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CHARACTERIZATIONS OF INCLINED CURVES WHICH IS CONCERNED WITH OSCULATING SPHERE IN $\mathbb{L}^n$ FOR SPACE-LIKE CURVES

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Abstract. In this study, we first calculate the higher ordered curvatures in terms of higher ordered harmonic curvatures for the space-like curves in $\mathbb{L}^n$. Thus, we give characterizations of inclined curves in $\mathbb{L}^n$ for space-like curves. Furthermore, we obtain the coordinates of the central point of the space-like curve's osculating sphere. Then we calculate the higher ordered curvatures of the space-like curves in terms of the coordinates of the central point of its osculating sphere. At the end, we give another characterization for inclined curves in $\mathbb{L}^n$, in terms of these coordinates.

1. Introduction

The characterizations of inclined curves in E^n is given in [3] and [4] that

1) γ is an inclined curve in $E^n \Leftrightarrow \sum\limits_{i=1}^{n-2} H_i^2$ =constant and

2) γ is an inclined curve in $E^{n-1} \Leftrightarrow \det\left(V_1^{'}, V_2^{'}, ..., V_n^{'}\right) = 0.$

In Section 3 and Section 5, we show that inclined curves which is space-like in $\mathbb{L}^n$ have the following characterizations:

1) γ is an inclined curve in $\mathbb{L}^n \Leftrightarrow \sum\limits_{i=1}^{n-2} \varepsilon_{i+1} H_i^2$ =constant and

2) γ is an inclined curve in $\mathbb{L}^{n-1} \Leftrightarrow \det\left(V_1^{'}, V_2^{'}, ..., V_n^{'}\right) = 0.$

In Section 4, we obtain the relation between the higher ordered curvatures and the higher ordered harmonic curvatures.

In addition, characterizations of inclined curves which is concerned with osculating sphere in E^n are given in [7]. In Section 6, we show that if M is a space-like

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curve in $\mathbb{L}^n$ then the coordinates of the central point of the osculating sphere of M are

$$m_i : I \to \mathbb{R}$$
$$m_i = \begin{cases} 0 & , i = 1 \\ \frac{\varepsilon_1}{k_1(s)} & , i = 2 \\ \{m'_{i-1}(s) + k_{i-2}(s)m_{i-2}(s)\}\frac{\varepsilon_{i-1}\varepsilon_{i-2}}{k_{i-1}(s)} & , 2 < i \leq n. \end{cases}$$

In Section 7, we give the relation between the functions of m_i and k_i. In Section 8, we write the relationship between m'_i and m_i values in the matrix form. Let M be a space-like curve in $\mathbb{L}^{n+1}$, $n \geq 4$, and let n be an even number. If we denote the coefficient matrix as B_n, we get the following characterizations:

1) $\det B_n = 0 \Leftrightarrow \sum\limits_{i=2}^{n} \varepsilon_{i-1} m_i^2$ =constant

2) $\det B_n = 0 \Leftrightarrow M \subset \mathbb{L}^{n+1}$ is an inclined curve in $\mathbb{L}^n$, where $m'_i = \frac{dm_i}{ds}$ and s denotes the arc length parameter of M.

2. Preliminaries

2.1. **Symmetric bilinear forms.** Let V be a real vector space. A bilinear form on V is an r-bilinear function:

$$\langle\ ,\ \rangle : \mathbb{V}x\mathbb{V} \to \mathbb{R}.$$

We also consider only the symmetric case. A symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{V}$ is :

(a) positive [negative] definite providing that $v \neq 0$ implies $\langle v, v\rangle > 0$ [< 0] ,

(b) positive [negative] semidefinite providing that $\langle v, v\rangle \geq 0$ [≤ 0] for all $v \in \mathbb{V}$.

(c) nondegenerate providing that $\langle v, w\rangle = 0$ for all $w \in \mathbb{V}$ implies $v \neq 0$.

If $\langle\ ,\ \rangle$ is a symmetric bilinear form on $\mathbb{V}$, then, for any subspace $\mathbb{W}$ of $\mathbb{V}$, the restriction $\langle\ ,\ \rangle\,|_{\mathbb{W}X\mathbb{W}}$denoted merely by $\langle\ ,\ \rangle\,|_{\mathbb{W}}$ is also symmetric and bilinear.

The index q of a symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{V}$ is the largest integer that is the dimension of a subspace $\mathbb{W} \subset \mathbb{V}$ on which $\langle\ ,\ \rangle\,|_{\mathbb{W}}$ is negative definite.

Thus, $0 \leq q \leq \dim \mathbb{V}$ and $q = 0$ if and only if $\langle\ ,\ \rangle$ is positive semidefinite [5].

2.2. **Scalar product.** A scalar product $\langle\ ,\ \rangle$ on a vector space $\mathbb{V}$ is a nondegenerate symmetric bilinear form on $\mathbb{V}$ [5].

Lemma 2.1. *A scalar product space $\mathbb{V} \neq 0$ has an orthonormal basis.The matrix of $\langle\ ,\ \rangle$ relative to an orthonormal basis $e_1, e_2, ..., e_n$ for $\mathbb{V}$ is diagonal. In fact,*

$$< e_i, e_j >= \delta_{i\,j}\varepsilon_j \text{ where } \varepsilon_j =< e_j, e_j >= \pm 1 \text{ [5].}$$

Lemma 2.2. *Let $e_1, e_2, ..., e_n$ be an orthonormal basis for $\mathbb{V}$, with $\varepsilon_j = < e_j, e_j >$. Then, each $v \in \mathbb{V}$ has a unique expression*

$$v = \sum_{i=1}^{n} \varepsilon_i < v, e_i > e_i \text{ [5].}$$

Lemma 2.3. *For any orthonormal basis* $e_1, e_2, ..., e_n$ *for* $\mathbb{V}$*, the number of negative signs in the signature* $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ *is the index* q *of* $\mathbb{V}$ *[5].*

Lemma 2.4. *For any orthonormal basis* $e_1, e_2, ..., e_n$ *for* $\mathbb{V}$*, the number of an integer* q *with* $0 \leq q \leq n$ *, changing the first* q *plus signs above the minus, gives tensor*

$$< v, w >= -\sum_{i=\ 1}^{q} v^i w^i + \sum_{j=\ q+1}^{n} v^j w^j$$

of index q*.*

The resulting semi-Euclidean space $\mathbb{R}_q^n$ reduces to $\mathbb{R}^n$ if q = 0. For n $\geq$ 2 , $\mathbb{R}_1^n$ is called Minkowski n-space; if n = 4, it is the simplest example of a relativistic spacetime.

Fix the notation as follows:

$$\varepsilon_i = -1 \quad for \quad 0 \leq i \leq q-1$$
$$\varepsilon_i = 1 \quad for \quad \mathrm{q} \leq i \leq n-1.$$

A Lorentz vector space is a scalar product space of index 1 and dimension ≥ 2 [5].

2.3. **Lorentzian space.** Let M be a smooth connected paracompact Hausdorff manifold, and let π : TM $\rightarrow$ M denote the tangent bundle of M. A Lorentzian metric $<,>$ for M is a smooth symmetric tensor field of type (0,2) on M such that, for each p $\in$ M, the tensor

$$<,>_p : T_P M x T_P M \rightarrow \mathbb{R}$$

is a nondegenerate inner product of signature (-, +, . . . ,+). In other words, a matrix representation of $<,>$ at p will have one negative eigenvalue, and all other eigenvalues will be positive.

A Lorentzian manifold (M, $<,>$) is a manifold M together with a Lorentzian metric $<,>$ for M. All noncompact manifolds admit Lorentzian metrics. However, a compact manifold admits a Lorentzian metric if its Euler characteristic vanishes [6].

Lorentzian space is the manifold M $=\mathbb{R}^n$ together with the metric

$$ds^2 = -dx_1^2 + \sum_{i=2}^{n} dx_i^2.$$

This space-time is time oriented by the vector field $\partial \ / \ \partial x_1$ [6].

Definition 2.5. A tangent vector v $\in \mathbb{L}^n$ is

(i) space-like if $\langle v, v\rangle > 0$ or $v = 0$,
(ii) null if $\langle v, v\rangle = 0$ and $v \neq 0$,
(iii) time-like if $\langle v, v\rangle < 0$ [6].

2.4. **Curves and curvatures.** A curve in a Lorentzian space, $\mathbb{L}^n$, is a smooth mapping:

$$\alpha : I \to \mathbb{L}^n$$

where I is open interval in the real line $\mathbb{R}$. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of α at t $\in I$ is

$$\alpha^{'} = \frac{d\alpha(u)}{d(u)}|_t.$$

A curve α is said to be regular if $\alpha^{'}(t)$ does not vanish for all t in I. $\alpha \in \mathbb{L}^n$ is space-like if its velocity vectors $\alpha^{'}$ are space-like for all $t \in I$, similar to time-like and null.

If α is a space-like or time-like curve, we can reparametrize it such that $< \alpha^{'}(t), \alpha^{'}(t) >= \varepsilon_0$ (where $\varepsilon_0 = +1$ if α is a space-like and ε_0 =-1 if α is time-like, respectively). In this case, α is said to be unit speed, or *it* has arc length parameterization. Here and in the sequel, we assume that α has arc length parametrization [5].

Definition 2.6. Let M $\subset \mathbb{L}^n$ be the curve with coordinate neighborhood (I ,α), and let$\{V_1, V_2, ..., V_r\}$be the Frenet r-frame at $\alpha(s)$ with s$\in I$. Then, the function

$$k_i : I \to \mathbb{R},\ 1 \leq i \leq r$$
$$s \to k_i(s) = \ < V_i^{'}(s), V_{i+1}(s) >$$

is called i.th curvature function of the curve M, and for s$\in I$, k$_i(s)$ is called i.th curvature of M at $\alpha(s)$.

Definition 2.7. Let M be curve in $\mathbb{L}^n$, parametrized by its own arc length. Let us denote the Frenet vector fields of this curve $\{V_1, V_2, ..., V_r\}$. Then, the equality

$$k_i = \varepsilon_i < V_i^{'}(s), V_{i+1}(s) >$$

is called the higher ordered curvatures of the curve M [2].

Theorem 2.8. *Let $M \subset \mathbb{L}^n$be a regular curve with coordinate neighborhood (I , α), and let $\{V_1, V_2, ..., V_r\}$be the Frenet r-frame at $\alpha(s)$ with s$\in I$. Then,*

a) $V_1^{'}(s) = k_1(s)V_2(s)$
b) $V_i^{'}(s) = -\varepsilon_{i-2}.\varepsilon_{i-1}.k_{i-1}(s).V_{i-1}(s) + k_i(s).V_{i+1}(s)$, $i = 1, 2, .., r$
c) $V_r^{'}(s) = -\varepsilon_{r-2}.\varepsilon_{r-1}.k_{r-1}(s).V_{r-1}(s)$ [2].

3. A New Characterization for Inclined Curves in Lorentzian Spaces for Sace-Like Curves

Definition 3.1. Let γ be a space-like curve in $\mathbb{L}^n$, and let V_1 be the first Frenet vector field of γ. $X \in \chi(\mathbb{L}^n)$ is a constant unit vector field. If

$$< V_1, X >= \cosh\ \varphi \quad \text{(constant)}$$

then γ is called a general helix (inclined curve) in $\mathbb{L}^n$. φ is called slope angle, and the space $Sp\{X\}$ is called slope axis [1].

Definition 3.2. Assume that γ is space-like or time-like curve in $\mathbb{L}^n$. If the higher ordered curvatures of γ are k_r , $1 < r \leq n-1$, then the higher ordered harmonic curvatures H_r, $1 \leq r \leq n-2$, are

$$H_i : I \rightarrow \mathbb{R}$$

$$H_1 = \varepsilon_0.\varepsilon_1 \frac{k_1}{k_2}$$

$$H_i = \left\{ H_{i-1}^{'} + \varepsilon_{i-1}.\varepsilon_i.k_i.H_{i-2} \right\} \frac{1}{k_{i+1}} \ ,\ 2 \leq i \leq n-2$$

[1].

Theorem 3.3. *Let $\gamma : I \rightarrow \mathbb{L}^n$ be a general helix (inclined curve), parametrized by its arc length. Let X be a unit and constant vector field of $\mathbb{L}^n$, and let $\{V_1, V_2, ..., V_r\}$ be Frenet r-frame at the point of $\gamma(s)$ of γ. If we consider the angle between $\gamma^{'}$ and X as φ, we have*

$$H_j : I \rightarrow \mathbb{R},\ < V_{j+2}, X >= H_j \cosh\ \varphi.$$

Then, the value of the H_j function at the point of $\gamma(s)$ is called as the j-th harmonic curvature according to X at the point of $\gamma(s)$ of γ [1].

Theorem 3.4. *Let γ be space-like curve in $\mathbb{L}^n$. Let the Frenet frame of γ be $F = \{V_1, V_2, ..., V_n\}$, and the higher ordered harmonic curvatures be $H_1, H_2, ..., H_{n-2}$. Then,*

$$\gamma \text{ is an inclined curve in } \mathbb{L}^n \Leftrightarrow \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j^2 = constant. \tag{1}$$

Proof. ($\Rightarrow$) Let γ be inclined curve in $\mathbb{L}^n$. We denote slope angle of γ with φ, and we also denote slope axis of γ with $Sp\{X\}$. From Definition3.1 we can write $< V_1, X >= \cosh \varphi$; and from Theorem 3.1, we can write
$< V_{i+2}, X >= H_i < V_1, X >$. If we take derivative of the equation

$$< V_1, X >= \cosh\ \varphi$$

we will get $< V_1^{'}, X >= 0$ or $< V_2, X >= 0$.

Since $X \in Sp\{V_1, V_2, ..., V_n\}$ we can write

$$X = \sum_{i=1}^{n} \varepsilon_{i-1} < V_i, X > V_i.$$

Thus, we have

$$X = \varepsilon_0 < V_1, X > V_1 + \varepsilon_1 < V_2, X > V_2 + \sum_{i=3}^{n} \varepsilon_{i-1} < V_i, X > V_i.$$

Since $< V_1, X >= \cosh \varphi$ and $< V_2, X >= 0$, we have

$$X = \varepsilon_0 \cosh \ \varphi V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} < V_{j+2}, X > V_{j+2}$$

or

$$X = \cosh \ \varphi \left\{ \varepsilon_0 V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j V_{j+2} \right\}. \tag{2}$$

Since X is a space-like and unit vector field, we can write

$$1 = \|X\|^2 =< X, X > .$$

Thus, using the equation (2), we obtain

$$< X, X >= \cosh^2 \varphi \left\{ < V_1, V_1 > + \sum_{j=1}^{n-2} H_j^2 < V_{j+2}, V_{j+2} > \right\}.$$

Since $< X, X >= 1$ and $< V_{j+2}, V_{j+2} >= \varepsilon_{j+1}$, we can write

$$1 = \cosh^2 \varphi \left\{ \varepsilon_0 + \sum_{j=1}^{n-2} H_j^2 \varepsilon_{j+1} \right\}$$

or since $\varepsilon_0 = 1$, we can write

$$1 = \cosh^2 \varphi \left\{ 1 + \sum_{j=1}^{n-2} H_j^2 \varepsilon_{j+1} \right\}. \tag{3}$$

Hence, we obtain

$$\sum_{j=1}^{n-2} H_j^2 \varepsilon_{j+1} = -\tanh^2 \varphi = \text{constant}.$$

($\Leftarrow$) Let us assume that $\sum_{j=1}^{n-2} H_j^2 \varepsilon_{j+1} = -\tanh^2 \varphi$(=constant), and let us show that γ is an inclined curve. We know that

$$X = \cosh \varphi \left\{ V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j V_{j+2} \right\} \tag{4}$$

and X is a space-like vector field.

Hence, we show that X is a constant unit vector field. If we take the derivative of (4), we will get

$$D_{V_1} X = \cosh\ \varphi \left\{ D_{V_1} V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} \left(D_{V_1} H_j V_{j+2} + H_j D_{V_1} V_{j+2} \right) \right\}.$$

Using the value of $D_{V_1} H_j$ in Definition 3.2 and the value of $D_{V_1} V_{j+2}$ at Theorem 2.5, we obtain

$$D_{V_1} X = \cosh\ \varphi \left\{ k_1 V_2 + \sum_{j=1}^{n-2} \varepsilon_{j+1} \left(\begin{array}{c} k_{j+2} H_{j+1} V_{j+2} - \varepsilon_j \varepsilon_{j+1} k_{j+1} H_{j-1} V_{j+2} \\ -\varepsilon_j \varepsilon_{j+1} k_{j+1} H_j V_{j+1} + k_{j+2} H_j V_{j+3} \end{array} \right) \right\}.$$

Hence, we also obtain $D_{V_1} X = 0$. As a result, X is a constant vector field. Furthermore, $< X, X >= +1$, and this means that X is a unit vector field. Finally, for space-like curve, γ and the constant unit vector field (space-like), X we can write

$$< V_1, X > = < V_1, \cosh\ \varphi \left\{ V_1 + \sum_{j=1}^{n-2} \varepsilon_{j+1} H_j V_{j+2} \right\} > .$$

Hence, we obtain

$$< V_1, X >= \cosh\ \varphi.$$

This completes the proof of theorem. ∎

4. Higher Ordered Curvatures in Terms of Higher Ordered Harmonic Curvatures

Theorem 4.1. *Let γ be an inclined curve (space-like) in $\mathbb{L}^n$. The relation between the higher ordered curvatures k_r , $2 < r \leq n-2$, and the higher ordered harmonic curvatures H_r, $1 \leq r \leq n-2$, is*

$$k_r = \varepsilon_{r-1} \frac{\sum_{i=1}^{r-2} \varepsilon_{i+1} (H_i^2)'}{2 H_{r-1} H_{r-2}}. \tag{5}$$

Proof. We will prove the theorem by induction method. From Definition 3.2, we have

$$k_{i+1} = \frac{H_{i-1}' + \varepsilon_{i-1} \varepsilon_i k_i H_{i-2}}{H_i},\ 1 < i \leq n-2. \tag{6}$$

For $i = 2$, (6) gives us

$$k_3 = \frac{H_1^{'} + \varepsilon_1 \varepsilon_2 k_2 H_0}{H_2}. \tag{7}$$

Since we assume that $H_0 = 0$, (5) is satisfied for $i = 2$. If we extend (7) by $2H_1$, then we have $k_3 = \frac{2H_1 H_1^{'}}{2H_1 H_2}$ or $k_3 = \frac{\sum_{i=1}^{1} (H_i^2)^{'}}{2H_1 H_2}$. This proves that theorem is true for $r = 3$.

Now,let us assume that the theorem is true for $r = p - 1$, and then let us prove that the theorem is also true for $r = p$.

As our assumption, we have that

$$k_{p-1} = \varepsilon_{p-2} \frac{\sum_{i=1}^{p-3} \varepsilon_{i+1} (H_i^2)^{'}}{2H_{p-2} H_{p-3}}. \tag{8}$$

In the equation (6) for $i = p - 1$, we have

$$k_p = \frac{H_{p-2}^{'} + \varepsilon_{p-2} \varepsilon_{p-1} k_{p-1} H_{p-3}}{H_{p-1}},$$

and if we replace here the value of k_{p-1}, from (8), we will obtain

$$k_p = \varepsilon_{p-1} \frac{\sum_{i=1}^{p-2} \varepsilon_{i+1} (H_i^2)^{'}}{2H_{p-1} H_{p-2}}.$$

Thus, the theorem is also true for $r = p$. This completes the proof of the theorem. □

5. Another Characterization for Inclined Curvatures in Lorentzian Spaces for Space-Like Curves

Theorem 5.1. *Let γ be space-like curve in $\mathbb{L}^n$, $n = 2k \geq 4$. Let us assume that the Frenet frame of γ be $F = \{V_1, V_2, ..., V_n\}$. Then,*

$$\gamma \text{ is an inclined curve in } \mathbb{L}^{n-1} \Leftrightarrow \det\left(V_1^{'}, V_2^{'}, ..., V_n^{'}\right) = 0.$$

Proof. In this proof, we use the induction method.

($\Rightarrow$) Let γ be an inclined curve in $\mathbb{L}^{n-1}$. Then, we show that $\det\left(V_1^{'}, V_2^{'}, ..., V_n^{'}\right) = 0$. From Theorem 2.5, we know that

$$\begin{aligned} V_1^{'} &= k_1 V_2 \\ V_2^{'} &= -\varepsilon_0 \varepsilon_1 k_1 V_1 + k_2 V_3 \\ V_3^{'} &= -\varepsilon_1 \varepsilon_2 k_2 V_2 + k_3 V_4 \\ V_4^{'} &= -\varepsilon_2 \varepsilon_3 k_3 V_3. \end{aligned}$$

Then, we can write

$$\det(V_1^{'}, V_2^{'}, V_3^{'}, V_4^{'}) = \begin{vmatrix} 0 & k_1 & 0 & 0 \\ -\varepsilon_0\varepsilon_1 k_1 & 0 & k_2 & 0 \\ 0 & -\varepsilon_1\varepsilon_2 k_2 & 0 & k_3 \\ 0 & 0 & -\varepsilon_2\varepsilon_3 k_3 & 0 \end{vmatrix}$$

or

$$\det(V_1^{'}, V_2^{'}, V_3^{'}, V_4^{'}) = \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3 k_1^2 k_3^2. \tag{9}$$

From the Theorem 4.1, we may write

$$k_3 = \frac{\sum_{i=1}^{1} (H_i^2)^{'}}{2H_1H_2}.$$

If we replace here the value of k_3, from (9), we will obtain

$$\det(V_1^{'}, V_2^{'}, V_3^{'}, V_4^{'}) = \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3 k_1^2 \left[\frac{\sum_{i=1}^{1} (H_i^2)^{'}}{2H_1H_2} \right]^2 .$$

According to hypothesis, if γ is an inclined curve in $\mathbb{L}^3$, then $\sum_{i=1}^{1} H_i^2 =$const. Thus, we have $\sum_{i=1}^{1} (H_i^2)^{'} = 0$ or $\det(V_1^{'}, V_2^{'}, V_3^{'}, V_4^{'}) = 0$.

As a result, the theorem is true for $n = 4$. Now, let us assume that the theorem is true for $n = p$.We show that the theorem is also true for $n = p + 2$.

As our assumption, we have

$$\det\left(V_1^{'}, ..., V_p^{'}\right) = \begin{vmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 & 0 \\ -\varepsilon_0\varepsilon_1 k_1 & 0 & k_2 & \cdots & 0 & 0 & 0 \\ 0 & -\varepsilon_1\varepsilon_2 k_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_{p-2} & 0 \\ 0 & 0 & 0 & \cdots & -\varepsilon_{p-3}\varepsilon_{p-2}k_{p-2} & 0 & k_{p-1} \\ 0 & 0 & 0 & \cdots & 0 & -\varepsilon_{p-2}\varepsilon_{p-1}k_{p-1} & 0 \end{vmatrix}$$

or

$$\det(V_1^{'}, ..., V_p^{'}) = \varepsilon_0\varepsilon_1\varepsilon_2 ... \varepsilon_{p-1} k_1^2 k_3^2 ... k_{p-1}^2. \tag{10}$$

Now, we show that the theorem is true for $n = p + 2$.

We have

$$\det(V'_1,...,V'_{p+2}) = \begin{vmatrix} 0 & k_1 & 0 & \cdots & 0 & 0 & 0 \\ -\varepsilon_0\varepsilon_1 k_1 & 0 & k_2 & \cdots & 0 & 0 & 0 \\ 0 & -\varepsilon_1\varepsilon_2 k_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_p & 0 \\ 0 & 0 & 0 & \cdots & -\varepsilon_{p-1}\varepsilon_p k_p & 0 & k_{p+1} \\ 0 & 0 & 0 & \cdots & 0 & -\varepsilon_p\varepsilon_{p+1}k_{p+1} & 0 \end{vmatrix}$$

.Calculation of this determinant gives us that

$$\begin{aligned} \det(V'_1,...,V'_{p+2}) &= \det(V'_1,...,V'_p).\varepsilon_p\varepsilon_{p+1}k^2_{p+1} \\ &= \varepsilon_0\varepsilon_1\varepsilon_2...\varepsilon_{p-1}\varepsilon_p\varepsilon_{p+1}k^2_1k^2_3...k^2_{p-1}k^2_{p+1} \end{aligned} \tag{11}$$

If we replace the value of k_{p+1} of Theorem 4.1, we will have

$$\det(V'_1,...,V'_{p+2}) = \varepsilon_0\varepsilon_1...\varepsilon_{p+1}k^2_1k^2_3...k^2_{p-1}\left[\varepsilon_p\frac{\sum\limits_{i=1}^{p-1}\varepsilon_{i+1}(H^2_i)'}{2H_pH_{p-1}}\right]^2.$$

According to the hypothesis, we may write

$$\sum_{i=1}^{p-1}\varepsilon_{i+1}H^2_i = \text{constant} \Rightarrow \sum_{i=1}^{p-1}\varepsilon_{i+1}(H^2_i)' = 0.$$

Thus, the last equation becomes

$$\det(V'_1,...,V'_{p+2}) = 0,$$

and it proves the necessity of the theorem.

($\Leftarrow$)Let us assume that $\det(V'_1,...,V'_n) = 0$. Then, we will show that γ is an inclined curve. For the case of $n = 4$, we can write

$$\det(V'_1,V'_2,V'_3,V'_4) = -\left[k_1\frac{\left(\sum\limits_{i=1}^{1}H^2_i\right)'}{2H_1H_2}\right]^2.$$

According to the hypothesis, $\det(V'_1,V'_2,V'_3,V'_4) = 0$.

Since $k_1 \neq 0$, we have $\left(\sum\limits_{i=1}^{1}H^2_i\right)' = 0$ or $\sum\limits_{i=1}^{1}H^2_i$ =constant.Thus, the theorem is true for $n = 4$.

Now, let us assume that the theorem is true for $n = p$.We show that the theorem is also true for $n = p + 2$.

In the case of $n = p$, we can write

$$\det(V_1^{'}, ..., V_p^{'}) = \varepsilon_0\varepsilon_1...\varepsilon_{p-1}k_1^2k_3^2...k_{p-1}^2$$
$$= 0.$$

Thus, for $n = p + 2$, we have

$$\begin{aligned}\det(V_1^{'}, ..., V_{p+2}^{'}) &= \det(V_1^{'}, ..., V_p^{'})\varepsilon_p\varepsilon_{p+1}k_{p+1}^2 \\ &= \varepsilon_0\varepsilon_1...\varepsilon_{p+1}\left[k_1k_3...k_{p+1}\right]^2 \\ &= 0.\end{aligned}$$

Since we assume that $k_1 \neq 0, k_3 \neq 0, ..., k_{p-1} \neq 0$, then we obtain that $k_{p+1} = 0$. This means that

$$k_{p+1} = \varepsilon_p \frac{\sum\limits_{i=1}^{p-1} \varepsilon_{i+1}(H_i^2)^{'}}{2H_{p-1}H_p} = 0$$

or

$$\sum_{i=1}^{p-1} \varepsilon_{i+1}H_i^2 = \text{constant}.$$

Thus, γ is an inclined curve in $\mathbb{L}^{p+1}$. As a result, the theorem is also true for $n = p + 2$. This means that the theorem is true. □

6. Finding the Coordinates of the Central Point of the Space-Like Curves Osculating Sphere

Let $M \subset \mathbb{L}^n$ be a space-like curve with coordinate neighborhood (I, γ), and let $\{V_1(s), ...V_n(s)\}$ be the Frenet n-frame at $\gamma(s)$ with $s \in I$. Let b be the center of osculating sphere. Then, we can write

$$b = \gamma(s) + m_1(s)V_1(s) + ... + m_n(s)V_n(s).$$

Thus, we have

$$b - \gamma(s) = m_1(s)V_1(s) + ... + m_n(s)V_n(s), \tag{12}$$

where m_i denotes the coordinate functions of the centers of osculating spheres of M.

Let r be the radius of osculating sphere of M. Then, we can write

$$|< b - \gamma(s), b - \gamma(s) >| = r^2. \tag{13}$$

If we take the derivative of (13) with respect to V_1, we will obtain

$$< V_1(s), b - \gamma(s) >= 0, \tag{14}$$

and from equation (12) , we have

$$< V_1(s), b - \gamma(s) >= m_1(s). \tag{15}$$

From (14) and (15), we get $m_1(s)=0$. If we take the derivative of the equation (14) with respect to V_1,

$$< V_1^{'}(s), b-\gamma(s) > - < V_1(s), V_1(s) >= 0$$

is obtained, and using the value of $V_1^{'}(s)$ from (13), we get

$$< k_1(s)V_2(s), b-\gamma(s) > -\varepsilon_0 = 0,$$

where $\varepsilon_0 = 1$ since $M \subset \mathbb{L}^n$ is a space-like curve. Therefore, we can write

$$< V_2(s), b-\gamma(s) >= \frac{1}{k_1(s)}. \tag{16}$$

Also from (12), we know that

$$< V_2(s), b-\gamma(s) >= \varepsilon_1 m_2(s). \tag{17}$$

Thus, from (16) and (17), we obtain

$$m_2(s) = \frac{\varepsilon_1}{k_1(s)}. \tag{18}$$

If we take the derivative of (17), we will get

$$< V_2^{'}(s), b-\gamma(s) > - < V_2(s), V_1(s) >= \varepsilon_1 m_2^{'}(s).$$

We know that $< V_1(s), V_2(s) >= 0$ and

$$V_2^{'}(s) = -\varepsilon_0\varepsilon_1 k_1(s)V_1(s) + k_2(s)V_3(s)$$

from Theorem 2.5. Thus, we obtain

$$-\varepsilon_0\varepsilon_1 k_1(s) < V_1(s), b-\gamma(s) > +k_2(s) < V_3(s), b-\gamma(s) >= \varepsilon_1 m_2^{'}(s).$$

Since $< V_3(s), b-\gamma(s) >= \varepsilon_2 m_3(s)$ from (12), we can write

$$m_3(s) = \varepsilon_1\varepsilon_2 \frac{m_2^{'}(s)}{k_2(s)}. \tag{19}$$

If we continue in this way for $i = 4, ...n$, we can write from (12)

$$< V_{i-1}(s), b-\gamma(s) >= \varepsilon_{i-2} m_{i-1}(s). \tag{20}$$

If we take the derivation of equation (20) with respect to V_1, we will obtain

$$< V_{i-1}^{'}(s), b-\gamma(s) > - < V_{i-1}(s), V_1(s) >= \varepsilon_{i-2} m_{i-1}^{'}(s).$$

Since $V_{i-1}^{'}(s) = -\varepsilon_{i-3}\varepsilon_{i-2}k_{i-2}(s)V_{i-2}(s) + k_{i-1}(s)V_i(s)$ from Theorem 2.5, the result is

$$m_i(s) = \frac{\varepsilon_{i-1}\varepsilon_{i-2}}{k_{i-1}(s)}\{m_{i-1}^{'}(s) + k_{i-2}(s)m_{i-2}(s)\}. \tag{21}$$

Finally we can give the following.

Definition 6.1. Let M be a space-like curve in $\mathbb{L}^{n+1}$, and let the coordinate functions of osculating sphere of M be $m_1, ..., m_{n+1}$.Then

$$m_i : I \to \mathbb{R}$$
$$m_i = \begin{cases} 0 & , i = 1 \\ \frac{\varepsilon_1}{k_1(s)} & , i = 2 \\ \frac{\varepsilon_{i-1}\varepsilon_{i-2}}{k_{i-1}(s)}\{m'_{i-1}(s) + m_{i-2}(s)\} & , 2 < i \leq n \end{cases} \tag{22}$$

where k_i, $1 \leq i \leq n$, are the higher ordered curvatures of the space-like curve.

7. The Relations Between the functions of m_i and k_i

Theorem 7.1. *Let M be a space-like curve in $\mathbb{L}^{n+1}$. Then, the relation between the functions of m_i and k_i is*

$$k_p = \varepsilon_p \frac{\sum_{i=2}^{p} \varepsilon_{i-1}(m_i^2)'}{2m_p m_{p+1}}, \ 2 \leq p < n+1 \tag{23}$$

where m_i and k_p denote the coordinate functions of the centers of osculating spheres of M and the curvature functions of $M \subset \mathbb{L}^{n+1}$, respectively.

Proof. We will use the induction method.

From Definition 6.1 we have

$$k_{i-1} = \{m'_{i-1} + k_{i-2}m_{i-2}\}\frac{\varepsilon_{i-1}\varepsilon_{i-2}}{m_i}, \ 2 < i < n+1$$

or for the case of $i = j+1$, we have

$$k_j = \{m'_j + k_{j-1}m_{j-1}\}\frac{\varepsilon_j\varepsilon_{j-1}}{m_{j+1}}, \ 1 < j < n. \tag{24}$$

If we get $j = 2$ in the equation (24), we will have

$$k_2 = \varepsilon_1\varepsilon_2\frac{m'_2}{m_3}.$$

On the other hand, if we write $p = 2$ in the equation (23), we will obtain

$$k_2 = \varepsilon_1\varepsilon_2\frac{m'_2}{m_3}.$$

Therefore, the theorem is true for $p = 2$. Now, we assume that the theorem is true for $p = r$, and let us prove it for $p = r+1$.

For $p = r$, we have

$$k_r = \varepsilon_r \frac{\sum_{i=2}^{r} \varepsilon_{i-1}(m_i^2)'}{2m_r m_{r+1}}. \tag{25}$$

Now, we get $j = r + 1$ in the equation (24). Then, we obtain

$$k_{r+1} = \varepsilon_r \varepsilon_{r+1} \frac{m'_{r+1} + k_r m_r}{m_{r+2}}. \tag{26}$$

If we replace k_r from (25) into (26), we will obtain

$$k_{r+1} = \varepsilon_{r+1} \frac{\sum_{i=2}^{r+1} \varepsilon_{i-1} (m_i^2)'}{2 m_{r+1} m_{r+2}}.$$

This completes the theorem. □

8. Characterization of Inclined Curves Which is Concerned With Osculating Sphere in $\mathbb{L}^n$ for Space-Like Curves

Theorem 8.1. *Let M be a space-like curve in $\mathbb{L}^{n+1}$, $n \geq 4$, and let n be an even number. Then, the relationship between m_i' and m_i values in the matrix form as follows:*

$$\begin{bmatrix} m_2' \\ m_3' \\ m_4' \\ \cdot \\ \cdot \\ m_{n-1}' \\ m_n' \\ m_{n+1}' \end{bmatrix} =$$

$$\begin{bmatrix} 0 & \varepsilon_1\varepsilon_2 k_2 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & 0 & \varepsilon_2\varepsilon_3 k_3 & \cdots & 0 & 0 & 0 \\ 0 & -k_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \varepsilon_{n-2}\varepsilon_{n-1}k_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -k_{n-1} & 0 & \varepsilon_{n-1}\varepsilon_n k_n \\ 0 & 0 & 0 & \cdots & 0 & -k_n & 0 \end{bmatrix} \begin{bmatrix} m_2 \\ m_3 \\ m_4 \\ \cdot \\ \cdot \\ m_{n-1} \\ m_n \\ m_{n+1} \end{bmatrix} \tag{27}$$

If we denote the coefficient matrix as B_n, we will get the following characterizations:

Theorem 8.2. (1) $\det B_n = 0 \Leftrightarrow \sum_{i=2}^{n} \varepsilon_{i-1} m_i^2 =$*constant*

(2) $\det B_n = 0 \Leftrightarrow M \subset \mathbb{L}^{n+1}$ *is an inclined curve in* $\mathbb{L}^n$*, where* $m_i' = \frac{dm_i}{ds}$ *and* s *denotes the arc length parameter of* M*.*

Proof. 1. ($\Rightarrow$)Let us assume that $\det B_n = 0$. We show that $\sum\limits_{i=2}^{n} \varepsilon_{i-1} m_i^2$ =constant. For $n = 4$, the theorem is true.

For n =4, we know that from (27)

$$\det B_4 = \begin{vmatrix} 0 & \varepsilon_1\varepsilon_2 k_2 & 0 & 0 \\ -k_2 & 0 & \varepsilon_2\varepsilon_3 k_3 & 0 \\ 0 & -k_3 & 0 & \varepsilon_3\varepsilon_4 k_4 \\ 0 & 0 & -k_4 & 0 \end{vmatrix} \tag{28}$$

or

$$\det B_4 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 k_2^2 k_4^2. \tag{29}$$

On the other hand, from the hypothesis, we know that $\det B_4 = 0$. Therefore, we can write $k_2 k_4 = 0$. Since we assume that $k_2 \neq 0$, from Theorem 7.1 we have

$$k_4 = \varepsilon_4 \frac{\sum\limits_{i=2}^{4} \varepsilon_{i-1} (m_i^2)'}{2 m_4 m_5} = 0, \tag{30}$$

that is,

$$\sum_{i=2}^{4} \varepsilon_{i-1} m_i^2 = \text{constant}.$$

This proves the theorem for $n = 4$.

Now, let us assume that the theorem is true for $n = p$, and let us show that the theorem is also true for $n = p + 2$. Then, from (27) we can write

$$\det B_p = \begin{vmatrix} 0 & \varepsilon_1\varepsilon_2 k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & 0 & \varepsilon_2\varepsilon_3 k_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_3 & 0 & \varepsilon_3\varepsilon_4 k_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -k_{p-1} & 0 & \varepsilon_{p-1}\varepsilon_p k_p \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_p & 0 \end{vmatrix} \tag{31}$$

or

$$\det B_p = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 ... \varepsilon_{p-1}\varepsilon_p k_2^2 k_4^2 ... k_p^2. \tag{32}$$

Now, we prove that the theorem is also true for $n = p+2$. From (27), we can write

$$\det B_{p+2} = \begin{vmatrix} 0 & \varepsilon_1\varepsilon_2 k_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -k_2 & 0 & \varepsilon_2\varepsilon_3 k_3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & k_p & 0 & 0 \\ 0 & 0 & 0 & \cdots & -k_p & 0 & \varepsilon_p\varepsilon_{p+1} k_{p+1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & -k_{p+1} & 0 & \varepsilon_{p+1}\varepsilon_{p+2} k_{p+2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -k_{p+2} & 0 \end{vmatrix} \tag{33}$$

or

$$\det B_{p+2} = \varepsilon_{p+1}\varepsilon_{p+2} k_{p+2}^2 \det B_P \tag{34}$$

or

$$\det B_{p+2} = \varepsilon_1 ... \varepsilon_p \varepsilon_{p+1}\varepsilon_{p+2} k_2^2 k_4^2 ... k_p^2 k_{p+2}^2. \tag{35}$$

Since $\det B_{p+2} = 0$, we have $k_2 k_4 ... k_p k_{p+2} = 0$. Here since $k_2 \neq 0$,.., $k_p \neq 0$, we obtain $k_{p+2} = 0$. Thus, from Theorem 7.1 we can write

$$k_{p+2} = \varepsilon_{p+2} \frac{\sum\limits_{i=2}^{p+2} \varepsilon_{i-1} (m_i^2)'}{2 m_{p+2} m_{p+3}} = 0 \tag{36}$$

or

$$\sum_{i=2}^{p+2} \varepsilon_{i-1} m_i^2 = \text{constant}.$$

This proves the necessity of the theorem.

($\Leftarrow$) Let us assume that $\sum\limits_{i=2}^{n} \varepsilon_{i-1} m_i^2$ =constant, and we show that $\det B_n = 0$. For $n = 4$, the theorem is true.

if we replace the value of k_4 in the equation (29) into the equation (30),

$$\det B_4 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 [k_2 \frac{\sum\limits_{i=2}^{4} \varepsilon_{i-1} (m_i^2)'}{2 m_4 m_5}]^2$$

is obtained. From the hypothesis,

$$\sum_{i=2}^{4} \varepsilon_{i-1} m_i^2 = \text{constant} \Rightarrow \sum_{i=2}^{4} \varepsilon_{i-1} (m_i^2)' = 0.$$

Thus, we get $\det B_4 = 0$. This proves that the theorem is true for $n = 4$.

Now, let us assume that theorem is true for $n = p$, and let us prove that the theorem is true for $n = p+2$.

We may write from (34),(35), and (36)

$$\det B_{p+2} = \varepsilon_{p+1}\varepsilon_{p+2}[\frac{\sum\limits_{i=2}^{p+2}\varepsilon_{i-1}(m_i^2)'}{2m_{p+2}m_{p+3}}]^2 \det B_p.$$

Since we know that $\sum\limits_{i=2}^{p+2}\varepsilon_{i-1}m_i^2$ =constant, we have $\sum\limits_{i=2}^{p+2}\varepsilon_{i-1}(m_i^2)' = 0$. Thus, we obtain $\det B_{p+2} = 0$, which proves the sufficiency of the theorem.

2. We will use the induction method:

($\Leftarrow$) Let us assume that M is an inclined curve in $\mathbb{L}^n$. Then, from (1), we know that $\sum\limits_{i=1}^{n-2}\varepsilon_{i+1}H_i^2$ =constant. We show that $\det B_n = 0$. Theorem is true for $n = 4$.

From (5), the value of k_4 is

$$k_4 = \varepsilon_3 \frac{\sum\limits_{i=1}^{2}\varepsilon_{i+1}(H_i^2)'}{2H_2H_3}.$$

Using this value in equation (29), we get

$$\det B_4 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 \left(k_2 \left[\varepsilon_3 \frac{\sum\limits_{i=1}^{2}\varepsilon_{i+1}(H_i^2)'}{2H_2H_3} \right] \right)^2.$$

According to the hypothesis, since M is an inclined curve in $\mathbb{L}^3$,

$$\sum_{i=1}^{2}\varepsilon_{i+1}H_i^2 = \text{constant} \Rightarrow \sum_{i=1}^{2}\varepsilon_{i+1}(H_i^2)' = 0$$

can be written, and thus we have $\det B = 0$. This proves that the theorem is true for $n = 4$. Now, let us assume that the theorem is true for $n = p$, and let us also prove it for $n = p + 2$. From (5), (31), (34) and (35) the following equation can be written

$$\det B_{p+2} = \varepsilon_1\varepsilon_2...\varepsilon_{p+2}k_2^2...k_p^2.\frac{\sum\limits_{i=1}^{p}\varepsilon_{i+1}(H_i^2)'}{2H_pH_{p+1}}.$$

According to the hypothesis, since M is an inclined curve in $\mathbb{L}^n$
$\sum\limits_{i=2}^{p}\varepsilon_{i+1}H_i^2$ =constant $\Rightarrow \sum\limits_{i=2}^{p}\varepsilon_{i+1}(H_i^2)' = 0$, thus, $\det B_{p+2} = 0$. This proves the sufficiency of the theorem.

($\Rightarrow$) Let us assume that $\det B_n = 0$, and let us show that the curve M is an inclined curve. Theorem is true for $n = 4$. Indeed, since we know that

$$\begin{aligned}\det B_4 &= \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 k_2^2 k_4^2 \\ &= [k_2 \frac{\sum\limits_{i=1}^{2} \varepsilon_{i+1}(H_i^2)'}{2H_2H_3}]^2,\end{aligned}$$

and according to the hypothesis, we may write

$$\det B_4 = 0.$$

Let us assume that $k_2 \neq 0$. Thus, we obtain

$$\sum_{i=1}^{2} \varepsilon_{i+1}(H_i^2)' = 0 \Rightarrow \sum_{i=1}^{2} \varepsilon_{i+1}H_i^2 = \text{constant}.$$

This means that M is an inclined curve in $\mathbb{L}^3$.

Now, let us assume that the theorem is true for $n = p$, and let us show that the theorem is true for $n = p + 2$.

Since the theorem is true for $n = p$, then we have

$$\det B_p = \varepsilon_1 ... \varepsilon_p k_2^2 ... k_p^2.$$

On the other hand, for $n = p + 2$, we may write

$$\begin{aligned}\det B_{p+2} &= \varepsilon_{p+1}\varepsilon_{p+2} k_{p+2}^2 \det B_p \\ &= \varepsilon_1 ... \varepsilon_p \varepsilon_{p+1}\varepsilon_{p+2}[k_2k_4...k_pk_{p+2}]^2.\end{aligned}$$

If we replace k_{p+2} into equation (5), we obtain

$$\det B_{p+2} = \varepsilon_1 ... \varepsilon_p \varepsilon_{p+1}\varepsilon_{p+2} \left[k_2k_4...k_p \frac{\sum\limits_{i=1}^{p} \varepsilon_{i+1}(H_i^2)'}{2H_pH_{p+1}} \right]^2.$$

According to the hypothesis, we have

$$\det B_{p+2} = 0.$$

Here, since $k_2 \neq 0, k_4 \neq 0, ..., k_p \neq 0$, the result is

$$\sum_{i=1}^{p} \varepsilon_{i+1}(H_i^2)' = 0 \Rightarrow \sum_{i=1}^{p} \varepsilon_{i+1}H_i^2 = \text{constant}.$$

This means that the curve M is an inclined curve in $\mathbb{L}^n$, which proves the necessity of the theorem. □

ÖZET: Bu çalışmamızda, önce $\mathbb{L}^{n}$' de uzay benzeri eğriler için yüksek mertebeden eğrilikleri, yüksek mertebeden harmonik eğrilikler cinsinden hesaplıyoruz. Böylece, $\mathbb{L}^{n}$' de uzay benzeri eğriler için eğilim çizgilerinin karakterizasyonlarını veriyoruz. Sonra da uzay benzeri eğrilerin oskülatör küresinin merkez koordinatlarını buluyoruz. Uzay benzeri eğrilerin yüksek mertebeden eğriliklerini oskülatör küresinin merkez koordinatları cinsinden hesaplıyoruz. Son olarak $\mathbb{L}^{n}$' de eğilim çizgilerinin bu koordinatlar cinsinden olan karakterizasyonlarını veriyoruz.

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