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g-NATURAL METRICS ON THE COTANGENT BUNDLE

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Abstract. The main aim of this paper is to investigate curvature properties and geodesics of the g-natural metric on the cotangent bundle of Riemannian manifold.

1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold, T^*M^n its cotangent bundle and π the natural projection $T^*M^n \to M^n$. A system of local coordinates $(U, x^i), i = 1, ..., n$ on M^n induces on T^*M^n a system of local coordinates $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$, $\bar{i} := n + i$ $(i = 1, ..., n)$, where $x^{\bar{i}} = p_i$ are the components of covectors p in each cotangent space $T^*_x M^n$, $x \in U$ with respect to the natural coframe $\{dx^i\}$, $i = 1, ..., n$.

We denote by $\Im^r_s(M^n)(\Im^r_s(T^*M^n))$ the module over $F(M^n)(F(T^*M^n))$ of C^∞ tensor fields of type (r, s), where $F(M^n)(F(T^*M^n))$ is the ring of real-valued C^∞ functions on $M^n(T^*M^n)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in $U \subset M^n$ of a vector and a covector (1-form) field $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively. Then the complete and horizontal lifts ${}^CX, {}^HX \in \Im^1_0(T^*M^n)$ of $X \in \Im^1_0(M^n)$ and the vertical lift ${}^V\omega \in \Im^1_0(T^*M^n)$ of $\omega \in \Im^0_1(M^n)$ are given, respectively, by

$$(1.1)\qquad {}^CX = X^i \frac{\partial}{\partial x^i} - \sum_i p_h \partial_i X^h \frac{\partial}{\partial x^{\bar{i}}},$$

$$(1.2)\qquad {}^HX = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}},$$

$$(1.3)\qquad {}^V\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}},$$

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with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, where Γ_{ij}^h are the components of the Levi-Civita connection ∇_g on M^n (see [19] for more details).

Theorem 1.1. *Let M^n be a Riemannian manifold with metric g, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M^n of M^n satisfies the following*

$$\begin{aligned} &i) \quad [{}^V\omega, {}^V\theta] = 0, \\ (1.4) \qquad &ii) \quad [{}^HX, {}^V\omega] = {}^V(\nabla_X\omega), \\ &iii) \quad [{}^HX, {}^HY] = {}^H[X,Y] + \gamma R(X,Y) = {}^H[X,Y] + {}^V(pR(X,Y)) \end{aligned}$$

for all $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$. (See [19, p.238, p.277] for more details)

Definition 1.1. Let M^n be a Riemannian manifold with metric g. A Riemannian metric $\bar{g}$ on cotangent bundle T^*M^n is said to be natural with respect to g on M^n if

$$\begin{aligned} &i) \quad \bar{g}({}^HX, {}^HY) = g(X,Y), \\ (1.5) \qquad &ii) \quad \bar{g}({}^HX, {}^V\omega) = 0 \end{aligned}$$

for all $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$.

Theorem 1.2. *Let M^n be a Riemannian manifold with metric g and T^*M^n be the cotangent bundle of M^n. If the Riemannian metric $\bar{g}$ on T^*M^n is natural with respect to g on M^n then the corresponding Levi-Civita connection $\bar{\nabla}$ satisfies*

$$\begin{aligned} &i) \quad \bar{g}(\bar{\nabla}_{{}^HX}{}^HY, {}^HZ) = g(\nabla_XY, Z), \\ &ii) \quad \bar{g}(\bar{\nabla}_{{}^HX}{}^HY, {}^V\omega) = \frac{1}{2}\bar{g}({}^V\omega, {}^V(pR(X,Y))), \\ &iii) \quad \bar{g}(\bar{\nabla}_{{}^HX}{}^V\omega, {}^HZ) = \frac{1}{2}\bar{g}({}^V(pR(Z,X)), {}^V\omega,), \\ &iv) \quad \bar{g}(\bar{\nabla}_{{}^HX}{}^V\omega, {}^V\theta) = \frac{1}{2}\big({}^HX(\bar{g}({}^V\omega, {}^V\theta)) - \bar{g}({}^V\omega, {}^V(\nabla_X\theta)) \\ &\qquad + \bar{g}({}^V\theta, {}^V(\nabla_X\omega))\big), \\ (1.6) \quad &v) \quad \bar{g}(\bar{\nabla}_{{}^V\omega}{}^HY, {}^HZ) = -\frac{1}{2}\bar{g}({}^V\omega, {}^V(pR(Y,Z))), \\ &vi) \quad \bar{g}(\bar{\nabla}_{{}^V\omega}{}^HY, {}^V\theta) = \frac{1}{2}\big({}^HY(\bar{g}({}^V\omega, {}^V\theta)) - \bar{g}({}^V\omega, {}^V(\nabla_Y\theta)) \\ &\qquad - \bar{g}({}^V\theta, {}^V(\nabla_Y\omega))\big), \\ &vii) \quad \bar{g}(\bar{\nabla}_{{}^V\omega}{}^V\theta, {}^HZ) = \frac{1}{2}\big(-{}^HZ(\bar{g}({}^V\omega, {}^V\theta)) + \bar{g}({}^V\omega, {}^V(\nabla_Z\theta)) \\ &\qquad + \bar{g}({}^V\theta, {}^V(\nabla_Z\omega))), \\ &viii) \quad \bar{g}(\bar{\nabla}_{{}^V\omega}{}^V\theta, {}^V\xi) = \frac{1}{2}({}^V\omega(\bar{g}({}^V\theta, {}^V\xi) + {}^V\theta(\bar{g}({}^V\xi, {}^V\omega) - {}^V\xi(\bar{g}({}^V\omega, {}^V\theta)) \end{aligned}$$

for all $X, Y \in \Im_0^1(M^n)$ and $\omega, \theta \in \Im_1^0(M^n)$ [4].

Corollary 1.1. *Let M^n be a Riemannian manifold with metric g and $\bar{g}$ be a natural metric on the cotangent bundle T^*M^n of M^n. Then the Levi-Civita connection $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{{}^HX}{}^HY = {}^H(\nabla_XY) + \frac{1}{2}{}^V(pR(X,Y)) \tag{1.7}$$

for all $X, Y \in \Im^1_0(M^n)$ *[4].*

For each $x \in M^n$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $\pi^{-1}(x) = T^*_x M^n$ by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im^0_1(M^n)$.

Definition 1.2. A g-natural metric $\tilde{g}$ is defined on T^*M^n by the following three equations

$$\tilde{g}\big({}^H X, {}^H Y\big) = {}^V\big(g(X,Y)\big) = g(X,Y) \circ \pi, \tag{1.8}$$

$$\tilde{g}\big({}^V \omega, {}^H Y\big) = 0, \tag{1.9}$$

$$\tilde{g}\big({}^V \omega, {}^V \theta\big) = \varphi(z) g^{-1}(\omega,\theta) + \psi(z) g^{-1}(\omega,p) g^{-1}(\theta,p) \tag{1.10}$$

for any $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^0_1(M^n)$. Here φ and ψ are some functions of argument $z = \frac{1}{2}|p| = \frac{1}{2}g^{-1}(p,p)$ such that $\varphi > 0$ and $\varphi + 2z\psi > 0$.

Since any tensor field of type (0,2) on T^*M^n is completely determined by its action on vector fields of type ${}^H X$ and ${}^V \omega$, it follows that $\tilde{g}$ is completely determined by its equations (1.8), (1.9) and (1.10).

The Sasaki metric is obtained for $\varphi(z) = 1$ and $\psi(z) = 0$, while the Cheeger-Gromoll metric for $\varphi(z) = \psi(z) = \frac{1}{1+r^2}$, $r^2 = g^{-1}(p,p)$. Sasaki, Cheeger-Gromoll and g-natural metrics are in the class of natural metric.

We now see, from (1.1) and (1.2), that the complete lift ${}^C X$ of $X \in \Im^1_0(M^n)$ is expressed by

$${}^C X = {}^H X - {}^V\big(p(\nabla X)\big), \tag{1.11}$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$.

Using (1.8), (1.9), (1.10) and (1.11), we have

$$\begin{aligned}\tilde{g}\big({}^C X, {}^C Y\big) &= {}^V\big(g(X,Y)\big) + \varphi(z)\big(g^{-1}(p(\nabla X), p(\nabla Y))\big) \\ &\quad + \psi(z) g^{-1}(p(\nabla X), p) g^{-1}(p(\nabla Y), p),\end{aligned} \tag{1.12}$$

where $g^{-1}(p(\nabla X), p(\nabla Y)) = g^{ij}(p_l \nabla_i X^l)(p_k \nabla_j Y^k)$, $g^{-1}(p(\nabla X), p) = g^{ij} p_i (p(\nabla X))_j$.

Since the tensor field $\tilde{g} \in \Im^0_2(T^*M^n)$ is completely determined also by its action on vector fields type ${}^C X$ and ${}^C Y$ (see[19, p.237]), we have an alternative characterization of $\tilde{g}$ on T^*M^n: $\tilde{g}$ is completely determined by the condition (1.12).

The main purpose of this paper is to introduce Levi-Civita connection of g-natural type metric on the cotangent bundle T^*M^n of Riemannian manifold M^n and investigate curvature properties and geodesics on T^*M^n with respect to the Levi-Civita connection of $\tilde{g}$. Since the construction of lifts to the cotangent bundle is not similar to the definition of lifts to the tangent bundle, we have some differences for g-natural metrics on cotangent bundles. g-natural metric includes the Sasaki metric ([7], [12], [13]) and the Cheeger-Gromoll metric (see also [2], [4], [5], [6], [9], [11], [14], [15], [16], [18]) as a special cases. In [1]-[3] Abbasi and Sarih characterized the g-natural metric on the tangent bundle. In [17] Sukhova studied a class of Riemannian almost product metrics on the tangent bundle of a smooth manifold and investigated the scalar curvature of the tangent bundle. In [10] Munteanu computed the Levi-Civita connection, the curvature tensor, the sectional curvature and the scalar curvature of the g-natural metric on the tangent bundle.

2. Levi-Civita connection of $\tilde{g}$

We put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \theta^{(i)} = dx^i, i = 1, ..., n.$$

Then from (1.2) and (1.3) we see that ${}^H X_{(i)}$ and ${}^V \theta^{(i)}$ have respectively local expressions of the form

$$\tilde{e}_{(i)} = {}^H X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{2.1}$$

$$\tilde{e}_{(\bar{i})} = {}^V \theta^{(i)} = \frac{\partial}{\partial x^{\bar{i}}}. \tag{2.2}$$

We call the set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\} = \{{}^H X_{(i)}, {}^V \theta^{(i)}\}$ the frame adapted to Levi-Civita connection ∇_g. The indices $\alpha, \beta, ... = 1, ..., 2n$ indicate the indices with respect to the adapted frame.

We now, from the equations (1.2), (1.3), (2.1) and (2.2) see that ${}^H X$ and ${}^V \omega$ have respectively components

$${}^H X = X^i \tilde{e}_{(i)}, \quad {}^H X = ({}^H X^\alpha) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{2.3}$$

$${}^V \omega = \sum_i \omega_i \tilde{e}_{(\bar{i})}, \quad {}^V \omega = ({}^V \omega^\alpha) = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{2.4}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, where X^i and ω_i being local components of $X \in \Im^1_0(M^n)$ and $\omega \in \Im^0_1(M^n)$, respectively.

From (1.8), (1.9) and (1.10) we see that

$$\begin{aligned}
\tilde{g}_{ij} &= \tilde{g}\big(\tilde{e}_{(i)}, \tilde{e}_{(j)}\big) = {}^V\big(g(\partial_i, \partial_j)\big) = g_{ij}, \\
\tilde{g}_{\bar{i}j} &= \tilde{g}\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(j)}\big) = 0, \\
\tilde{g}_{\bar{i}\bar{j}} &= \tilde{g}\big(\tilde{e}_{(\bar{i})}, \tilde{e}_{(\bar{j})}\big) = \varphi(z) g^{-1}(dx^i, dx^j) + \psi(z) g^{-1}(dx^i, p_k) g^{-1}(dx^j, p_l) \\
&= \varphi(z) g^{ij} + \psi(z) g^{ik} g^{lj} p_k p_l,
\end{aligned}$$

i.e. $\tilde{g}$ has components

$$\tilde{g} = \begin{pmatrix} g_{ij} & 0 \\ 0 & \varphi(z) g^{ij} + \psi(z) g^{ik} g^{lj} p_k p_l \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

For the Levi-Civita connection of the g-natural metric we have the following.

Theorem 2.1. *Let M^n be a Riemannian manifold with metric g and $\tilde{\nabla}$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the g-natural* *metric $\tilde{g}$. Then $\tilde{\nabla}$ satisfies*

$$
\begin{aligned}
&i)\quad \tilde{\nabla}_{^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}{}^V(pR(X,Y),\\
&ii)\quad \tilde{\nabla}_{^HX}{}^V\omega = {}^V(\nabla_X\omega) + \frac{\varphi(z)}{2}{}^H(p(g^{-1}\circ R(\ ,X)\tilde{\omega}),\\
(2.6)\qquad &iii)\quad \tilde{\nabla}_{^V\omega}{}^HY = \frac{\varphi(z)}{2}{}^H(p(g^{-1}\circ R(\ ,Y)\tilde{\omega}),\\
&iv)\quad \tilde{\nabla}_{^V\omega}{}^V\theta = -\frac{\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}\big(\tilde{g}({}^V\omega,\gamma\delta){}^V\theta + \tilde{g}({}^V\theta,\gamma\delta){}^V\omega\big)\\
&\qquad + \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}\tilde{g}({}^V\omega,{}^V\theta)\gamma\delta\\
&\qquad + \frac{\psi'(z)\varphi(z)-\varphi'(z)\psi(z)-2\psi^2(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))^3}\tilde{g}({}^V\omega,\gamma\delta)\tilde{g}({}^V\theta,\gamma\delta)\gamma\delta
\end{aligned}
$$

for all $X,Y\in\Im^1_0(M^n)$, $\omega,\theta\in\Im^0_1(M^n)$, where $\tilde{\omega}=g^{-1}\circ\omega\in\Im^1_0(M^n)$, $R(\ ,X)\tilde{\omega}\in\Im^1_1(M^n)$, $g^{-1}\circ R(\ ,X)\tilde{\omega}\in\Im^2_0(M^n)$, $z=\frac{1}{2}|p|=\frac{1}{2}g^{-1}(p,p)$, $\varphi>0$, $\varphi+2z\psi>0$, R and $\gamma\delta$ denotes respectively the curvature tensor of ∇ and the canonical vertical vector field on T^*M^n with expression $\gamma\delta=p_ie_{(\bar{i})}$.

Proof. i) The first statement is just Corollary 1.1.
ii) Following Definition 1.1 and Theorem 1.2 we see that

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{^HX}{}^V\omega,{}^HY) &= \tilde{g}\big({}^V(pR(Y,X)),{}^V\omega\big)\\
&= \varphi(z)g^{-1}(pR(Y,X),\omega)+\psi(z)g^{-1}(pR(Y,X),p)g^{-1}(\omega,p)\\
&= \varphi(z)\tilde{g}\big({}^H(p(g^{-1}\circ R(\ ,X)\tilde{\omega})),{}^HY\big)
\end{aligned}
$$

and

$$
\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{^HX}{}^V\omega,{}^V\theta) &= \big({}^HX(\tilde{g}({}^V\omega,{}^V\theta))-\tilde{g}({}^V\omega,{}^V(\nabla_X\theta))+\tilde{g}({}^V\theta,{}^V(\nabla_X\omega))\big)\\
&= \tilde{g}({}^V\omega,{}^V(\nabla_X\theta))+\tilde{g}({}^V\theta,{}^V(\nabla_X\omega))-\tilde{g}({}^V\omega,{}^V(\nabla_X\theta))+\tilde{g}({}^V\theta,{}^V(\nabla_X\omega))\\
&= 2\tilde{g}({}^V\theta,{}^V(\nabla_X\omega)) = 2\tilde{g}({}^V(\nabla_X\omega),{}^V\theta)
\end{aligned}
$$

Using

$$
\begin{aligned}
g^{-1}(pR(Y,X),\omega) &= (g^{kl}(pR(Y,X))_k\omega_l)\\
&= (g^{kl}p_sR_{ijk}{}^sY^iX^j\omega_l) = (p_sR_{ijk}{}^sY^iX^jg^{kl}\omega_l)\\
&= (p_sR_{ijk}{}^sY^iX^j\tilde{\omega}^k) = (g_{ai}p_sR^a_{.jk}{}^sY^iX^j\tilde{\omega}^k)\\
&= g\big(p(g^{-1}\circ R(\ ,X)\tilde{\omega}),Y\big)\\
&= \tilde{g}\big({}^H(p(g^{-1}\circ R(\ ,X)\tilde{\omega})),{}^HY\big),
\end{aligned}
$$

$$
\begin{aligned}
g^{-1}(pR(Y,X),p) &= (g^{ij}p_sR_{abi}{}^sY^aX^bp_j)\\
&= (p_sg^{ts}R_{abit}Y^aX^b\tilde{p}^i) = (R_{abit}Y^aX^b\tilde{p}^i\tilde{p}^t)\\
&= (R_{itab}Y^aX^b\tilde{p}^i\tilde{p}^t) = (g_{fb}R_{ita.}{}^fY^aX^b\tilde{p}^i\tilde{p}^t)\\
&= g\big(R(\tilde{p},\tilde{p})Y,X\big) = 0,
\end{aligned}
$$

${}^HX(\varphi(z))=0$, ${}^HX(\psi(z))=0$
and
${}^HX(\tilde{g}({}^V\omega,{}^V\theta)=\tilde{g}({}^V\omega,{}^V(\nabla_X\theta))+\tilde{g}({}^V\theta,{}^V(\nabla_X\omega))$
we have

$\tilde{\nabla}_{^H X}{}^V\omega = {}^V(\nabla_X\omega) + \frac{\varphi(z)}{2}{}^H(p(g^{-1}\circ R(\ ,X)\tilde{\omega})$
iii) Calculations similar to those in *ii)* give

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{^V\omega}{}^HY, {}^V\theta) &= \big({}^HY(\tilde{g}({}^V\omega, {}^V\theta)) - \tilde{g}({}^V\omega, {}^V(\nabla_Y\theta)) - \tilde{g}({}^V\theta, {}^V(\nabla_Y\omega))\big) \\
&= \tilde{g}({}^V\omega, {}^V(\nabla_Y\theta)) + \tilde{g}({}^V\theta, {}^V(\nabla_Y\omega)) - \tilde{g}({}^V\omega, {}^V(\nabla_Y\theta)) - \tilde{g}({}^V\theta, {}^V(\nabla_Y\omega)) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{^V\omega}{}^HY, {}^HZ) &= -\tilde{g}\big({}^V\omega, {}^V(pR(Y,Z))\big) \\
&= -\varphi(z)g^{-1}(\omega, pR(Y,Z)) - \psi(z)g^{-1}(pR(Y,Z),p)g^{-1}(\omega,p) \\
&= \varphi(z)\tilde{g}\big({}^H(p(g^{-1}\circ R(\ ,Y)\tilde{\omega})), {}^HZ\big).
\end{aligned}$$

Thus we have

$$\tilde{\nabla}_{^V\omega}{}^HY = \frac{\varphi(z)}{2}{}^H(p(g^{-1}\circ R(\ ,Y)\tilde{\omega}).$$

iv) Appliying Theorem 1.2 we yield

$$\begin{aligned}
&2\tilde{g}(\tilde{\nabla}_{^V\omega}{}^V\theta, {}^HZ) = \big(-{}^HZ(\tilde{g}({}^V\omega, {}^V\theta)) + \tilde{g}({}^V\omega, {}^V(\nabla_Z\theta)) + \tilde{g}({}^V\theta, {}^V(\nabla_Z\omega))\big) \\
&= -\tilde{g}({}^V\omega, {}^V(\nabla_Z\theta)) - \tilde{g}({}^V\theta, {}^V(\nabla_Z\omega)) + \tilde{g}({}^V\omega, {}^V(\nabla_Z\theta)) + \tilde{g}({}^V\theta, {}^V(\nabla_Z\omega)) \\
&= 0
\end{aligned}$$

Using ${}^V\omega(\varphi(z)) = \varphi'(z)g^{-1}(\omega,p)$, ${}^V\omega(\psi(z)) = \psi'(z)g^{-1}(\omega,p)$

$$\begin{aligned}
{}^V\omega(\tilde{g}({}^V\theta, {}^V\xi)) &= \varphi'(z)g^{-1}(\omega,p)g^{-1}(\theta,\xi) \\
&\quad +\psi'(z)g^{-1}(\omega,p)g^{-1}(\theta,p)g^{-1}(\xi,p) \\
&\quad +\psi(z)\big(g^{-1}(\omega,\theta)g^{-1}(\xi,p) + g^{-1}(\theta,p)g^{-1}(\omega,\xi)\big)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{g}({}^V\omega, \gamma\delta) &= \varphi(z)g^{-1}(\omega,p) + \psi(z)g^{-1}(\omega,p)g^{-1}(p,p) \\
&= g^{-1}(\omega,p)\big(\varphi(z) + 2z\psi(z)\big),
\end{aligned}$$

we have

$$\begin{aligned}
\tilde g(\tilde\nabla_{{}^V\omega}{}^V\theta,{}^V\xi) &= {}^V\omega(\tilde g({}^V\theta,{}^V\xi)) + {}^V\theta(\tilde g({}^V\xi,{}^V\omega)) - {}^V\xi(\tilde g({}^V\omega,{}^V\theta))\big) \\
&= \varphi'(z)g^{-1}(\omega,p)g^{-1}(\theta,\xi) + \varphi'(z)g^{-1}(\theta,p)g^{-1}(\xi,\omega) \\
&\quad -\varphi'(z)g^{-1}(\omega,\theta)g^{-1}(\xi,p) + \psi'(z)g^{-1}(\theta,p)g^{-1}(\omega,p)g^{-1}(\xi,p) \\
&\quad +\psi'(z)g^{-1}(\theta,p)g^{-1}(\xi,p)g^{-1}(\omega,p) - \psi'(z)g^{-1}(\theta,p)g^{-1}(\xi,p)g^{-1}(\omega,p) \\
&\quad +2\psi(z)g^{-1}(\xi,p)g^{-1}(\theta,\omega) \\
&= \frac{\varphi'(z)}{\varphi(z)}g^{-1}(\omega,p)\tilde g({}^V\theta,{}^V\xi) - \frac{\varphi'(z)}{\varphi(z)}g^{-1}(\xi,p)\tilde g({}^V\omega,{}^V\theta) \\
&\quad +\frac{\varphi'(z)}{\varphi(z)}g^{-1}(\theta,p)\tilde g({}^V\xi,{}^V\omega) - \psi'(z)g^{-1}(\theta,p)g^{-1}(\omega,p)g^{-1}(\xi,p) \\
&\quad +2\psi(z)g^{-1}(\omega,\theta)g^{-1}(\xi,p) \\
&= \tilde g\big(\frac{\varphi'(z)}{\varphi(z)(\varphi(z)+2z\psi(z))}(\tilde g({}^V\omega,\gamma\delta){}^V\theta + \tilde g({}^V\theta,\gamma\delta){}^V\omega) \\
&\quad +\frac{2\psi(z)-\varphi'(z)}{\varphi(z)(\varphi(z)+2z\psi(z))}\tilde g({}^V\omega,{}^V\theta)\gamma\delta \\
&\quad +\frac{\psi'(z)\varphi(z)-\varphi'(z)\psi(z)-2\psi^2(z)}{\varphi(z)(\varphi(z)+2z\psi(z))^3}\tilde g({}^V\omega,\gamma\delta)\tilde g({}^V\theta,\gamma\delta)\gamma\delta,{}^V\xi\big).
\end{aligned}$$

Thus

$$\begin{aligned}
\tilde\nabla_{{}^V\omega}{}^V\theta &= \frac{\varphi'(z)}{\varphi(z)(\varphi(z)+2z\psi(z))}(\tilde g({}^V\omega,\gamma\delta){}^V\theta + \tilde g({}^V\theta,\gamma\delta){}^V\omega) \\
&\quad +\frac{2\psi(z)-\varphi'(z)}{\varphi(z)(\varphi(z)+2z\psi(z))}\tilde g({}^V\omega,{}^V\theta)\gamma\delta \\
&\quad +\frac{\psi'(z)\varphi(z)-\varphi'(z)\psi(z)-2\psi^2(z)}{\varphi(z)(\varphi(z)+2z\psi(z))^3}\tilde g({}^V\omega,\gamma\delta)\tilde g({}^V\theta,\gamma\delta)\gamma\delta. \qquad \square
\end{aligned}$$

We write $\tilde\nabla_{e_\alpha}e_\beta = \tilde\Gamma^\delta_{\alpha\beta}e_\delta$ with respect to the adapted frame $\{e_\alpha\}$ of T^*M^n, where $\tilde\Gamma^\delta_{\alpha\beta}$ denote the Christoffel symbols constructed by $\tilde g$. From Theorem 2.1, we immediately have

Corollary 2.1. *Let M^n be a Riemannian manifold with metric g and $\tilde\nabla$ be the Levi-Civita connection of the cotangent bundle T^*M^n equipped with the g-natural metric $\tilde g$. The particular values of $\tilde\Gamma^\delta_{\alpha\beta}$ for different indices, on taking account of (2.6) are then found to be*

$$\begin{aligned}
&\tilde\Gamma^k_{ij} = \Gamma^k_{ij}, && \tilde\Gamma^k_{\bar i\bar j} = \tilde\Gamma^{\bar k}_{\bar i j} = 0, \\
&\tilde\Gamma^{\bar k}_{i\bar j} = -\Gamma^j_{ik}, && \tilde\Gamma^{\bar k}_{ij} = \frac{1}{2}p_aR_{ijk}{}^a, \\
&\tilde\Gamma^k_{i\bar j} = \frac{\varphi(z)}{2}p_aR^{k\ ia}_{.j.}, && \tilde\Gamma^k_{\bar i j} = \frac{\varphi(z)}{2}p_aR^{k\ ja}_{.i.}, \qquad (2.7) \\
&\tilde\Gamma^{\bar k}_{\bar i\bar j} = \frac{\varphi'(z)}{2\varphi(z)}(p^i\delta^j_k + p^j\delta^i_k) + \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}g^{ij}p_k \\
&\qquad +\frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}p^ip^jp_k \\
&\qquad = L(p^i\delta^j_k + p^j\delta^i_k) + Mg^{ij}p_k + Np^ip^jp_k.
\end{aligned}$$

with respect to the adapted frame, where $p^i = g^{it}p_t$, $R^{k\ ia}_{\cdot j\cdot} = g^{kt}g^{is}R_{tjs}{}^a$.

3. Curvature properties of $\tilde{g}$

We now consider local 1-forms $\tilde{\omega}^\alpha$ in $\pi^{-1}(U)$ defined by

$$\tilde{\omega}^\alpha = \bar{A}^\alpha{}_B dx^B,$$

where

$$A^{-1} = (\bar{A}^\alpha{}_B) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_j & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -p_a\Gamma^a_{ij} & \delta^j_i \end{pmatrix} \tag{3.1}$$

The matrix (3.1) is the inverse of the matrix

$$A = (A_\beta{}^A) = \begin{pmatrix} A_j{}^i & A_{\bar{j}}{}^i \\ A_j{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ p_a\Gamma^a_{ij} & \delta^j_i \end{pmatrix} \tag{3.2}$$

of the transformation $\tilde{e}_\beta = A_\beta{}^A\partial_A$ (see (2.1) and (2.2)). We easily see that the set $\{\tilde{\omega}^\alpha\}$ is the coframe dual to the adapted frame $\{\tilde{e}_{(\beta)}\}$, i.e. $\tilde{\omega}^\alpha(\tilde{e}_{(\beta)}) = \bar{A}^\alpha{}_B A_\beta{}^B = \delta^\alpha_\beta$.

Since the adapted frame $\{\tilde{e}_{(\beta)}\}$ is non-holonomic, we put

$$[\tilde{e}_\gamma, \tilde{e}_\beta] = \Omega_{\gamma\beta}{}^\alpha \tilde{e}_\alpha$$

from which we have

$$\Omega_{\gamma\beta}{}^\alpha = (\tilde{e}_\gamma A_\beta{}^A - \tilde{e}_\beta A_\gamma{}^A)\bar{A}^\alpha{}_A.$$

According to (2.1), (2.2), (3.1) and (3.2), the components of non-holonomic object $\Omega_{\gamma\beta}{}^\alpha$ are given by

$$\begin{cases} \Omega_{l\bar{j}}{}^{\bar{i}} = -\Omega_{\bar{j}l}{}^{\bar{i}} = \Gamma^j_{li}, \\ \Omega_{lj}{}^{\bar{i}} = p_a R_{lji}{}^a, \end{cases} \tag{3.3}$$

all the others being zero, where $R_{lji}{}^a$ being local components of the curvature tensor R of ∇_g.

Let $\tilde{R}$ be a curvature tensor of $\tilde{\nabla}$. Then we obtain
$\tilde{R}(\tilde{e}_{(\alpha)}, \tilde{e}_{(\beta)})\tilde{e}_{(\gamma)} = \tilde{\nabla}_\alpha\tilde{\nabla}_\beta\tilde{e}_{(\gamma)} - \tilde{\nabla}_\beta\tilde{\nabla}_\alpha\tilde{e}_{(\gamma)} - \Omega_{\alpha\beta}{}^\varepsilon\tilde{\nabla}_\varepsilon\tilde{e}_{(\gamma)}$,

where $\tilde{\nabla}_\alpha = \tilde{\nabla}_{\tilde{e}_{(\alpha)}}$. The curvature tensor $\tilde{R}$ has components

$\tilde{R}_{\alpha\beta\gamma}{}^\sigma = \tilde{e}_\alpha\tilde{\Gamma}^\sigma_{\beta\gamma} - \tilde{e}_\beta\tilde{\Gamma}^\sigma_{\alpha\gamma} + \tilde{\Gamma}^\sigma_{\alpha\varepsilon}\tilde{\Gamma}^\varepsilon_{\beta\gamma} - \tilde{\Gamma}^\sigma_{\beta\varepsilon}\tilde{\Gamma}^\varepsilon_{\alpha\gamma} - \Omega_{\alpha\beta}{}^\varepsilon\tilde{\Gamma}^\sigma_{\varepsilon\gamma}$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

Taking account of (2.7) and (3.3), we find

$$\begin{aligned}
\tilde{R}_{kij}{}^{l} &= R_{kij}{}^{l} - \frac{\varphi(z)}{2} p_m p_a R_{kit}{}^{a} R^{l\,tm}_{.j.} \\
&\quad + \frac{\varphi(z)}{4} p_m p_a (R^{l\,tm}_{.k.} R_{ijt}{}^{a} - R^{l\,tm}_{.i.} R_{kjt}{}^{a}), \\
\tilde{R}_{\bar{k}ij}{}^{l} &= \frac{\varphi(z)}{2} p_m \nabla_k R^{l\,im}_{.j.}, \\
\tilde{R}_{ki\bar{j}}{}^{l} &= \frac{\varphi(z)}{2} p_m (\nabla_k R^{l\,jm}_{.i.} - \nabla_i R^{l\,jm}_{.k.}), \\
\tilde{R}_{kij}{}^{\bar{l}} &= \frac{1}{2} p_m (\nabla_k R_{ijl}{}^{m} - \nabla_i R_{kjl}{}^{m}), \\
\tilde{R}_{ki\bar{j}}{}^{\bar{l}} &= R_{ikl}{}^{j} + \frac{\varphi(z)}{4} p_m p_a (R_{ktl}{}^{a} R^{t\,ja}_{.i.} - R_{itl}{}^{m} R^{t\,ja}_{.k.}) \\
&\quad - \frac{\varphi'(z)}{2\varphi(z)} p_a p^j R_{kil}{}^{a} - \frac{2\psi(z)-\varphi'(z)}{2(\psi(z)+2z\varphi(z))} p_l p_a R_{ki.}{}^{ja}, \\
\tilde{R}_{\bar{k}ij}{}^{\bar{l}} &= \frac{1}{2} R_{ijl}{}^{k} - \frac{\varphi(z)}{4} p_m p_a R_{itl}{}^{m} R^{t\,ka}_{.j.} \\
&\quad + \frac{\varphi'(z)}{4\varphi(z)} p_a p^k R_{ijl}{}^{a} + \frac{2\psi(z)-\varphi'(z)}{4(\psi(z)+2z\varphi(z))} p_l p_a R_{ij.}{}^{ka}, \\
\tilde{R}_{\bar{k}\bar{i}j}{}^{l} &= \frac{\varphi'(z)}{2} p_a (p^k R^{l\,ia}_{.j.} - p^i R^{l\,ka}_{.j.}) + \frac{\varphi(z)}{2} (R^{l\,ik}_{.j.} - R^{l\,ki}_{.j.}) \\
&\quad + \frac{\varphi^2(z)}{4} p_m p_a (R^{l\,km}_{.t.} R^{t\,ia}_{.j.} - R^{l\,im}_{.t.} R^{t\,ka}_{.j.}), \\
\tilde{R}_{\bar{k}i\bar{j}}{}^{l} &= \frac{\varphi(z)}{2} R^{l\,jk}_{.i.} + \frac{\varphi'(z)}{4} p_a (p^k R^{l\,ja}_{.i.} - p^j R^{l\,ka}_{.i.}) \\
&\quad + \frac{\varphi^2(z)}{4} p_m p_a R^{l\,km}_{.t.} R^{t\,ja}_{.i.}, \\
\tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}} &= [L - M(1+2zL)](g^{jk}\delta^i_l - g^{ij}\delta^k_l) \\
&\quad + [N - (2M' + M^2 + 2zMN)](g^{kj} p^i p_l - g^{ij} p^k p_l) \\
&\quad + [2L' - L^2 - N(1+2zL)](\delta^i_l p^k p^j - \delta^k_l p^i p^j),
\end{aligned}$$

$$\tilde{R}_{k\bar{i}\bar{j}}{}^{\bar{l}} = \tilde{R}_{\bar{k}i\bar{j}}{}^{\bar{l}} = \tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{l} = \tilde{R}_{\bar{k}i\bar{j}}{}^{\bar{l}} = 0, \tag{3.4}$$

where $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $N = \frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$.

It is known (see [8, p.200]) that the sectional curvature on $(T^*M^n, {}^{CG}g)$ for $P(U,V)$ is given by

$$\tilde{K}(P) = -\frac{\tilde{R}_{kmij} U^k V^m U^i V^j}{(\tilde{g}_{ki}\tilde{g}_{mj} - \tilde{g}_{kj}\tilde{g}_{mi}) U^k V^m U^i V^j}, \tag{3.5}$$

where $P(U,V) = (U,V)$ denotes the plane spanned by (U,V). Let $\{X_i\}$ and $\{\omega^i\}$, $i = 1,...,n$ be a local ortonormal frame and coframe on M^n, respectively. Then from (8)-(10) we see that $\{{}^H X_1, ..., {}^H X_n, {}^V\omega^1, ..., {}^V\omega^n\}$ is a local ortonormal frame on T^*M^n. Let $\tilde{K}({}^H X, {}^H Y)$, $\tilde{K}({}^H X, {}^V\theta)$ and $\tilde{K}({}^V\omega, {}^V\theta)$ denote the sectional curvature of the plane spanned by $({}^H X, {}^H Y)$, $({}^H X, {}^V\theta)$ and $({}^V\omega, {}^V\theta)$ on $(T^*M^n, \tilde{g})$, respectively. Then, using (2.3), (2.4), (2.5) and (3.4), we have from (3.5)

$$
\begin{aligned}
&i)\quad \tilde{K}({}^{H}X,{}^{H}Y) = -\frac{\tilde{R}_{kijs}{}^{H}\tilde{X}^{k\,H}\tilde{Y}^{i\,H}\tilde{X}^{j\,H}\tilde{Y}^{s}}{(\tilde{g}_{kj}\tilde{g}_{is}-\tilde{g}_{ks}\tilde{g}_{ij})^{H}\tilde{X}^{k\,H}\tilde{Y}^{i\,H}\tilde{X}^{j\,H}\tilde{Y}^{s}}\\
&= -\frac{\tilde{R}_{kij}{}^{l}\tilde{g}_{sl}{}^{H}\tilde{X}^{k\,H}\tilde{Y}^{i\,H}\tilde{X}^{j\,H}\tilde{Y}^{s}+\tilde{R}_{kij}{}^{\bar{l}}\tilde{g}_{s\bar{l}}{}^{H}\tilde{X}^{k\,H}\tilde{Y}^{i\,H}\tilde{X}^{j\,H}\tilde{Y}^{s}}{(\tilde{g}_{kj}\tilde{g}_{is}-\tilde{g}_{ks}\tilde{g}_{ij})^{H}\tilde{X}^{k\,H}\tilde{Y}^{i\,H}\tilde{X}^{j\,H}\tilde{Y}^{s}}\\
&= K(X,Y)+\frac{\frac{\varphi(z)}{2}g^{tf}\big(pR(X,Y)\big)_{t}\big(pR(X,Y)\big)_{f}}{g(X,X)g(Y,Y)-g(X,Y)g(X,Y)}\\
&\quad -\frac{\frac{\varphi(z)}{4}g^{tf}\big(pR(X,Y)\big)_{t}\big(pR(X,Y)\big)_{f}}{g(X,X)g(Y,Y)-g(X,Y)g(X,Y)}+\frac{\frac{\varphi(z)}{4}g^{tf}\big(pR(Y,Y)\big)_{t}\big(pR(X,X)\big)_{f}}{g(X,X)g(Y,Y)-g(X,Y)g(X,Y)}\\
&\quad = K(X,Y)-\frac{3\varphi(z)}{4}\big|(pR(X,Y)\big|^{2}.
\end{aligned}
$$

$$
\begin{aligned}
ii)\quad \tilde{K}({}^{H}X,{}^{V}\theta) &= -\frac{\tilde{R}_{k\bar{i}j\bar{s}}{}^{H}\tilde{X}^{k\,V}\tilde{\omega}^{\bar{i}\,H}\tilde{X}^{j\,V}\tilde{\omega}^{\bar{s}}}{(\tilde{g}_{kj}\tilde{g}_{\bar{i}\bar{s}}-\tilde{g}_{k\bar{s}}\tilde{g}_{\bar{i}j})^{H}\tilde{X}^{k\,H}\tilde{\omega}^{\bar{i}\,H}\tilde{X}^{j\,V}\tilde{\omega}^{\bar{s}}}\\
&= -\frac{\tilde{R}_{k\bar{i}j}{}^{l}\tilde{g}_{\bar{s}l}X^{k}\omega_{i}X^{j}\omega_{s}+\tilde{R}_{k\bar{i}j}{}^{\bar{l}}\tilde{g}_{\bar{s}\bar{l}}X^{k}\omega_{i}X^{j}\omega_{s}}{\big(g_{kj}\big(\varphi(z)g^{is}+\psi(z)g^{ia}g^{sb}p_{a}p_{b}\big)\big)X^{k}\omega_{i}X^{j}\omega_{s}}\\
&= \frac{\frac{\varphi^{2}(z)}{4}g^{tf}\big(pR(\;,X)\tilde{\omega}\big)_{t}\big(pR(\;,X)\tilde{\omega}\big)_{f}}{(\varphi(z)g(X,X)g^{-1}(\omega,\omega)+\psi(z)g(X,X)(g^{-1}(\omega,p))^{2})}\\
&= \frac{\varphi^{2}(z)}{4\big(\varphi(z)+\psi(z)(g^{-1}(\omega,p))^{2}\big)}\big|(pR(\;,X)\tilde{\omega})\big|^{2}
\end{aligned}
$$

$$
\begin{aligned}
&iii)\quad \tilde{K}({}^{V}\omega,{}^{V}\theta) = -\frac{\tilde{R}_{\bar{k}\bar{i}\bar{j}\bar{s}}{}^{V}\tilde{\omega}^{\bar{k}\,V}\tilde{\theta}^{\bar{i}\,V}\tilde{\omega}^{\bar{j}\,V}\tilde{\theta}^{\bar{s}}}{(\tilde{g}_{\bar{k}\bar{j}}\tilde{g}_{\bar{i}\bar{s}}-\tilde{g}_{\bar{k}\bar{s}}\tilde{g}_{\bar{i}\bar{j}})^{V}\tilde{\omega}^{\bar{k}\,V}\tilde{\theta}^{\bar{i}\,V}\tilde{\omega}^{\bar{j}\,V}\tilde{\theta}^{\bar{s}}}\\
&= -\frac{\tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{l}\tilde{g}_{\bar{s}l}\omega_{k}\theta_{i}\omega_{j}\theta_{s}+\tilde{R}_{\bar{k}\bar{i}\bar{j}}{}^{\bar{l}}\tilde{g}_{\bar{s}\bar{l}}\omega_{k}\theta_{i}\omega_{j}\theta_{s}}{(\tilde{g}_{\bar{k}\bar{j}}\tilde{g}_{\bar{i}\bar{s}}-\tilde{g}_{\bar{k}\bar{s}}\tilde{g}_{\bar{i}\bar{j}})\omega_{k}\theta_{i}\omega_{j}\theta_{s}}\\
&= -\Big[\frac{A(\delta_{l}^{i}p^{k}p^{j}-\delta_{l}^{k}p^{i}p^{j})+B(g^{kj}p^{i}p_{l}-g^{ij}p^{k}p_{l})}{P}\\
&\quad +\frac{C(g^{jk}\delta_{l}^{i}-g^{ij}\delta_{l}^{k})}{P}\Big]\big(\varphi(z)(g^{sl}+\psi(z)g^{sa}g^{lb}p_{a}p_{b})\big)\omega_{k}\theta_{i}\omega_{j}\theta_{s}\\
&= -\frac{A(g^{-1}(\omega,p))^{2}}{\varphi(z)+\psi(z)[(g^{-1}(\theta,p))^{2}+(g^{-1}(\omega,p))^{2}]}\\
&\quad -\frac{B(\varphi(z)+2z\psi(z))(g^{-1}(\theta,p))^{2}+C\big(\varphi(z)+\psi(z)(g^{-1}(\theta,p))^{2}\big)}{\varphi^{2}(z)+\varphi(z)\psi(z)[(g^{-1}(\theta,p))^{2}+(g^{-1}(\omega,p))^{2}]},
\end{aligned}
$$

where

$$\begin{aligned}
P &= (\tilde{g}_{\bar{k}\bar{j}}\tilde{g}_{\bar{i}\bar{s}} - \tilde{g}_{\bar{k}\bar{s}}\tilde{g}_{\bar{i}\bar{j}})\omega_k\theta_i\omega_j\theta_s \\
&= \Big[(\varphi(z)g^{kj} + \psi(z)g^{ka}g^{jb}p_ap_b)(\varphi(z)g^{is} + \psi(z)g^{it}g^{sf}p_tp_f) \\
&\quad -(\varphi(z)g^{ks} + \psi(z)g^{kc}g^{sd}p_cp_d)(\varphi(z)g^{ij} + \psi(z)g^{iu}g^{jv}p_up_v)\Big]\omega_k\theta_i\omega_j\theta_s \\
&= \varphi^2(z)g^{-1}(\omega,\omega)g^{-1}(\theta,\theta) + \varphi(z)\psi(z)g^{-1}(\omega,\omega)\big(g^{-1}(\theta,p)\big)^2 \\
&\quad +\varphi(z)\psi(z)g^{-1}(\theta,\theta)\big(g^{-1}(\omega,p)\big)^2 + \psi^2(z)\big(g^{-1}(\omega,p)\big)^2\big(g^{-1}(\theta,p)\big)^2 \\
&\quad -\varphi^2(z)\big(g^{-1}(\omega,\theta)\big)^2 - \varphi(z)\psi(z)g^{-1}(\omega,\theta)g^{-1}(\omega,p)g^{-1}(\theta,p) \\
&\quad -\varphi(z)\psi(z)g^{-1}(\omega,\theta)g^{-1}(\omega,p)g^{-1}(\theta,p) - \psi^2(z)\big(g^{-1}(\omega,p)\big)^2\big(g^{-1}(\theta,p)\big)^2 \\
&= \varphi^2(z) + \varphi(z)\psi(z)\big[\big(g^{-1}(\omega,p)\big)^2 + \big(g^{-1}(\theta,p)\big)^2\big].
\end{aligned}$$

and $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $N = \frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $A = 2L' - L^2 - N(1+2zL)$, $B = N - (2M' + M^2 + 2zMN)$, $C = L - M(1+2zL)$

Thus we have

Theorem 3.1. *Let (M^n, g) be a Riemannian manifold and T^*M^n be its cotangent bundle equipped with the g-natural metric $\tilde{g}$. Then the sectional curvature $\tilde{K}$ of $(T^*M^n, \tilde{g})$ satisfy the following:*

$$\begin{aligned}
&i) K({}^H\tilde{X}, {}^HY) = K(X,Y) - \frac{3\varphi(z)}{4}\big|(pR(X,Y)\big|^2, \\
&ii) \tilde{K}({}^HX, {}^V\omega) = \frac{\varphi^2(z)}{4\big(\varphi(z) + \psi(z)(g^{-1}(\omega,p))\big)}\big|(pR(\ ,X)\tilde{\omega})\big|^2, \\
&iii) \tilde{K}({}^V\omega, {}^V\theta) = -\frac{A(g^{-1}(\omega,p))^2}{\varphi(z) + \psi(z)[(g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2]} \\
&\qquad - \frac{B(\varphi(z) + 2z\psi(z))(g^{-1}(\theta,p))^2 + C\big(\varphi(z) + \psi(z)(g^{-1}(\theta,p))^2\big)}{\varphi^2(z) + \varphi(z)\psi(z)[(g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2]},
\end{aligned}$$

where K is a sectional curvature of (M^n, g) and $\tilde{\omega} = g^{-1}\circ\omega = (g^{ij}\omega_j) \in \Im^1_0(M^n)$, $R(\ ,X)\tilde{\omega} \in \Im^1_1(M^n)$, $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $N = \frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $A = 2L' - L^2 - N(1+2zL)$, $B = N - (2M' + M^2 + 2zMN)$ and $C = L - M(1+2zL)$.

Theorem 3.2. *Let (M^n, g) be a Riemannian manifold of constant sectional curvature K. Let T^*M^n be its cotangent bundle equipped with the g-natural metric $\tilde{g}$.*

Then the sectional curvature $\tilde{K}$ of $(T^*M^n, \tilde{g})$ satisfy the following:

$$
\begin{aligned}
& i)\tilde{K}({}^HX, {}^HY) = K - \frac{3\varphi^2(z)}{4}K^2\big((g^{-1}(p,\tilde{X}))^2 + (g^{-1}(p,\tilde{Y}))^2\big), \\
& ii)\tilde{K}({}^HX, {}^V\omega) = \begin{cases} \frac{\varphi^2(z)K^2\left(2z - 2g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + \left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\left(\varphi(z)+\psi(z)(g^{-1}(\omega,p))^2\right)}, & g(X,\tilde{\omega}) = 1, \\ \frac{\varphi^2(z)K^2\left(\left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\left(\varphi(z)+\psi(z)(g^{-1}(\omega,p))^2\right)}, & g(X,\tilde{\omega}) = 0, \end{cases} \\
& iii)\tilde{K}({}^V\omega, {}^V\theta) = -\frac{A\big(g^{-1}(\omega,p)\big)^2}{\varphi(z) + \psi(z)\big[(g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2\big]} \\
& \qquad -\frac{B(\varphi(z) + 2z\psi(z))(g^{-1}(\theta,p))^2 + C\big(\varphi(z) + \psi(z)(g^{-1}(\theta,p))^2\big)}{\varphi^2(z) + \varphi(z)\psi(z)\big[(g^{-1}(\theta,p))^2 + (g^{-1}(\omega,p))^2\big]},
\end{aligned}
$$

where $\tilde{\omega} = g^{-1} \circ \omega = (g^{ij}\omega_j) \in \Im_0^1(M^n)$, $X^i = g^{ij}X_j = g^{-1} \circ \tilde{X} \in \Im_0^1(M^n)$, $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z) - \varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $N = \frac{\psi'(z)\varphi(z) - 2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $A = 2L' - L^2 - N(1+2zL)$, $B = N - (2M' + M^2 + 2zMN)$ *and* $C = L - M(1+2zL)$.

Proof. Let ${R_{kmj}}^s = K(\delta_k^s g_{mj} - \delta_m^s g_{kj})$.

$$
\begin{aligned}
i)\tilde{K}({}^HX, {}^HY) &= K(X,Y) - \frac{3\varphi(z)}{4}\big|(pR(X,Y)\big|^2 \\
&= K - \frac{3\varphi(z)}{4}g^{ij}(pR(X,Y))_i(pR(X,Y))_j \\
&= K - \frac{3\varphi(z)}{4}g^{ij}p_aK(\delta_k^a g_{li} - \delta_l^a g_{ki})p_bK(\delta_f^b g_{mj} - \delta_m^b g_{fj})X^kY^lX^fY^m \\
&= K - \frac{3\varphi(z)}{4}K^2\big[g^{-1}(\tilde{Y},\tilde{Y})g^{-1}(\tilde{X},p)g^{-1}(\tilde{X},p) \\
&\quad -g^{-1}(X,Y)g^{-1}(X,p)g^{-1}(Y,p) - g^{-1}(X,Y)g^{-1}(\tilde{X},p)g^{-1}(\tilde{Y},p) \\
&\quad +g^{-1}(X,X)g^{-1}(\tilde{Y},p)g^{-1}(\tilde{Y},p)\big] \\
&= K - \frac{3\varphi(z)}{4}K^2\big((g^{-1}(p,\tilde{X}))^2 + (g^{-1}(p,\tilde{Y}))^2\big)
\end{aligned}
$$

ii) Using Theorem 3.1, we have

$$
\begin{aligned}
\tilde{K}({}^HX, {}^V\omega) &= \frac{\varphi^2(z)}{4\big(\varphi(z) + \psi(z)\big(g^{-1}(\omega,p)\big)^2\big)}\big|(pR(\ ,X)\tilde{\omega})\big|^2 \\
&= \frac{\varphi^2(z)g^{tf}\big(pR(\ ,X)\tilde{\omega}\big)_t\big(pR(\ ,X)\tilde{\omega}\big)_f}{4\big(\varphi(z) + \psi(z)\big(g^{-1}(\omega,p)\big)^2\big)} \\
&= \frac{\varphi^2(z)g^{tf}p_a{R_{tij}}^aX^i\tilde{\omega}^jp_b{R_{fkm}}^bX^k\tilde{\omega}^m}{4\big(\varphi(z) + \psi(z)\big(g^{-1}(\omega,p)\big)^2\big)}
\end{aligned}
$$

$$= \frac{\varphi^2(z)g^{tf}p_a\big(K(\delta_t^a g_{ij} - \delta_i^a g_{tj})\big)X^i\tilde{\omega}^j p_b\big(K(\delta_f^b g_{km} - \delta_k^b g_{fm})\big)X^k\tilde{\omega}^m}{4\big(\varphi(z) + \psi(z)\big(g^{-1}(\omega,p)\big)^2\big)}$$

$$= \frac{\varphi^2(z)K^2\big(2z(g(X,\tilde{\omega}))^2 - 2g(X,\tilde{\omega})g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + g(\tilde{\omega},\tilde{\omega})\big(g^{-1}(\tilde{X},p)\big)^2\big)}{4\big(\varphi(z) + \psi(z)\big(g^{-1}(\omega,p)\big)^2\big)}$$

$$= \begin{cases} \frac{\varphi^2(z)K^2\left(2z - 2g^{-1}(\tilde{X},p)g^{-1}(\omega,p) + \left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\left(\varphi(z)+\psi(z)\left(g^{-1}(\omega,p)\right)^2\right)}, & g(X,\tilde{\omega}) = 1, \\ \frac{\varphi^2(z)K^2\left(\left(g^{-1}(\tilde{X},p)\right)^2\right)}{4\left(\varphi(z)+\psi(z)(g^{-1}(\omega,p))^2\right)}, & g(X,\tilde{\omega}) = 0, \end{cases}$$

where

$$\begin{aligned} g(X_a,\tilde{\omega}^b) &= g_{ij}X_a^i(\tilde{\omega}^b)^j = g_{ij}X_a^i g^{jk}\omega_k^b = \delta_i^k X_a^i \omega_k^b \\ &= X_a^k\omega_k^b = \omega^b(X_a) = \delta_a^b = \begin{cases} 1, & a = b, \\ 0, & a \neq b, \end{cases} \end{aligned}$$

$$\begin{aligned} g(\tilde{\omega},\tilde{\omega}) &= g_{ij}\tilde{\omega}^i\tilde{\omega}^j = g_{ij}g^{is}\omega_s g^{jk}\omega_k = \delta_j^s\omega_s g^{jk}\omega_k \\ &= g^{sk}\omega_s\omega_k = g^{-1}(\omega,\omega) = 1. \end{aligned}$$

iii) The statement is obtained by *iii)* of Theorem 3.1. □

Let (x,p) be a point on T^*M^n with $p \neq 0$ and $\{e_1,...,e_n\}$ be an orthonormal basis for the tangent space T_xM^n of M^n at x. Also, let $\{\omega^1,...,\omega^n\}$ be a dual orthonormal basis for the cotangent spaces $T_x^*M^n$ of M^n at x such that $\omega^1 = \frac{p}{|p|}$, where $|p|$ is the norm of p with respect to the metric g on M^n. Then for $i \in \{1,...,n\}$ and $k \in \{2,...,n\}$ define the horizontal and vertical lifts by $f_i = {}^He_i, \quad f_{n+1} = \frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}$ and $f_{n+k} = \frac{1}{\sqrt{\varphi(z)}}({}^V\omega^k)$, $z = \frac{1}{2}|p| = g^{-1}(p,p)$. Then $\{f_1,...,f_{2n}\}$ is an orthonormal basis for the cotangent space $T^*_{(x,p)}M^n$ with respect to the g-natural metric $\tilde{g}$.

Using Theorem 3.1, we have

$$i)\tilde{K}(f_i,f_j) = \tilde{K}({}^He_i,{}^He_j) = K(e_i,e_j) - \frac{3\varphi(z)}{4}\big|pR(e_i,e_j)\big|^2,$$

$$\begin{aligned} ii)\tilde{K}(f_i,f_{n+1}) &= \tilde{K}\Big({}^He_i,\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}\Big) \\ &= \frac{\varphi^2(z)}{4\Big(\varphi(z)+\psi(z)\Big(g^{-1}\big(\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}},p\big)\Big)^2\Big)}\Big|\Big(pR(\ ,e_i)\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}\Big)\Big|^2 \\ &= 0 \end{aligned}$$

by virtue of
$pR(\ ,e_i)\tilde{\omega}^1 = \big(p_mR_{\ ,ks}{}^m e_i^k(\frac{p}{|p|})^s\big) = \big(R_{\ ,ksl}e_i^k(\frac{p}{|p|})^s p^l\big) = \frac{1}{|p|}(R_{\ ,ksl}e_i^k p^s p^l) = 0.$

$$\begin{aligned} iii)\tilde{K}(f_i,f_{n+k}) &= \tilde{K}\Big({}^He_i,\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}\Big) = \frac{\varphi^2(z)\big|\big(pR(\ ,e_i)\frac{\tilde{\omega}^k}{\sqrt{\varphi(z)}}\big)\big|^2}{4\Big(\varphi(z)+\psi(z)\Big(g^{-1}\big(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}},p\big)\Big)^2\Big)} \\ &= \frac{1}{4}\big|(pR(\ ,e_i)\tilde{\omega}^k)\big|^2, \end{aligned}$$

$$iv)\tilde{K}(f_{n+1}, f_{n+k}) = \tilde{K}\Big(\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}, \frac{{}^V\omega^k}{\sqrt{\varphi(z)}}\Big)$$

$$= -\frac{A\big(g^{-1}(\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}, p)\big)^2}{\varphi(z)+\psi(z)\big[\big(g^{-1}(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2 + \big(g^{-1}(\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}, p)\big)^2\big]}$$

$$- \frac{B(\varphi(z)+2z\psi(z))\big(g^{-1}(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2 + C\big(\varphi(z)+\psi(z)\big(g^{-1}(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2\big)}{\varphi^2(z)+\varphi(z)\psi(z)\big[\big(g^{-1}(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2 + \big(g^{-1}(\frac{{}^V\omega^1}{\sqrt{\varphi(z)+2z\psi(z)}}, p)\big)^2\big]}$$

$$= -\frac{2zA + (\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)},$$

$$v)\tilde{K}(f_{n+k}, f_{n+l}) = \tilde{K}\Big(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, \frac{{}^V\omega^l}{\sqrt{\varphi(z)}}\Big)$$

$$= -\frac{A\big(g^{-1}(\frac{{}^V\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2}{\varphi(z)+\psi(z)\big[\big(g^{-1}(\frac{\omega^l}{\sqrt{\varphi(z)}}, p)\big)^2 + \big(g^{-1}(\frac{\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2\big]}$$

$$- \frac{B(\varphi(z)+2z\psi(z))\big(g^{-1}(\frac{\omega^l}{\sqrt{\varphi(z)}}, p)\big)^2 + C\big(\varphi(z)+\psi(z)\big(g^{-1}(\frac{\omega^l}{\sqrt{\varphi(z)}}, p)\big)^2\big)}{\varphi^2(z)+\varphi(z)\psi(z)\big[\big(g^{-1}(\frac{\omega^l}{\sqrt{\varphi(z)}}, p)\big)^2 + \big(g^{-1}(\frac{\omega^k}{\sqrt{\varphi(z)}}, p)\big)^2\big]}$$

$$= -\frac{C}{\varphi(z)} = \frac{2\varphi(z)\psi(z) - 2\varphi(z)\varphi'(z) - z\varphi'^2(z)}{2\varphi^2(z)(\varphi(z)+2z\psi(z))},$$

where $A = 2L' - L^2 - N(1+2zL)$, $B = N - (2M' + M + 2zMN)$ and $C = L - M(1+2zL)$.
Thus we have

Theorem 3.3. *Let (x,p) be a point on T^*M^n and $\{f_1, ..., f_{2n}\}$ be an orthonormal basis for the cotangent spaces $T_x^*M^n$ as above. Then the sectional curvature $\tilde{K}$ satisfy the following equation*

$$i)\quad \tilde{K}(f_i, f_j) = K(e_i, e_j) - \frac{3\varphi(z)}{4}\big|pR(e_i, e_j)\big|^2,$$

$$ii)\quad \tilde{K}(f_i, f_{n+1}) = 0,$$

$$iii)\quad \tilde{K}(f_i, f_{n+k}) = \frac{1}{4}\big|(pR(\ , e_i)\tilde{\omega}^k)\big|^2,$$

$$iv)\quad \tilde{K}(f_{n+1}, f_{n+k}) = -\frac{2zA + (\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)},$$

$$v)\quad \tilde{K}(f_{n+k}, f_{n+l}) == -\frac{C}{\varphi(z)} = \frac{2\varphi(z)\psi(z) - 2\varphi(z)\varphi'(z) - z\varphi'^2(z)}{2\varphi^2(z)(\varphi(z)+2z\psi(z))}$$

where K is a sectional curvature of (M^n, g) and $\tilde{\omega}^k = g^{-1} \circ \omega^k$, for $i \in \{1, ..., n\}$ and $k, l \in \{2, ..., n\}$.

Corollary 3.1. *Let (M^n, p) be a Riemannian manifold and the cotangent bundle T^*M^n be equipped with the g-natural metric $\tilde{g}$. If we have constant sectional curvature, then T^*M^n flat and $A = C = 0$ for any z. It follows $N = \frac{2L' - L^2}{1+2zL}$ and*

$M = \frac{L}{1+2zL}$. *We use* $C = \frac{2\varphi(z)\varphi'(z)+z\varphi'^2(z)-2\varphi(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))} = 0$, *then*

$$2\varphi(z)\varphi'(z) + z\varphi'^2(z) - 2\varphi(z)\psi(z) = 0.$$

So,

$$\psi(z) = \varphi'(z)\big(1 + \frac{z\varphi'(z)}{2\varphi(z)}\big).$$

i) $\psi(z) = k\varphi'(z)$ *where* k *is a real constant.*
If $\varphi'(z) = 0$ *then and* $\varphi(z)$ *is constant.*
If $\varphi'(z) \neq 0$ *then* $\varphi(z) = az^{2(k-1)}$ $(k > 1 \quad or \quad k \leq 0, \quad a > 0)$.
ii) $\psi(z) = \varphi(z)$, *then we obtain* $\frac{\varphi(z)}{\varphi'(z)} = \frac{-1\pm\sqrt{1+2z}}{z}$ *which gives*

$$\varphi(z) = a\frac{e^{2\sqrt{1+2z}}}{\big(1+\sqrt{1+2z}\big)^2}, \quad a > 0$$

or

$$\varphi(z) = a\frac{e^{-2\sqrt{1+2z}}}{\big(\sqrt{1+2z}-1\big)^2}.$$

So we have to deal with non zero vector.

Let now $\{f_1, ..., f_{2n}\}$ be an orthonormal basis for the cotangent space $T^*_xM^n$ as above, then the scalar curvature $\tilde{r} = \sum_{i\neq j}\tilde{K}(f_i, f_j)$ is given by

$$\begin{aligned}
\tilde{r} &= \sum_{i\neq j}\tilde{K}(f_i, f_j) \\
&= 2\sum_{\substack{i,j=1\\ i<j}}^{n}\tilde{K}(f_i, f_j) + 2\sum_{i,j=1}^{n}{}^{CG}K(f_i, f_{n+j}) + 2\sum_{\substack{i,j=1\\ i<j}}^{n}{}^{CG}K(f_{n+i}, f_{n+j}) \\
&= \sum_{i\neq j}^{n}K(e_i, e_j) - \frac{3\varphi(z)}{4}\sum_{i,j=1}^{n}\big|pR(e_i, e_j)\big|^2 + \frac{1}{2}\sum_{i,j=1}^{n}\big|(pR(\ , e_i)\tilde{\omega}^j)\big|^2 \\
&\quad -2\sum_{i=2}^{n}\frac{2zA + (\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)} + \sum_{\substack{i,j=2\\ i\neq j}}^{n}\frac{C}{\varphi(z)} \\
&= r - \frac{3\varphi(z)}{4}\sum_{i,j=1}^{n}\big|pR(e_i, e_j)\big|^2 + \frac{1}{2}\sum_{i,j=1}^{n}\big|(pR(\ , e_i)\tilde{\omega}^j)\big|^2 \\
&\quad -2(n-1)\frac{2zA + (\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)} - (n-1)(n-2)\frac{C}{\varphi(z)}
\end{aligned}$$

from which we have

Theorem 3.4. *Let* (M^n, g) *be a Riemannian manifold and* T^*M^n *be its cotangent bundle equipped with the* g*-natural metric* $\tilde{g}$. *Let* r *be the scalar curvature of* g *and*

$\tilde{r}$ be the scalar curvature of $\tilde{g}$. Then the following equation holds

$$\tilde{r} = r - \frac{3\varphi(z)}{4}\sum_{i,j=1}^{n}\left|pR(e_i,e_j)\right|^2 + \frac{1}{2}\sum_{i,j=1}^{n}\left|(pR(\ ,e_i)\tilde{\omega}^j)\right|^2$$
$$- 2(n-1)\frac{2zA + (\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)} - (n-1)(n-2)\frac{C}{\varphi(z)}.$$

Theorem 3.5. *Let (M^n, g), $n > 2$ be a Riemannian manifold of constant scalar curvatura κ. Then the scalar curvature $\tilde{r}$ of (T^*M^n, g) is*

$$\tilde{r} = (n-1)\big[n\kappa + z(2-3\varphi(z))\kappa^2$$
$$-\Big(\frac{4zA + 2(\varphi(z)+2z\psi(z))C}{(\varphi(z)+2z\psi(z))\varphi(z)+2z\psi(z)} - \frac{(n-2)C}{\varphi(z)}\Big)\big],$$

where $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $N = \frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$, $A = 2L' - L^2 - N(1+2zL)$ and $C = L - M(1+2zL)$.

Proof. Using the formulas $R_{kmj}{}^s = \kappa(\delta_k^s g_{mj} - \delta_m^s g_{kj})$, $r = n(n-1)\kappa$ and Theorem 8, we get the conclusion. □

Example. Using the Theorem 3.5. If $\varphi(z) = \frac{2}{3}$ and $\psi(z) = 0$, then (T^*M^n, g) has constant scalar curvature $\tilde{r} = n(n-1)\kappa$.

4. Geodesics of $\tilde{g}$

Let C be a curve in M^n expressed locally by $x^h = x^h(t)$ and $\omega_h(t)$ be a covector field along C. Then, in the cotangent bundle T^*M^n, we defined a curve $\tilde{C}$ by

$$x^h = x^h(t), \quad x^{\bar{h}} \overset{def}{=} p_h = \omega_h(t) \tag{4.1}$$

If the curve C satisfies at all the points the relation

$$\frac{\delta\omega_h}{dt} = \frac{d\omega_h}{dt} - \Gamma_{jh}^i\frac{dx^j}{dt}\omega_i = 0,$$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C in M^n. Thus, if the initial condition $\omega_h = \omega_h^0$ for $t = t_0$ is given, there exists a unique horizontal lift expressed by (4.1).

We now consider differential equations of the geodesic in the cotangent bundle T^*M^n with the metric $\tilde{g}$. If t is the arc length of a curve $x^A = x^A(t)$, $\quad A = (i, \bar{i})$ in T^*M^n, then equations of geodesic in T^*M^n have the usual form

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2x^A}{dt^2} + \tilde{\Gamma}_{CB}^A\frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{4.2}$$

with respect to the induced coordinates $(x^i, x^{\bar{i}}) = (x^i, p_i)$ in T^*M^n, where $\tilde{\Gamma}_{CB}^A$ are components of $\tilde{\nabla}$ defined by (2.7).

We find it more convenient to refer equations (4.2) to the adapted frame $\{e_\alpha\}$. From (2.1) and (2.2) we see that the matrix of change of frames $e_\beta = A_\beta{}^H\partial_H$ has components of the form (3.2).

Using (3.1), now we write

$$\theta^\alpha = \bar{A}^\alpha{}_A dx^A,$$

i.e.

$$\theta^h = \bar{A}^h{}_A dx^A = \delta_i^h dx^i = dx^h$$

for $\alpha = h$ and

$$\theta^{\bar{h}} = \bar{A}^{\bar{h}}{}_A dx^A = -p_a \Gamma^a_{hj} dx^j + \delta^h_j dx^j = \delta p_h$$

for $\alpha = \bar{h}$. Also we put

$$\frac{\theta^h}{dt} = \bar{A}^h{}_A \frac{dx^A}{dt} = \frac{dx^h}{dt},$$
$$\frac{\theta^{\bar{h}}}{dt} = \bar{A}^{\bar{h}}{}_A \frac{dx^A}{dt} = \frac{\delta p_h}{dt}$$

along a curve $x^A = x^A(t)$ in T^*M^n.

If we therefore write down the form equivalent to (4.2), namely,

$$\frac{d}{dt}\Big(\frac{\theta^\alpha}{dt}\Big) + \tilde{\Gamma}^\alpha_{\gamma\beta}\frac{\theta^\gamma}{dt}\frac{\theta^\beta}{dt} = 0$$

with respect to adapted frame and taking account of (2.7), then we have

$$(4.3) \quad \begin{cases} (a) \quad \frac{\delta^2 x^h}{dt^2} + \varphi(z) p_a R^{k\ ja}_{.i.} \frac{dx^i}{dt}\frac{\delta p_j}{dt} = 0, \\ \\ (b) \quad \frac{\delta^2 p_h}{dt^2} + \left[L(p^i\delta^j_h + p^j\delta^i_h) + Mg^{ij}p_h + Np^ip^jp_h\right]\frac{\delta p_i}{dt}\frac{\delta p_j}{dt} = 0. \end{cases}$$

where $L = \frac{\varphi'(z)}{2\varphi(z)}$, $M = \frac{2\psi(z)-\varphi'(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$ and $N = \frac{\psi'(z)\varphi(z)-2\varphi'(z)\psi(z)}{2\varphi(z)(\varphi(z)+2z\psi(z))}$.

Thus the equations (4.3) are the equations of the geodesic in T^*M^n with the metric $\tilde{g}$. Let now $\tilde{C} : x^h = x^h(t), \quad x^{\bar{h}} = p_h(t) = \omega_h(t)$ be a horizontal lift $\big(\frac{\delta p_h}{dt} = \frac{\delta\omega_h}{dt} = 0\big)$ of the geodesic $C : x^h = x^h(t)$ $(\frac{\delta^2 x^h}{dt^2} = 0)$ in M^n of ∇_g. Then by virtue of (4.3), we have

Theorem 4.1. *The horizontal lift of a geodesic in (M^n, g) is always geodesic in T^*M^n with the g-natural metric $\tilde{g}$.*

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