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INTERVAL OSCILLATION CRITERIA FOR SECOND-ORDER DELAY AND ADVANCED DIFFERENCE EQUATIONS

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Abstract. Interval oscillation criteria are established for second-order difference equations in the form

$$\Delta\left(k\left(n\right)\Delta x(n)\right)+p\left(n\right)x\left(g\left(n\right)\right)+q\left(n\right)\left|x\left(g\left(n\right)\right)\right|^{\gamma-1}x\left(g\left(n\right)\right)=e\left(n\right), \quad (E_\gamma)$$

where $n\geq n_0$, $n_0\in\mathbb{N}=\{0,1,...\}$, $\gamma>1$; k, p, q, e and g are sequences of real numbers; $k\left(n\right)>0$ is nondecreasing; $g(n)$ is nondecreasing, $\lim_{n\to\infty}g(n)=\infty$. Several oscillation criteria are given for equation (E_γ) considered as to separate delay and advanced difference equations when $g(n)<n$ and $g(n)>n$ respectively. Illustrative examples are included.

1. Introduction

We consider second-order difference equations of the form,

$$\Delta\left(k\left(n\right)\Delta x(n)\right)+p\left(n\right)x\left(g\left(n\right)\right)+q\left(n\right)\left|x\left(g\left(n\right)\right)\right|^{\gamma-1}x\left(g\left(n\right)\right)=e(n) \qquad (E_\gamma)$$

where $n\geq n_0$, $n_0\in\mathbb{N}=\{0,1,...\}$, $\gamma>1$; k, p, q, e and g are sequences of real numbers; $k\left(n\right)>0$ is nondecreasing; $g(n)$ is nondecreasing, $\lim_{n\to\infty}g(n)=\infty$. Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$. As is customary, we assume that solutions of (E_γ) exist on some set $\{n_0,\,n_0+1,\,...\}$. For the theory of existence of solutions of such equations, we refer [1]. A nontrivial solution $\{x(n)\}$ of (E_γ) is called oscillatory if for any given $\tilde{n}_0\geq n_0$ there exists an integer $n_1\geq\tilde{n}_0$ such that $x(n_1)x(n_1+1)\leq 0$, otherwise it is called nonoscillatory. The equation will be called oscillatory if every solution is oscillatory. Taking $g(n)$ as $\tau(n)$ with $\tau(n)<n$ and $\lim_{n\to\infty}\tau(n)=\infty$, $\gamma=\alpha$, equation (E_γ) is considered as a delay difference equation

$$\Delta\left(k\left(n\right)\Delta x(n)\right)+p_1\left(n\right)x\left(\tau\left(n\right)\right)+q_1\left(n\right)\left|x\left(\tau\left(n\right)\right)\right|^{\alpha-1}x\left(\tau\left(n\right)\right)=e\left(n\right) \qquad (E_D)$$

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or taking $g(n)$ as $\sigma(n)$ with $\sigma(n) > n$ and $\gamma = \beta$,equation (E_γ) is considered as an advanced difference equation

$$\Delta\left(k\left(n\right)\Delta x(n)\right)+p_2\left(n\right)x\left(\sigma\left(n\right)\right)+q_2\left(n\right)\left|x\left(\sigma\left(n\right)\right)\right|^{\beta-1}x\left(\sigma\left(n\right)\right)=e\left(n\right). \qquad (E_A)$$

In literature, there isn't enough work dealing with the oscillation of difference equations (E_D) and (E_A). Equation (E_γ), when $k(n) \equiv 1$, $p\left(n\right) \equiv 0$ or $q\left(n\right) \equiv 0$ and $g(n) = n$, $n+1$, $n-\tau$ has been studied by many authors, see [6, 7, 12, 13, 15] and the references cited therein.

Using Riccatti tecnique, Saker[9] obtained some oscillation criteria for forced Emden-Fowler superlinear difference equation of the form

$$\Delta^2 x(n)+q\left(n\right)x^{\gamma}\left(n+1\right)=e(n)$$

when $q(n)$ and $e(n)$ are sequences of positive real numbers.

Zhang and Chen [14] established some oscillation criteria

$$\Delta^2 x(n)+q\left(n\right)f\left(x\left(n+1\right)\right)=0$$

when$f$ is nondecreasing and $uf(u) > 0$ for $u \neq 0$.

The first result concerning the interval oscillation of $(E\gamma)$ when $g(n) = n+1$, $q(n) \equiv 0$, $e(n) \equiv 0$ has been studied by Kong and Zettl [7]. They have applied the telescoping principle for equation of the form

$$\Delta\left(k\left(n\right)\Delta x(n)\right)+p\left(n\right)x\left(n+1\right)=0.$$

Recently, Güvenilir and Zafer [4] has presented some sufficient conditions about oscillation of second-order differential equation

$$(k(t)x'(t))'+p\left(t\right)\left|x\left(\tau\left(t\right)\right)\right|^{\alpha-1}x\left(\tau\left(t\right)\right)+q\left(t\right)\left|x\left(\sigma\left(t\right)\right)\right|^{\beta-1}x\left(\sigma\left(t\right)\right)=e\left(t\right). \qquad (1.1)$$

where $n \geq 0$. Later, in [2] Anderson generalized the results of Güvenilir and Zafer [4] to the dynamic equation

$$(kx^{\Delta})^{\Delta}(t)+p\left(t\right)\left|x\left(\tau\left(t\right)\right)\right|^{\alpha-1}x\left(\tau\left(t\right)\right)+q\left(t\right)\left|x\left(\sigma\left(t\right)\right)\right|^{\beta-1}x\left(\sigma\left(t\right)\right)=e\left(t\right) \qquad (1.2)$$

where $n \geq 0$ for arbitrary time scales.

In this work, our purpose is to derive interval oscillation criteria as discrete analogues of the ones contained [3]. The difference between (E_γ) and (1.2) is the appearence of both linear and nonlinear terms. Therefore, the results in [2] fails to apply for (E_γ).

For our purpose, we denote

$$D\left(a_k, b_k\right) = \left\{u : u\left(a_k\right) = u\left(b_k\right) = 0,\ k = 1, 2,\ u\left(n\right) \not\equiv 0,\ n \in \mathbb{N}(a_k, b_k)\right\},$$

where $\mathbb{N}(a_k, b_k) = \{a_k, a_k+1, ..., b_k\}$. As in [4], we define

$$P_*\left(n\right) = *\left(*-1\right)^{1/*-1}q\left(n\right)^{1/*}\left|e\left(n\right)\right|^{1-1/*}. \qquad (1.3)$$

2. Delay Difference Equations

Suppose that for any given $N \geq 0$ there exist a_1,a_2,b_1,$b_2 \geq N$ such that $a_1 < b_1$, $a_2 < b_2$ and

$$p_1\left(n\right) \geq 0,\ \ q_1(n) \geq 0 \text{ for } n \in \mathbb{N}(\tau\left(a_1\right), b_1) \cup \mathbb{N}(\tau\left(a_2\right), b_2). \tag{2.1}$$

Let $e\left(n\right)$ satisfies

$$\begin{array}{c} e\left(n\right) \leq 0, \ \text{ for } n \in \mathbb{N}(\tau\left(a_1\right), b_1) \\ e\left(n\right) \geq 0, \ \text{ for } n \in \mathbb{N}(\tau\left(a_2\right), b_2). \end{array} \tag{2.2}$$

Theorem 2.1. *Suppose that* (2.1) *and* (2.2) *hold. If there exist an* $H_1 \in D\left(a_i, b_i\right),\ i = 1, 2,$ *such that*

$$\sum_{n=a_i}^{b_i-1}\left[H_1^2\left(n+1\right)\left(p_1\left(n\right)+P_\alpha\left(n\right)\right)\frac{\tau\left(n\right)-\tau\left(a_i\right)}{n+1-\tau\left(a_i\right)} - \left(\Delta H_1\left(n\right)\right)^2 k\left(n\right)\right] \geq 0, \tag{2.3}$$

for $i = 1, 2,$ *then* (E_D) *is oscillatory.*

Proof. To get a contradiction, let us suppose that $x\left(n\right)$ is a nonoscillatory solution of equation $\left(E_D\right)$. First, assume $x\left(n\right) > 0$, $x\left(\tau\left(n\right)\right) > 0$ for all $n \geq n_1$ for some $n_1 > 0$.

We may say

$$F(x) = Ax^\mu - \mu\left(\mu-1\right)^{1/\mu-1} A^{1/\mu} B^{1-1/\mu} x + B \geq 0 \text{ for } x \in [0, \infty) \tag{2.4}$$

where A, B are nonnegative constants and $\mu > 1$, [10].

If we choose $A = q_1(t)$, $B = -e(n)$ and $\mu = \alpha$ in (2.4), we have

$$q_1\left(t\right) x^\alpha\left(\tau\left(n\right)\right) - e\left(n\right) \geq \alpha\left(\alpha-1\right)^{1/\alpha-1} q_1\left(n\right)^{\frac{1}{\alpha}} \left|e\left(n\right)\right|^{1-\frac{1}{\alpha}} x\left(\tau\left(n\right)\right). \tag{2.5}$$

for $n \in \mathbb{N}(\tau(a_1), b_1)$

See also [8, 10].

Define

$$w\left(n\right) = -\frac{k\left(n\right)\Delta x\left(n\right)}{x\left(n\right)},\ \ n \geq n_1,\ \ n_1 > 0. \tag{2.6}$$

In view of $\left(E_D\right)$, we see that

$$\begin{array}{c} \Delta w\left(n\right) = \frac{x(n)}{k(n)x(n+1)} w^2\left(n\right) + p_1\left(n\right) \frac{x(\tau(n))}{x(n+1)} \\ + \left[q_1\left(n\right) x^\alpha\left(\tau\left(n\right)\right) - e\left(n\right)\right] \frac{1}{x(n+1)}. \end{array} \tag{2.7}$$

Using (2.1) and (2.5), we see from (2.7) that

$$\Delta w\left(n\right) \geq \frac{x(n)}{k\left(n\right)x(n+1)} w^2\left(n\right) + \left[p_1\left(n\right) + P_\alpha\left(n\right)\right]\frac{x\left(\tau\left(n\right)\right)}{x\left(n+1\right)},\ \ n \in \mathbb{N}(\tau\left(a_1\right), b_1).$$

Moreover

$$x(n+1) = x(n) + \Delta x(n),$$

$$\frac{x(n+1)}{x(n)} = 1 + \frac{\Delta x(n)}{x(n)}$$

and then

$$\frac{x(n)}{k\left(n\right)x(n+1)} = \frac{1}{k\left(n\right) - w\left(n\right)}.$$

Therefore

$$\Delta w\left(n\right) \geq \frac{1}{k\left(n\right) - w\left(n\right)} w^{2}\left(n\right) + \left[p_{1}\left(n\right) + P_{\alpha}\left(n\right)\right] \frac{x\left(\tau\left(n\right)\right)}{x\left(n+1\right)},\ n \in \mathbb{N}(\tau\left(a_{1}\right), b_{1}). \tag{2.8}$$

Now by the Mean Value Theorem in [1]

$$x(n) - x\left(\tau\left(a_{1}\right)\right) \geq \frac{k\left(\xi\right)\Delta x\left(\xi\right)}{k\left(\xi\right)}(n - \tau\left(a_{1}\right))$$

for some $\xi \in \mathbb{N}(\tau\left(a_{1}\right), n)$. From which, for any $n \in \mathbb{N}(a_1, b_1)$,we have

$$x(n) \geq \Delta x\left(n\right)(n - \tau\left(a_{1}\right)),\ n \in \mathbb{N}(a_{1}, b_{1})$$

and hence,

$$\frac{\Delta x\left(n\right)}{x(n)} \leq \frac{1}{n - \tau\left(a_{1}\right)},\ n \in \mathbb{N}(a_{1}, b_{1}).$$

Moreover, following the arguments in [2], since

$$x(m) - \Delta x\left(m\right)(m - \tau\left(a_{1}\right)) \geq 0,\ m \in \mathbb{N}(\tau(n), n+1),\ n \in \mathbb{N}(a_{1}, b_{1})$$

we have

$$\frac{x(m) - \Delta x\left(m\right)(m - \tau\left(a_{1}\right))}{x(m)x(m+1)} \geq 0.$$

Therefore,

$$\Delta\left(\frac{m - \tau\left(a_{1}\right)}{x(m)}\right) \geq 0.$$

It follows that

$$\sum_{m=\tau(n)}^{n} \Delta\left(\frac{m - \tau\left(a_{1}\right)}{x(m)}\right) = \frac{n + 1 - \tau\left(a_{1}\right)}{x(n+1)} - \frac{\tau\left(n\right) - \tau\left(a_{1}\right)}{x(\tau\left(n\right))},$$

in other words

$$\frac{x(\tau\left(n\right))}{x(n+1)} \geq \frac{\tau\left(n\right) - \tau\left(a_{1}\right)}{n + 1 - \tau\left(a_{1}\right)},\ n \in \mathbb{N}(a_{1}, b_{1}). \tag{2.9}$$

In view of (2.9), it follows from (2.8) that

$$\Delta w(n) \geq \frac{1}{k(n) - w(n)} w^2(n) + [p_1(n) + P_\alpha(n)] \frac{\tau(n) - \tau(a_1)}{n + 1 - \tau(a_1)}, \ n \in \mathbb{N}(\tau(a_1), b_1). \tag{2.10}$$

Let $H_1 \in D(a_1, b_1)$ be given as in the hypothesis. Multiplying $H_1^2(n+1)$ through (2.10) we find

$$\begin{aligned} \Delta w(n) H_1^2(n+1) \quad & \geq \tfrac{1}{k(n)-w(n)} w^2(n) H_1^2(n+1) \\ & + [p_1(n) + P_\alpha(n)] \tfrac{\tau(n)-\tau(a_1)}{n+1-\tau(a_1)} H_1^2(n+1) \end{aligned}$$

for $n \in \mathbb{N}(\tau(a_1), b_1)$. Since

$$\Delta(H_1^2(n) w(n)) = H_1^2(n+1)\Delta w(n) + w(n) \Delta H_1^2(n)$$

$$\begin{aligned} \Delta(H_1^2(n)) &= \Delta(H_1(n) H_1(n)) \\ &= H_1(n+1)\Delta H_1(n) + H_1(n) \Delta H_1(n) \\ &= \Delta H_1(n) (H_1(n+1) + H_1(n)) \end{aligned}$$

and

$$\Delta(H_1^2(n)) = \Delta H_1(n) [2H_1(n+1) - \Delta H_1(n)]$$

then taking the sum from a_1 to $(b_1 - 1)$ we obtain

$$\begin{aligned} & \sum_{n=a_1}^{b_1-1} \left\{ [p_1(n) + P_\alpha(n)] \frac{\tau(n) - \tau(a_1)}{n + 1 - \tau(a_1)} H_1^2(n+1) - k(n) (\Delta H_1(n))^2 \right\} \\ \leq \quad & -\Delta H_1^2 w(a_1) - \sum_{n=a_1}^{b_1-1} \left[\frac{w(n)H_1(n+1)}{\sqrt{k(n) - w(n)}} + \sqrt{k(n) - w(n)} \Delta H_1(n) \right]^2 . \end{aligned}$$

$$\begin{aligned} & \sum_{n=a_1}^{b_1-1} \left\{ [p_1(n) + P_\alpha(n)] \frac{\tau(n) - \tau(a_1)}{n + 1 - \tau(a_1)} H_1^2(n+1) - k(n) (\Delta H_1(n))^2 \right\} \\ \leq \quad & - \sum_{n=a_1}^{b_1-1} \left[\frac{w(n)H_1(n+1)}{\sqrt{k(n) - w(n)}} + \sqrt{k(n) - w(n)} \Delta H_1(n) \right]^2 < 0. \end{aligned} \tag{2.11}$$

Note that

$$\sum_{n=a_1}^{b_1-1} \left[\frac{w(n)H_1(n+1)}{\sqrt{k(n) - w(n)}} + \sqrt{k(n) - w(n)} \Delta H_1(n) \right]^2 = 0$$

is possible only if

$$\frac{w(n)H_1(n+1)}{\sqrt{k\,(n)-w\,(n)}}+\sqrt{k\,(n)-w\,(n)}\Delta H_1\,(n)=0.$$

Therefore

$$-\frac{w(n)H_1(n+1)}{\sqrt{k\,(n)-w\,(n)}}=\sqrt{k\,(n)-w\,(n)}\Delta H_1\,(n)$$

$$-w(n)H_1(n+1)=(k\,(n)-w\,(n))\Delta H_1\,(n)$$

and then

$$\frac{k\,(n)\,\Delta x\,(n)}{x(n)}H_1(n+1)=\frac{k\,(n)\,x\,(n+1)}{x(n)}\Delta H_1(n)$$

$$\Delta x\,(n)\,H_1(n+1)=x\,(n+1)\,\Delta H_1\,(n)\,.$$

Hence

$$\Delta(\frac{H_1\,(n)}{x\,(n)})=0$$

which implies

$$H_1\,(n)=cx\,(n)\,,$$

where c is a constant. This, however, contradicts the positivity of $x\,(n)$. Now (2.11) contradicts (2.3). Thus, the proof is complete, when $x\,(n)$ is eventually positive. The proof can be accomplished similarly by working with $\mathbb{N}(a_2,b_2)$ instead of $\mathbb{N}(a_1,b_1)$ when $x\,(n)$ is eventually negative.

Example 2.1. Consider the forced delay difference equation,

$$\Delta^2 x\,(n)+m_1\sin\left(\frac{\pi n}{60}\right)x\,(n-2)+m_2\cos\left(\frac{\pi n}{60}\right)x^3\,(n-2)=\cos\left(\frac{\pi n}{10}\right) \quad (2.12)$$

where $m_1,\ m_2>0$. Let

$$\begin{aligned} a_1 &= 8+120k,\ b_1=11+120k,\\ a_2 &= 17+120k,\ b_2=20+120k \end{aligned}$$

for any nonnegative integer k and let $H_1\,(n)=\sin\left(\pi\frac{(n+1)}{3}\right)$. It is easy to check that (2.1) is satisfied, namely

$$p_1(n)=m_1\sin(\frac{\pi n}{60})\geq 0,\ \ for\ n\in\mathbb{N}(6+120k,11+120k)\cup(15+120k,20+120k).$$

$$q_1(n)=m_2\cos(\frac{\pi n}{60})\geq 0,\ \ for\ n\in\mathbb{N}(6+120k,11+120k)\cup(15+120k,20+120k).$$

and

$$e(n) = \cos(\frac{\pi n}{10}) \leq 0, \ for \ n \in \mathbb{N}(6 + 120k, 11 + 120k)$$

$$e(n) = \cos(\frac{\pi n}{10}) \geq 0, \ for \ n \in \mathbb{N}(15 + 120k, 20 + 120k)$$

where $\tau(n) = n - 2$.

By Theorem 2.1, the equation (2.12) is oscillatory when $m_1 = 1$, $m_2 > 79$; when $m_2 = 1$, $m_1 > 14$.

3. Advanced Difference Equations

Consider

$$\Delta\left(k\left(n\right)\Delta x(n)\right) + p_2\left(n\right)x\left(\sigma\left(n\right)\right) + q_2\left(n\right)\left|x\left(\sigma\left(n\right)\right)\right|^{\beta-1}x\left(\sigma\left(n\right)\right) = e\left(n\right). \qquad (E_A)$$

where $n \geq n_0$, $n_0 \in \mathbb{N} = \{0, 1, ...\}$, $\beta > 1$, k, p_2, q_2, e and σ are sequences of real numbers, $k\left(n\right) > 0$ is nondecreasing; $\sigma(n) > n$, σ is nondecreasing. Suppose that for any given $N \geq 0$ there exist $c_1, c_2, d_1, d_2 \geq N$ such that $c_1 < d_1$, $c_2 < d_2$ and

$$p_2\left(n\right) \geq 0, \ q_2\left(n\right) \geq 0, for \ n \in \mathbb{N}(c_1, \sigma(d_1)) \cup \mathbb{N}(c_2, \sigma(d_2)). \qquad (3.1)$$

Let $e\left(n\right)$ satisfies

$$\begin{array}{c} e\left(n\right) \leq 0, \ for \ n \in \mathbb{N}(c_1, \sigma(d_1)) \\ e\left(n\right) \geq 0, \ for \ n \in \mathbb{N}(c_2, \sigma(d_2)). \end{array} \qquad (3.2)$$

Now, we can give the following .

Theorem 3.1. *Suppose that* (3.1) *and* (3.2) *hold. If there exist an* $H_2 \in D\left(c_i, d_i\right)$ *such that*

$$\sum_{n=c_i}^{d_i-1}\left[H_2^2\left(n+1\right)\left(p_2\left(n\right) + P_\beta\left(n\right)\right)\frac{\sigma\left(d_i\right) - \sigma\left(n\right)}{\sigma\left(d_i\right) - \left(n+1\right)} - \left(\Delta H_2\left(n\right)\right)^2 k\left(n\right)\right] \geq 0 \quad (3.3)$$

for $i = 1, 2$, *then* (E_A) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that $x\left(n\right)$ is a nonoscillatory solution of equation (E_A). First, assume $x\left(n\right)$, $x\left(\sigma\left(n\right)\right)$ are positive for all $n \geq n_1$ for some $n_1 > 0$.

Considering (2.6), in view of (E_A), we see that

$$\begin{aligned} \Delta w(n) \quad &= \quad \frac{x(n)}{k\left(n\right)x(n+1)}w^2\left(n\right) + p_2\left(n\right)\frac{x\left(\sigma\left(n\right)\right)}{x\left(n+1\right)} \\ &\quad + \left[q_2\left(n\right)x^\beta\left(\sigma\left(n\right)\right) - e\left(n\right)\right]\frac{1}{x\left(n+1\right)}. \end{aligned}$$

In (2.5) instead of $\tau(n)$, α and q_1 we take $\sigma(n)$, β and q_2 respectively, we get

$$\Delta w\left(n\right)\geq\frac{x(n)}{k\left(n\right)w\left(n+1\right)}w^{2}\left(n\right)+\left[p_{2}\left(n\right)+P_{\beta}\left(n\right)\right]\frac{x\left(\sigma\left(n\right)\right)}{x\left(n+1\right)},\ n\in\mathbb{N}(c_{1},\sigma(d_{1})).$$

By the same steps in Theorem 2.1, we obtain

$$\Delta w\left(n\right)\geq\frac{1}{k\left(n\right)-w\left(n\right)}w^{2}\left(n\right)+\left[p_{2}\left(n\right)+P_{\beta}\left(n\right)\right]\frac{x\left(\sigma\left(n\right)\right)}{x\left(n+1\right)},\ n\in\mathbb{N}(c_{1},\sigma(d_{1})). \tag{3.4}$$

Note that $\Delta\left(k\left(n\right)\Delta x(n)\right)\leq 0$ on $\left[c_{1},\sigma(d_{1})\right]$. In a similar manner as in the proof of Theorem (2.1) we get

$$\frac{x(\sigma\left(n\right))}{x(n+1)}\geq\frac{\sigma\left(d_{1}\right)-\sigma\left(n\right)}{\sigma\left(d_{1}\right)-\left(n+1\right)},\ n\in\mathbb{N}(c_{1},\sigma(d_{1})). \tag{3.5}$$

Applying inequality (3.5) to (3.4), we obtain

$$\Delta w\left(n\right)\geq\frac{1}{k\left(n\right)-w\left(n\right)}w^{2}\left(n\right)+\left[p_{2}\left(n\right)+P_{\beta}\left(n\right)\right]\frac{\sigma\left(d_{1}\right)-\sigma\left(n\right)}{\sigma\left(d_{1}\right)-\left(n+1\right)},n\in\mathbb{N}(c_{1},\sigma(d_{1})).$$

Using the same steps in the proof of Theorem (2.1) we get

$$\begin{aligned}&\sum_{n=c_{1}}^{d_{1}-1}\left\{\left[p_{2}\left(n\right)+P_{\beta}\left(n\right)\right]\frac{\sigma\left(d_{1}\right)-\sigma\left(n\right)}{\sigma\left(d_{1}\right)-\left(n+1\right)}H_{2}^{2}\left(n+1\right)-k\left(n\right)\left(\Delta H_{2}\left(n\right)\right)^{2}\right\}\\ \leq\quad&-\sum_{n=c_{1}}^{d_{1}-1}\left[\frac{w(n)H_{2}(n+1)}{\sqrt{k\left(n\right)-w\left(n\right)}}+\sqrt{k\left(n\right)-w\left(n\right)}\Delta H_{2}\left(n\right)\right]^{2}<0.\end{aligned} \tag{3.6}$$

(3.6) contradicts (3.3). Thus the proof is complete, when $x(n)$ is eventually positive. The proof can be accomplished similarly by working with $\mathbb{N}(c_2,d_2)$ instead of $\mathbb{N}(c_1,d_1)$ when $x(n)$ is eventually negative.

Example 3.1. Consider the advanced difference equation,

$$\Delta^{2}x\left(n\right)+m_{1}\sin\left(\frac{\pi n}{60}\right)x\left(n+2\right)+m_{2}\cos\left(\frac{\pi n}{60}\right)x^{3}\left(n+2\right)=\cos\left(\frac{\pi n}{10}\right) \tag{3.7}$$

where $m_1,\, m_2\geq 0$. Let

$$\begin{aligned}c_{1}&=&6+120k,\ d_{1}=9+120k,\\ c_{2}&=&15+120k,\ d_{2}=18+120k\end{aligned}$$

for any nonnegative integer k and let $H_2\left(n\right)=\sin\left(\frac{n\pi}{3}\right)$. It is easy to check that (3.1) is satisfied, namely

$$p_2(n) = m_1 \sin(\frac{\pi n}{60}) \geq 0, \ for \ n \in \mathbb{N}(6+120k, 11+120k) \cup (15+120k, 20+120k).$$

$$q_2(n) = m_2 \cos(\frac{\pi n}{60}) \geq 0, \ for \ n \in \mathbb{N}(6+120k, 11+120k) \cup (15+120k, 20+120k).$$

and

$$e(n) = \cos(\frac{\pi n}{10}) \leq 0, \ for \ n \in \mathbb{N}(6+120k, 11+120k)$$

$$e(n) = \cos(\frac{\pi n}{10}) \geq 0, \ for \ n \in \mathbb{N}(15+120k, 20+120k)$$

where $\sigma(n) = n+2$.

By Theorem 3.1, the equation (3.7) is oscillatory when $m_1 = 1$, $m_2 > 10$; when $m_2 = 1$, $m_1 > 1$.

4. Delay and Advanced Difference Equations

We obtain the delay and advanced difference equations as follows:

$$\begin{aligned} &\Delta\left(k\left(n\right)\Delta x(n)\right) + p_1\left(n\right)x\left(\tau\left(n\right)\right) + q_1\left(n\right)\left|x\left(\tau\left(n\right)\right)\right|^{\alpha-1}x\left(\tau\left(n\right)\right) \\ &\quad + p_2\left(n\right)x\left(\sigma\left(n\right)\right) + q_2\left(n\right)\left|x\left(\sigma\left(n\right)\right)\right|^{\beta-1}x\left(\sigma\left(n\right)\right) = e\left(n\right), \end{aligned} \qquad (E_{A,D})$$

where $n \geq n_0$, $n_0 \in \mathbb{N} = \{0, 1, ...\}$, $\beta > 1$, k, p_1, p_2, q_1, q_2, e, τ and σ are sequences of real numbers, $k\left(n\right) > 0$ is nondecreasing; $\tau\left(n\right) < n$, $\sigma(n) > n$, τ and σ are nondecreasing and $\lim_{\mathbf{t}\to\infty} \tau\left(t\right) = \infty$.

Suppose that for any given $N \geq 0$ there exist a_1, a_2, b_1, b_2 , $c_1, c_2, d_1, d_2 \geq N$ such that $a_1 < b_1$, $a_2 < b_2$ and $c_1 < d_1$, $c_2 < d_2$.

Theorem 4.1. Suppose that $(2.1), (2.2)$ and $(3.1), (3.2)$ hold. If there exists an $H_1 \in D\left(a_i, b_i\right)$ and $H_2 \in D\left(c_i, d_i\right)$ such that either

$$\sum_{n=a_i}^{b_i-1}\left[H_1^2\left(n+1\right)\left(p_1\left(n\right)+P_\alpha\left(n\right)\right)\frac{\tau\left(n\right)-\tau\left(a_i\right)}{n+1-\tau\left(a_i\right)} - \left(\Delta H_1\left(n\right)\right)^2 k\left(n\right)\right] \geq 0,$$

or

$$\sum_{n=c_i}^{d_i-1}\left[H_2^2\left(n+1\right)\left(p_2\left(n\right)+P_\beta\left(n\right)\right)\frac{\sigma\left(d_i\right)-\sigma\left(n\right)}{\sigma\left(d_i\right)-\left(n+1\right)} - \left(\Delta H_2\left(n\right)\right)^2 k\left(n\right)\right] \geq 0$$

for $i = 1, 2$, then $(E_{A,D})$ is oscillatory.