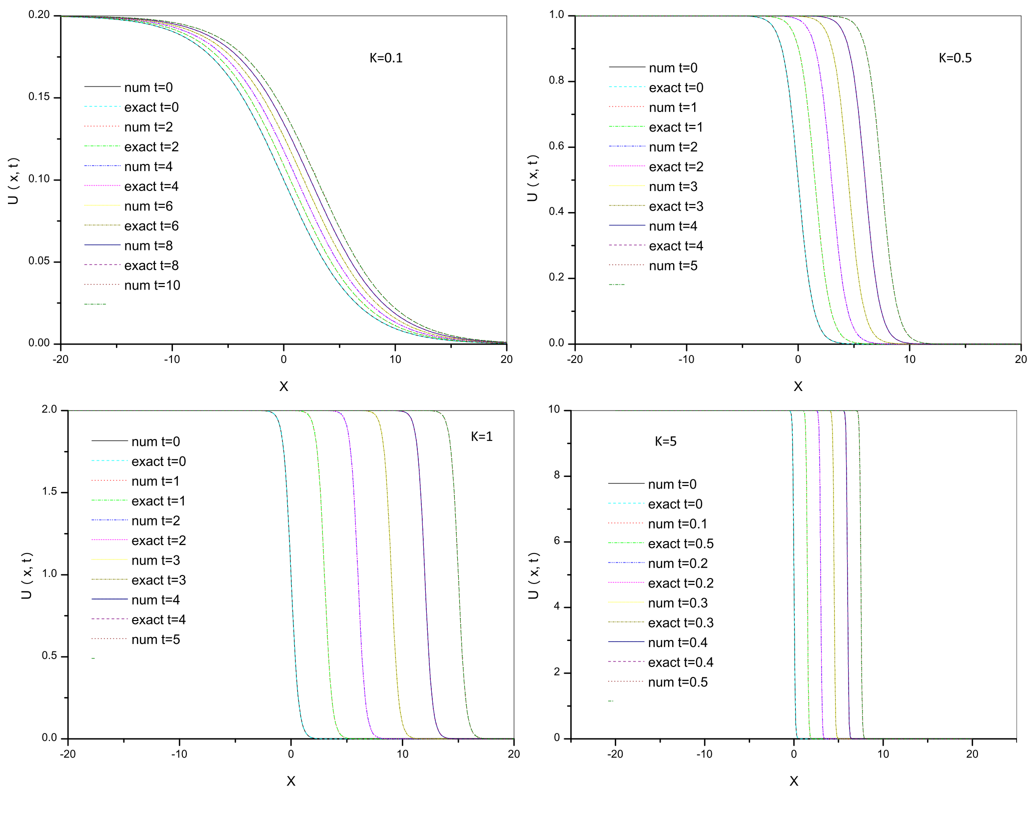

Figure 6.2: The behaviour of the solution for Problem 3 for values of $K = 0.1, 0.5, 1$ and 5 at different times.

7. Conclusion

In this study, the four viscous Burgers equation was split into four sub-equilibria. Each sub-equation finite element cubic B-spline collocation method was applied and numerical schemes were obtained. Strang splitting and Ext4 and Ext6 methods were used to solve these numerical schemes. In general, Ext4 is better than Strang splitting method and Ext6 is better than Ext4 method. In addition, the calculated results are consistent with the literature. Thus, it is seen that the used methods are suitable and effective numerical systems for nonlinear equations and equation systems.

Acknowledgement

We would like to express our sincere thanks to referees and editor for their many valuable suggestions and comments to improve the quality of the paper.

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