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ON THE EXPLICIT CHARACTERIZATION OF CURVES ON A $(n-1)$-SPHERE IN $\mathbb{S}^n$

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Abstract. In $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, harmonic curvatures and focal curvatures of a non-degenerate curve were defined by Özdamar and Hacisalihoğlu in [7] and by Uribe-Vargas in [9], respectively.

In this paper, we investigate the relations between the harmonic curvatures of a non-degenerate curve and the focal curvatures of tangent indicatrix of the curve. Also we give the relationship between the Frenet apparatus (vectors and the curvature functions) of a curve α in $\mathbb{E}^{n+1}$ and the Frenet apparatus of tangent indicatrix α_T of the curve α. In the main theorem of the paper, we give a characterization for a curve to be a $(n-1)$-spherical curve in $\mathbb{S}^n$ by using focal curvatures of the curve. Furtermore we give that harmonic curvature of the curve is focal curvature of the tangent indicatrix.

1. Introduction.

In the differential geometry of a regular curve in Euclidean 3-space, it is well known that, the functions k_1 (curvature) and k_2 (torsion) play an important role to determine the shape and size of the curve ([3]). For example: the condition for a curve to be a spherical curve, i.e., for it to lie on a sphere of $\mathbb{E}^3$, is usually given in the form

$$(1/k_2(1/k_1)')' + k_2/k_1 = 0. \tag{1.1}$$

In [8, 9], Wong, using a differential equation derived from (1.1), give the following explicit characterization of spherical curves:

$$(A\cos(\int k_2 ds) + B\sin(\int k_2 ds))k_1 = 1,$$

where A, B non-zero constants.

Another important example is helix: a helix is a geometric curve with non-vanishing constant curvature k_1 and non-vanishing constant torsion k_2. Indeed a helix is a special case of the general helix. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$, is defined by the property that the tangent makes a

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constant angle with a fixed straight line (the axis of the general helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 ([3]) is: a necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of k_1 and k_2 are non-zero constants it is, of course, a general helix. We call it a circular helix or simply helix.

In this paper, we investigate relations between harmonic curvatures of a non-degenerate curve and focal curvatures of tangent indicatrix of the curve. In the main theorem of the paper, we give a characterization for a curve to be a $(n-1)$-spherical curve in $\mathbb{S}^n$ by using focal curvatures of the curve.

2. Preliminaries.

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^{n+1}$ be arbitrary curve in the Euclidean $(n+1)$- space $\mathbb{E}^{n+1}$. Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $<\alpha'(s), \alpha'(s)>= 1$, where $<\cdot,\cdot>$ is the standard scalar product of $\mathbb{E}^{n+1}$ given by

$$< X, Y >= \sum_{i=1}^{n+1} x_i y_i,$$

for each $X = (x_1, x_2, ..., x_{n+1})$, $Y = (y_1, y_2, ..., y_{n+1}) \in \mathbb{E}^{n+1}$. In particular, the norm of a vector $X \in \mathbb{E}^{n+1}$ is given by $||X||^2 =< X, X >$. Let $\{V_1, V_2, ..., V_{n+1}\}$ be the moving Frenet frame along the unit speed curve α, where V_i $(i = 1, 2, ..., n+1)$ denote ith Frenet vector fields. Then the Frenet formulas are given by

$$\left\{ \begin{array}{c} V_1'(s) = k_1(s)\ V_2\ (s) \\ V_i'(s) = -k_{i-1}(s)\ V_{i-1}\ (s) +\ k_i(s)\ V_{i+1}\ (s), \quad i = 2, 3, ..., n \\ V_{n+1}'(s) = -k_n(s)\ V_n\ (s) \end{array} \right. \tag{2.1}$$

where k_i $(i = 1, 2, ..., n)$ denote ith curvature functions of the curve [2, 5]. If all curvatures k_i $(i = 1, 2, ..., n)$ of the curve nowhere vanish in $I \subset \mathbb{R}$, then the curve is called a non-degenerate curve.

In this paper, we assume that all curvatures of the curve are positive smooth functions of itself arc length. That is, Frenet frame of the curve are given by Gram-Schmidt method ([2]). If the curve lies in a hyperplane of $\mathbb{E}^{n+1}$, then it is said that α is n-flat curve ([6]). It is well known that α is n-flat curve in $\mathbb{E}^{n+1}$ if and only if $k_n(s) = 0$ ([6]).

Proposition 2.1. *([8]) A curve $\alpha : \mathbb{R} \to \mathbb{E}^{n+1}$ is a generalized helix if and only if the function* $\det\left(\alpha''(t), \alpha'''(t), ..., \alpha^{(n+2)}(t)\right)$ *is identically zero, where $\alpha^{(i)}$ represents the ith derivative of α with respect to its arc length. Equivalently, α is generalized helix if and only if α_T is n-flat curve, where $\alpha_T : I \subset \mathbb{R} \to \mathbb{S}^n$ is tangent indicatrix of the curve.*

Definition 2.1. ([5]) Let α be a unit curve in $\mathbb{E}^{n+1}$. Harmonic curvatures of α is defined by

$$H_i : I \subset \mathbb{R} \to \mathbb{R}\,,\ i = 0, 1, 2, ..., n-1,$$

$$H_i = \begin{cases} 0, & i=0 \\ \frac{k_1}{k_2} & i=1 \\ \{V_1[H_{i-1}] + H_{i-2}k_i\}\frac{1}{ki+1}, & i=2,3,...,n-1. \end{cases}$$

Definition 2.2. ([7]) Focal curvatures of the curve α in E^{n+1} is defined by

$$m_i(s) = \begin{cases} 0, & \text{i=1;} \\ \frac{1}{k_1}, & \text{i=2;} \\ \{V_1[m_{i-1}] + m_{i-2}k_{i-2}\}\frac{1}{k_{i-1}}, & \text{i=3,4,...,n+1.} \end{cases}$$

where $m_i : I \subset \mathbb{R} \to \mathbb{R}, i = 1, 2, ..., n+1$.

In the following theorem, Camci at.al.([1]), give the explicit characterization for a non-degenerate curve to be a generalized helix by using the harmonic curvatures of the curve:.

Theorem 2.1. *Let $\alpha(s)$ be a unit speed non-degenerate curve in n-dimensional Euclidean space $\mathbb{E}^n$ with Frenet vectors $\{V_1, V_2, ..., V_n\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-2}\}$. Then α a is a generalized helix if and only if*

$$V_1[H_{n-2}] + k_{n-1}H_{n-3} = 0. \tag{2.2}$$

Consequently, If we consider the results given in [1] , [7] and [6], we can give the following corollary:

Corollary 2.1. *Let $\alpha(s)$ be a unit speed non-degenerate curve in n-dimensional Euclidean space $\mathbb{E}^{n+1}$ with Frenet vectors $\{V_1, V_2, ..., V_{n+1}\}$, and harmonic curvatures $\{H_1, H_2, ..., H_{n-1}\}$. Then the following statements are equivalent:*

(i) *α is generalized helices in $\mathbb{E}^{n+1}$,*

(ii) *α_T is n-flat curve,*

(iii) *$k_n^T(s) = 0$,*

(iv) *$V_1[H_{n-1}] + H_{n-2}k_n = 0$,*

(v) *$\alpha_T : I \subset \mathbb{R} \to \mathbb{S}_r^{n-1} \subset \mathbb{S}^n$ is intersection of n-dimensional hyperplane and n-dimensional sphere $\mathbb{S}^n$, where $\mathbb{S}_r^{n-1}$ is a $(n-1)$- dimensional sphere in $\mathbb{S}^n$ and $0 < r \leq 1$.*

In particular, tangent indicatrix of the curve lies in $(n-1)$-dimensional sphere $\mathbb{S}_r^{n-1}$ and $\mathbb{S}_r^{n-1}$ is of maximum radius (i.e. equator) if and only if α is n-flat curve, in the sense tangent of the curve is orthogonal to normal of the hyperplane ([6]).

Let $\bar{\nabla}$ be Levi-Civita connection of R^{n+1} and ∇ be Riemannian connection of the induced Euclidian metric on S^n. Then, we can write Gauss-Weingarten formulas by

$$\bar{\nabla}_X Y = \nabla_X Y - < S(X), Y > Z,$$

where $X, Y \in \chi(S^n)$, Z is a unit normal vector field of the sphere and $S(X) = \bar{\nabla}_X Z$ is shape operator with respect to Z.

Let α be a non-degenerate curve in $\mathbb{E}^{n+1}$ with Frenet vectors $\{V_1, V_2, ..., V_{n+1}\}$ and with curvature functions $\{k_1, k_2, ..., k_n\}$ .Suppose that, s_T is arclenght function of the tangent indicatrix α_T of the curve α. By $\{V_1^T, V_2^T, ..., V_{n+1}^T\}$ and $\{k_1^T, k_2^T, ..., k_n^T\}$, we denote the Frenet vectors and curvature functions for tangent indicatrix α_T of the curve in $\mathbb{E}^{n+1}$ and by $\{\bar{V}_1^T, \bar{V}_2^T, ..., \bar{V}_n^T\}$ and $\{\bar{k}_1^T, \bar{k}_2^T, ..., \bar{k}_{n-1}^T\}$

we denote Frenet vectors and curvature functions for tangent indicatrix α_T of the curve in $\mathbb{S}^n$, respectively.

The focal curvatures of α_T in $\mathbb{S}^n$ is defined by

$$\bar{m}_i(s) = \begin{cases} 0, & \text{i=1;} \\ \frac{1}{\bar{k}_1^T}, & \text{i=2;} \\ \{\bar{V}_1^T[\bar{m}_{i-1}] + \bar{m}_{i-2}\bar{k}_{i-2}^T\}\frac{1}{\bar{k}_{i-1}^T}, & \text{i=2,3,4,...,n.} \end{cases}$$

where $\bar{m}_i : I \subset R \to R$, $i = 1, 2, ..., n$. In this notation, we can easily see that

$$\bar{V}_1^T = V_1^T = V_2, \tag{2.3}$$

and

$$\frac{ds_T}{ds} = k_1. \tag{2.4}$$

3. Characterization of curves in a $(n-1)$-sphere in $\mathbb{S}^n$

Theorem 3.1. *If $\alpha : I \subset \mathbb{R} \to \mathbb{E}^{n+1}$ is a non-degenerate curve in $\mathbb{E}^{n+1}$ and α_T is the tangent indicatrix of the curve, then we have*

$$\bar{V}_i^T = V_{i+1}, (i = 1, 2, 3, ..., n), \tag{3.1}$$

and

$$\bar{k}_{i-1}^T = \frac{k_i}{k_1}, (i = 2, 3, ..., n). \tag{3.2}$$

Proof. It is well known that the unit vector field of the unit sphere centered at the origin is the position vector itself. From Gauss-Weingarten formulas, we have

$$\bar{\nabla}_{\bar{V}_1^T}\bar{V}_1^T = \nabla_{\bar{V}_1^T}\bar{V}_1^T - < S(\bar{V}_1^T), \bar{V}_1^T > \alpha_T(s). \tag{3.3}$$

In S^n, it is well known that $S(\bar{V}_1^T) = \bar{V}_1^T$. From equation (2.3), (2.4), we obtain

$$\begin{aligned} \bar{\nabla}_{\bar{V}_1^T}\bar{V}_1^T &= \frac{d\bar{V}_1^T}{ds_T} \\ &= \frac{dV_2}{ds}\frac{ds}{ds_T} \\ &= -V_1 + \frac{k_2}{k_1}V_3. \end{aligned}$$

Thus from equation (3.3), we have

$$\bar{V}_2^T = V_3$$

and

$$\bar{k}_1^T = \frac{k_2}{k_1}.$$

Now suppose that, we have

$$\bar{V}_i^T = V_{i+1}$$

and

$$\bar{k}_{i-1}^T = \frac{k_i}{k_1}.$$

Then, from Gauss-Weingarten formulas, we obtain

$$\bar{\nabla}_{\bar{V}_1^T}\bar{V}_i^T = \nabla_{\bar{V}_1^T}\bar{V}_i^T - < S(\bar{V}_1^T), \bar{V}_i^T > \alpha_T(s), \tag{3.4}$$

where

$$\begin{aligned}\bar{\nabla}_{\bar{V}_1^T}\bar{V}_i^T &= \frac{d\bar{V}_i^T}{ds_T}\\ &= \frac{dV_{i+1}}{ds}\frac{ds}{ds_T}\\ &= -\frac{k_i}{k_1}V_i + \frac{k_{i+1}}{k_1}V_{i+2}.\end{aligned}$$

From equation (3.4), we have

$$\bar{V}_{i+1}^T = V_{i+2}$$

and

$$\bar{k}_{i+1}^T = \frac{k_{i+2}}{k_1}.$$

□

Theorem 3.2. *If $\alpha : I \subset \mathbb{R} \to \mathbb{E}^{n+1}$ is a non-degenerate curve in $\mathbb{E}^{n+1}$ and α_T is the tangent indicatrix of the curve, then we have*

$$\bar{m}_i(s) = H_{i-1} \quad (i = 1, ..., n). \tag{3.5}$$

Proof. For $i = 1$, we have

$$\bar{m}_2 = \frac{1}{\bar{k}_1}.$$

From equation (3.2), we get

$$\bar{m}_2 = \frac{k_1}{k_2} = H_1.$$

Suppose that, for $i = 0, 1, 2, ..., p$, the following equation is holds

$$\bar{m}_i(s) = H_{i-1}.$$

From assumption and equation (3.2) , we have

$$\begin{aligned}\bar{m}_{p+1}(s) &= \{\bar{V}_1^T[\bar{m}_p] + \bar{m}_{p-1}\bar{k}_{p-1}^T\}\frac{1}{\bar{k}_p^T}\\ &= \{\frac{d}{ds_T}[H_{p-1}] + H_{p-2}\frac{k_p}{k_1}\}\frac{k_1}{p+1}\\ &= \{\frac{d}{ds}[H_{p-1}]\frac{1}{k_1} + H_{p-2}\frac{k_p}{k_1}\}\frac{k_1}{k_{p+1}}\\ &= \{\frac{d}{ds}[H_{p-1}] + H_{p-2}k_p\}\frac{1}{k_{p+1}}\\ &= H_p\end{aligned}$$

So we see that harmonic curvature of the curve is focal curvature of the tangent indicatrix of the curve. □

Assume that the curve β lies in $\mathbb{S}^n$ (i.e. $\beta : I \subset \mathbb{R} \to \mathbb{S}^n$). Therefore, we define a curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^{n+1}$ such that $\alpha_T(s) = \beta(s)$. So, we have following theorem.

Theorem 3.3. ***(Main Theorem)*** *The curve β lies in $\mathbb{S}^{n-1}$ if and only if*

$$\bar{V}_1^T[\bar{m}_n] + \bar{m}_{n-1}\bar{k}_{n-1}^T = 0. \tag{3.6}$$

Proof. Assume that the curve β lies in $\mathbb{S}^{n-1}$ (i.e. $\beta : I \subset \mathbb{R} \to \mathbb{S}^{n-1}$). From assumption, α is generalized helices in $\mathbb{E}^{n+1}$. From equation (2.2), we have

$$V_1[H_{n-1}] + H_{n-2}k_n = 0$$

From equation (3.5) and (2.4), we have

$$\frac{d}{ds^T}[\bar{m}_n]k_1 + \bar{m}_{n-1}\bar{k}^T_{n-1}k_1 = 0$$

Thus we obtain

$$\bar{V}^T_1[\bar{m}_n] + \bar{m}_{n-1}\bar{k}^T_{n-1} = 0.$$

Conversely, let $\beta : I \subset \mathbb{R} \to \mathbb{S}^n$ be satisfy condition of equation (3.6). If we define a curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^{n+1}$ such that $\alpha_T(s) = \beta(s)$, then by similar method, from equation (3.6), we can easily see that

$$V_1[H_{n-1}] + H_{n-2}k_n = 0$$

for α. From Theorem 2.1, we see that α is generalized helices in $\mathbb{E}^{n+1}$. Thus β lies in $\mathbb{S}^{n-1}$. □

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