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# FUZZY CONTRACTIBILITY

ERDAL GÜNER

Abstract. In this paper, firstly some fundamental concepts are included relating to fuzzy topological spaces. Secondly, the fuzzy connected set is introduced. Finally, defining fuzzy contractible space, it is shown that $X$ is a fuzzy contractible space if and only if $X$ is fuzzy homotopic equivalent with a fuzzy single-point space.

## 1. Introduction

The concept of a fuzzy set was discovered by Zadeh [7]. The theory of fuzzy topology was developed by Chang [1] and others. The main problem in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case. In this paper, we construct the fuzzy contractible space and we give some characterizations of fuzzy contractible spaces.

Let $X$ be a set and $I$ the unit interval $[0,1]$. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A$ which associates with each point $x \in X$ its "grade of membership" $\mu_A(x) \in I$.

**Definition 1.1.** Let $A$ be a fuzzy set in $X$. The set

$$Supp\, A = A_0 = \{x \in X : A(x) > 0\}$$

is called the support of fuzzy set $A$.

**Definition 1.2.** A fuzzy point in $X$ is a fuzzy set with membership function $\mu_{a_\lambda}$ defined by

$$\mu_{a_\lambda}(x) = \begin{cases} \lambda, \, x = a \\ 0, \text{ otherwise} \end{cases}$$

for all $x \in X$, where $0 < \lambda \leq 1$.

We denoted by $k_\lambda$ the fuzzy set in $X$ with the constant membership function $\mu_{k_\lambda}(x) = \lambda$ for all $x \in X$.

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**Definition 1.3.** A fuzzy topology on a set $X$ is a family $\tau$ of fuzzy sets in $X$ which satisfies the following conditions:

$(i)$ $k_0, k_1 \in \tau$

$(ii)$ If $A, B \in \tau$, then $A \cap B \in \tau$

$(iii)$ If $A_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \tau$ [1, 2].

The pair $(X, \tau)$ is called a fuzzy topological space. Every member of $\tau$ is called an open fuzzy set. The complement of an open fuzzy set is called a closed fuzzy set.

**Definition 1.4.** Let $A$ be a fuzzy set in $X$ and $\tau$ is a fuzzy topology on $X$. Then the induced fuzzy topology on $A$ is the family of subsets of $A$ which are the intersections with $A$ of open fuzzy sets in $X$. The induced fuzzy topology is denoted by $\tau_A$, and the pair $(A, \tau_A)$ is called a fuzzy subspace of $(X, \tau)$.

**Definition 1.5.** Let $(X, \tau_1)$, $(Y, \tau_2)$ be two fuzzy topological spaces. A mapping $f$ of $(X, \tau_1)$ into $(Y, \tau_2)$ is fuzzy continuous iff for each open fuzzy set $V$ in $\tau_2$ the inverse image $f^{-1}(V)$ is in $\tau_1$. Conversely, $f$ is fuzzy open iff for each open fuzzy set $U$ in $\tau_1$, the image $f(U)$ is in $\tau_2$.

The mapping $f$ is fuzzy continuous at a point $a_\lambda \in X$ iff for each open fuzzy set $V$ in $\tau_2$ containing the fuzzy point $b_\delta = (f(a))_\delta$, $0 < \delta \leq 1$, the inverse image $f^{-1}(V)$ is an open fuzzy set in $\tau_1$ containing $a_\lambda$, $0 < \lambda \leq \delta$ [3, 4].

**Lemma 1.6.** *Let $\{(X_j, \tau_j)\}$, $j \in J$, be a family of fuzzy topological spaces and $(X, \tau)$ the product fuzzy topological space. The product fuzzy topology $\tau$ on $X$ has as a base the set of finite intersections of fuzzy sets of the form $p^{-1}(U_j)$, where $U_j \in \tau_j$ and the projections $p_j$ of $X$ onto $X_j$ are fuzzy continuous for each $j \in J$* [5].

**Definition 1.7.** Two fuzzy sets $A$ and $B$ in a fuzzy topological space $(X, \tau)$ are said to be $Q$-separated if there are closed fuzzy sets $F$ and $H$ such that $A \subset F$, $B \subset H$, $F \cap B = \emptyset$ and $A \cap H = \emptyset$.

**Definition 1.8.** A fuzzy set $D$ in a fuzzy topological space $(X, \tau)$ is called disconnected if there are non-empty fuzzy sets $A$ and $B$ in the subspace $(D_0, \tau_{D_0})$ such that $A$ and $B$ are $Q$-separated and $A \cup B = D$.

A fuzzy set is called connected if it is not disconnected.

## 2. Fuzzy Contractibility

In this section,we construct the fuzzy contractible and give some characterizations of fuzzy contractible spaces.

We begin the following theorem.

**Theorem 2.1.** *Let $(X, \tau)$ be a topological space. The collection*

$$\widetilde{T} = \{D : D \text{ is a fuzzy set in } X \text{ and } D_0 \in \tau\}$$

*is a fuzzy topology on $X$, called the fuzzy topology on $X$ introduced by $T$. $\left(X, \widetilde{T}\right)$ is called the fuzzy topological space introduced by $(X,T)$* [6].

Let $\widetilde{\varepsilon}_I$ denote Euclidean subspace topology on $I$ and $(I, \widetilde{\varepsilon}_I)$ denote the fuzzy topological space introduced by the topological space $(I, \varepsilon_I)$.

**Theorem 2.2.** *Let $(X,\tau)$ and $(Y,\sigma)$ be two topological spaces. Suppose that the fuzzy sets $A$ and $B$ taking only the values $0$ and $1$ on $X$ are two closed fuzzy sets in $(X,\tau)$ and $A \cup B = X$. Let*

$$f : (A, \tau_A) \to (Y, \sigma)$$

*and*

$$g : (B, \tau_B) \to (Y, \sigma)$$

*be two fuzzy continuous functions. If $f|_{A\cap B} = g|_{A\cap B}$, then $h : (X,\tau) \to (Y,\sigma)$ defined by*

$$h(x) = \begin{cases} f(x),\ x \in A \\ g(x),\ x \in B \end{cases}$$

*is a fuzzy continuous function* [8].

**Theorem 2.3.** *Let the fuzzy sets $E$ and $H$ be connected in the fuzzy topological space $(I, \widetilde{\varepsilon}_I)$ with $E(0) = \lambda > 0$, $E(1) = H(0) = \mu > 0$ and $H(1) = \delta > 0$. Then the fuzzy set $M$ defined by*

$$M(t) = \begin{cases} E(2t) & ,\ 0 \le t \le \frac{1}{2} \\ H(2t-1) & ,\ \frac{1}{2} \le t \le 1 \end{cases}$$

*is connected and $Q$-connected in $(I, \widetilde{\varepsilon}_I)$ with $M(0) > 0$ and $M(1) > 0$* [8].

**Definition 2.4.** Let $f, g : (X,\tau) \to (Y,\sigma)$ be two fuzzy continuous mappings. If there exists a fuzzy continuous mapping

$$F : (X,\tau) \times (J, \widetilde{\varepsilon}_J) \to (Y,\sigma)$$

such that $F(x_\lambda, 0) = f(x_\lambda)$ and $F(x_\lambda, 1) = g(x_\lambda)$ for every fuzzy point $x_\lambda$ in $(X,\tau)$, then we say that $f$ is fuzzy homotopic to $g$.

The mapping $F$ is called a fuzzy homotopy between $f$ and $g$, and we write $f \simeq g$.

**Theorem 2.5.** *The relation "$\simeq$" is an equivalence relation.*

*Proof.* (1)

($i$) It is $f \simeq f$. Indeed, let us now define a mapping

$$F : (X,\tau) \times (I, \widetilde{\varepsilon}_I) \to (Y,\sigma)$$

such that $F(x_\lambda, t) = f(x_\lambda)$ for every fuzzy point $x_\lambda$ in $(X,\tau)$. Then, by Theorem 2.2, $F$ is a fuzzy continuous function and $F(x_\lambda, 0) = f(x_\lambda)$, $F(x_\lambda, 1) = f(x_\lambda)$.

(*ii*) If $f \simeq g$, then $g \simeq f$. Because,

$$f \simeq g \Rightarrow \exists F : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $F$ is a fuzzy continuous function and $F(x_\lambda, 0) = f(x_\lambda)$ and $F(x_\lambda, 1) = g(x_\lambda)$. Then, $G : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$ defined by $G(x_\lambda, t) = F(x_\lambda, 1-t)$ is a fuzzy continuous function by Theorem 2.2 and $G(x_\lambda, 0) = F(x_\lambda, 1) = g(x_\lambda)$, $G(x_\lambda, 1) = F(x_\lambda, 0) = f(x_\lambda)$.

(*iii*) If $f \simeq g$ and $g \simeq h$, then $f \simeq h$. Because,

$$f \simeq g \Rightarrow \exists F : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $F$ is a fuzzy continuous function and $F(x_\lambda, 0) = f(x_\lambda)$ and $F(x_\lambda, 1) = g(x_\lambda)$.

$$g \simeq h \Rightarrow \exists G : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $G$ is a fuzzy continuous function and $G(x_\lambda, 0) = g(x_\lambda)$ and $G(x_\lambda, 1) = h(x_\lambda)$. Then, $H : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$ defined by

$$H(x_\lambda, t) = \begin{cases} F(x_\lambda, 2t) & , 0 \le t \le \frac{1}{2} \\ H(x_\lambda, 2t-1) & , \frac{1}{2} \le t \le 1 \end{cases}$$

is a fuzzy continuous function by Theorem 2.2 and $H(x_\lambda, 0) = F(x_\lambda, 0) = f(x_\lambda)$, $H(x_\lambda, 1) = G(x_\lambda, 1) = h(x_\lambda)$. Thus, the proof is completed.

□

The fuzzy homotopy equivalence of $f$ is denoted by $[f]$.

**Definition 2.6.** Let $f, g : (X, \tau) \to (Y, \sigma)$ be fuzzy continuous mappings and $f \simeq g$. If $g$ is a constant, then $f$ is called fuzzy homotopic to a constant.

**Definition 2.7.** Let $1_X : (X, \tau) \to (X, \tau)$ be an identity mapping. If $1_X$ is fuzzy homotopic to a constant, then $(X, \tau)$ is called a fuzzy contractible space.

**Theorem 2.8.** *Let $(Y, \sigma)$ be a fuzzy contractible space. Then every fuzzy continuous function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy homotopic to a constant.*

*Proof.* Since $(Y, \sigma)$ is a fuzzy contractible space, there exists a constant mapping $g : (Y, \sigma) \to (Y, \sigma)$ such that $1_Y \simeq g$ and $g(y_\lambda) = (y_0)_\lambda \in Y$ every fuzzy point $y_\lambda$ in $(Y, \sigma)$.

$$1_Y \simeq g \Rightarrow \exists F : (Y, \sigma) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $F$ is a fuzzy continuous function and $F(y_\lambda, 0) = 1_Y(y_\lambda)$, $F(y_\lambda, 1) = g(y_\lambda) = (y_0)_\lambda$.

Now, let $f : (X, \tau) \to (Y, \sigma)$ a fuzzy continuous mapping. Then $G : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$ defined by $G(x_\lambda, t) = F(f(x_\lambda), t)$ is a fuzzy continuous mapping and has the following properties:

$$\begin{aligned} G(x_\lambda, 0) &= F(f(x_\lambda), 0) = f(x_\lambda) \\ G(x_\lambda, 1) &= F(f(x_\lambda), 1) = (y_0)_\lambda. \end{aligned}$$

Therefore, it is $f \simeq g$. □

**Theorem 2.9.** *Let $f, g : (X, \tau) \to (Y, \sigma)$ be fuzzy continuous functions such that $f \simeq g$. If $h : (Y, \sigma) \to (Z, \delta)$ is a fuzzy continuous function, then $hf, hg : (X, \tau) \to (Z, \delta)$ are fuzzy continuous functions and $hf \simeq hg$.*

*Proof.* Since $h, f, g$ are fuzzy continuous functions, $hf, hg$ are fuzzy continuous functions. Furthermore,

$$f \simeq g \Rightarrow \exists F : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Y, \sigma)$$

such that $F$ is a fuzzy continuous function and $F(x_\lambda, 0) = f(x_\lambda)$, $F(x_\lambda, 1) = g(x_\lambda)$.

Now, $G : (X, \tau) \times (J, \widetilde{\varepsilon}_J) \to (Z, \delta)$ is given by $G(x_\lambda, t) = h(F(x_\lambda, t))$. Then, since $h, F$ are fuzzy continuous functions, $G = h \circ f$ is a fuzzy continuous function. Moreover, $G$ satisfies the following conditions:

$$\begin{aligned} G(x_\lambda, 0) &= h(F(x_\lambda, 0)) = h(f(x_\lambda)) = (hf)(x_\lambda) \\ G(x_\lambda, 1) &= h(F(x_\lambda, 1)) = h(g(x_\lambda)) = (hg)(x_\lambda). \end{aligned}$$

Therefore, $hf \simeq hg$. □

**Definition 2.10.** Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy continuous function. If there is a fuzzy continuous function $f^{'}$ satisfies the following conditions:

$(i)$ $ff^{'} \simeq 1_Y$
$(ii)$ $f^{'}f \simeq 1_X$

then, $f$ is called a fuzzy homotopy equivalence. Further, fuzzy topological spaces are called fuzzy homotopic equivalent spaces and denoted by $X \simeq Y$.

It is easily seen that this relation is an equivalence relation.

**Theorem 2.11.** *If $X$ and $Y$ are fuzzy topological equivalent spaces, then $X$ and $Y$ are fuzzy homotopic equivalent spaces.*

*Proof.* Since $X, Y$ are fuzzy topological equivalent spaces, there exists a function $f : X \to Y$ such that $f$ is one to one and surjective. Moreover, $f : X \to Y$ and $f^{-1} : Y \to X$ are fuzzy continuous functions. Therefore, $ff^{-1} = 1_Y$, $f^{-1}f = 1_X$. Since "$\simeq$" relation is an equivalence relation, $ff^{-1} \simeq 1_Y$, $f^{-1}f \simeq 1_X$. Thus, it is $X \simeq Y$. □

**Theorem 2.12.** *Let $X$ be any fuzzy topological space. $X$ is a fuzzy contractible space if and only if $X$ is fuzzy homotopic equivalent with a fuzzy single-point space.*

*Proof.* Let $X$ be a fuzzy contractible topological space. Then there exists a constant function $h : X \to X$ by defined $h(x_\lambda) = (x_0)_\lambda$ for every $x_\lambda \in X$ such that $1_X \simeq h$. Now, let $Y = \{(x_0)_\lambda\}$ be fuzzy single-point space, $f : X \to Y$ be fuzzy continuous function and $i : Y \to X$ be inclusion function. Then, $i$ is a fuzzy continuous function and $if = h$, $fi = 1_Y$. Since $1_X \simeq h$ and "$\simeq$" is an equivalence relation, $if \simeq 1_X$, $fi \simeq 1_Y$. Therefore, $X \simeq Y$.

Conversely, let us suppose that $Y$ is a fuzzy single-point space and $X \simeq Y$. Then, there exists a fuzzy continuous function $f : X \to Y$ such that $f$ is a fuzzy homotopy equivalence and so $f^{'}: Y \to X$ is a fuzzy continuous function, $f^{'}f \simeq 1_X$, $ff^{'} \simeq 1_Y$. Since $Y$ is a fuzzy single-point space, $Y$ can be chosen as $Y = \{(y_0)_\lambda\}$. Then, $f(x_\lambda) = (y_0)_\lambda$ for every $x_\lambda \in X$. Now, let $f^{'}((y_0)_\lambda) = (x_0)_\lambda$. It is clear that $f^{'}f = h$ and $1_X \simeq h$. Thus, $X$ is a fuzzy contractible space. □

## FUZZY BÜZÜLEBİLME

**ÖZET:** Bu çalışmada ilk olarak fuzzy topolojik uzaylarla ilgili bazı temel kavramlar verilmiştir. Daha sonra fuzzy irtibatlı cümle kavramı üzerinde incelemeler yapılmıştır. Son kısımda fuzzy irtibatlı uzay tanımlanarak, bir fuzzy uzayın fuzzy irtibatlı uzay olması için gerek ve yeter şartın o uzayın bir fuzzy tek nokta uzayı ile fuzzy homotopik eşdeğer olduğu gösterilmiştir.

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*Current address*: Ankara University, Faculty of Science, Department of Mathematics, Tandoğan, Ankara, Turkey

*E-mail address*: guner@science.ankara.edu.tr