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POINT-LINE GEOMETRY AND EQUIFORM KINEMATICS IN MINKOWSKI THREE-SPACE

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Abstract. The present paper studies the relation between the point-line displacement and the equiform transformation in Minkowski 3-space $\mathbb{R}^3_1$. A point-line can be transformed into another point-line via an equiform transformation. Observing that a point-line is nothing but a line element when its reference point is the origin of the coordinate system, we modelled that this transformation can also be performed by using dual split quaternions.

1. Introduction

In kinematics, a point-line is represented by an oriented (directed) line and an incident point on this line. The point-line in kinematics has many implementation areas in manufacturing. Yi Zhang and Kwun-Lon Ting examine [8] the point-line positions and displacement with the help of dual quaternion algebra. And O. Aydogmus, L. Kula and Y. Yayli [1] built up point-line displacements of a given line in $\mathbb{R}^3_1$. On the other hand, B. Odehnal, H. Pottmann, J. Wallner [7] investigate Plücker coordinates of the line elements in Euclidean 3-space $\mathbb{R}^3$.

Our interest in this paper is to investigate the relation between point-line representations and equiform kinematics in Minkowski three-space $\mathbb{R}^3_1$. In Section 2, we give dual split quaternions and some of their algebraic properties. Then in Section 3, we give the point-line operator in $\mathbb{R}^3_1$ and also we introduce the equiform transformation and the Plücker coordinates of line elements in Minkowski three-space $\mathbb{R}^3_1$. We examined the similarity between a point-line and a line element. Finally, we introduce the point-line operator which transforms one point-line to another.

2. Preliminaries

In this section, we give some definitions and fundamental facts about Minkowski three-space $\mathbb{R}^3_1$, that will be used throughout the paper.

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2.1. **Some properties of Minkowski three-space $\mathbb{R}_1^3$.** The Minkowski three-space $\mathbb{R}_1^3$ is the Euclidean three-space $\mathbb{R}^3$ endowed with the standard flat metric

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1,\ x_2,\ x_3)$ is a rectangular coordinate system of $\mathbb{R}_1^3$. Since g is an indefinite metric, a vector $v \in \mathbb{R}_1^3$ can have one of the three Lorentzian causal characters: it can be spacelike if $g(\vec{v},\vec{v}) > 0$ or $\vec{v} = 0$, timelike if $g(\vec{v},\vec{v}) < 0$ and null (lightlike) if $g(\vec{v},\vec{v}) = 0$ and $\vec{v} \neq 0$. In particular, the norm (length) of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$, and two vectors $\vec{v}$ and $\vec{w}$ are said to be orthogonal, if $g(\vec{v},\vec{w}) = 0$.

Theorem 2.1. *([6]) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in Minkowski three-space $\mathbb{R}_1^3$. Then,*
i. $\vec{u} \times (\vec{v} \times \vec{w}) = -g(\vec{u},\ \vec{w})\,\vec{v} + g(\vec{u},\ \vec{v})\,\vec{w}$,
ii. $g(\vec{u} \times \vec{v},\ \vec{u} \times \vec{v}) = -g(\vec{u},\ \vec{u})\,g(\vec{v},\ \vec{v}) + g(\vec{u},\ \vec{v})^2$,
where $\vec{u} = (u_1,\ u_2,\ u_3)$, $\vec{v} = (v_1,\ v_2,\ v_3)$ *and*

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) \end{aligned}$$

is the Lorentzian cross product in $\mathbb{R}_1^3$.

Let $\mathbb{R}_n^m$ be the set of matrices of m rows and n columns.

Definition 2.1. *([3]) Let $A = [a_{ij}] \in \mathbb{R}_n^m$ and $B = [b_{jk}] \in \mathbb{R}_p^n$. Lorentzian matrix multiplication is defined as*

$$AB = \left[-a_{i1}b_{1k} + \sum_{j=1}^{n} a_{ij}b_{jk} \right].$$

Note that AB is an $m \times p$ matrix. $\mathbb{R}_n^m$ with Lorentzian matrix multiplication is denoted by L_n^m.

Definition 2.2. *([3]) An $n \times n$ Lorentzian identity matrix with respect to Lorentzian multiplication, denoted by I_n, is given by*

$$I_n = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}.$$

Note that for every $A \in L_n^m$, $I_m A = A I_n = A$.

Definition 2.3. *([3]) A matrix $A \in L_n^n$ is called Lorentzian invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then B is called the Lorentzian inverse of A and is denoted by A^{-1}.*

Definition 2.4. *([3]) The transpose of a matrix $A = [a_{ij}] \in L_n^m$ is denoted by A^T and defined as $A^T = [a_{ji}] \in L_m^n$.*

Definition 2.5. *([3]) A matrix $A \in L_n^n$ is called Lorentzian orthogonal matrix if $A^{-1} = A^T$. The set of Lorentzian orthogonal matrices is denoted by $O_1(3)$.*

2.2. **Dual split quaternions.** In analogy with the complex numbers, W. K. Clifford, defined [2] the dual numbers and showed that they form an algebra. The dual numbers are defined by

$$\begin{aligned} D &= \{A = a + \varepsilon a^* \mid a, a^* \in R\} \\ &= \{A = (a,\ a^*) \mid a, a^* \in R\}, \end{aligned}$$

where ε is the dual symbol subjected to the rules

$$\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0.$$

The set D of dual numbers is a commutative ring with the operations $(+)$ and $(\cdot)$. The algebra

$$H = \{q = q_0 + q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3 \mid q_0,\ q_1,\ q_2,\ q_3 \in R\}$$

of split quaternions is defined as the four-dimensional vector space over R having basis $\{1,\ \vec{e}_1,\ \vec{e}_2,\ \vec{e}_3\}$ with the following properties:

$$\begin{array}{l} 1)\ (\vec{e}_1)^2 = -1,\ (\vec{e}_2)^2 = (\vec{e}_3)^2 = 1, \\ 2)\ \vec{e}_1\vec{e}_2 = -\vec{e}_2\vec{e}_1 = \vec{e}_3,\ \vec{e}_2\vec{e}_3 = -\vec{e}_3\vec{e}_2 = -\vec{e}_1,\ \vec{e}_3\vec{e}_1 = -\vec{e}_1\vec{e}_3 = \vec{e}_2. \end{array} \tag{2.1}$$

It is clear that H is an associative and not commutative algebra and 1 is the identity element of H. H is called split quaternion algebra (see [5] for split quaternions).

Similarly, as a consequence of this definition, a dual split quaternion Q can also be written as

$$Q = q + \varepsilon q^*,$$

where q and q^* are split quaternions.
A dual split quaternion

$$Q = q + \varepsilon q^*$$

is characterized by the following properties in [5]:

Scalar and vector parts of a dual split quaternion $Q = A_0 + A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$ are denoted by $S_Q = A_0$ and $\vec{V}_Q = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$, respectively. The basis $\{1,\ \vec{e}_1,\ \vec{e}_2,\ \vec{e}_3\}$ have the same multiplication properties of basis elements in real quaternions.

Two dual split quaternions Q and P obey the following multiplication rule,

$$QP = (qp) + \varepsilon\,(qp^* + pq^*)$$

where $P = p + \varepsilon p^*$, p and p^* are split quaternions.

Scalar product of dual split quaternions Q and P is given by

$$\begin{aligned} g\,(Q,\ P) &= g\,(P,\ Q) \\ &= g(q,\ p) + \varepsilon\,(g\,(q,\ p^*) + g\,(q^*,\ p))\,. \end{aligned} \tag{2.2}$$

Definition 2.6. *([1]) Let $\vec{V}_Q = \vec{q} + \varepsilon\vec{q}^* = (q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3) + \varepsilon\,(q_1^*\vec{e}_1 + q_2^*\vec{e}_2 + q_3^*\vec{e}_3)$ be a unit dual split vector. Then the dual split vector is called*

$$\begin{array}{ll} spacelike,\ if & g(\vec{q},\ \vec{q}) > 0,\ and\ \vec{q} = \vec{0}, \\ timelike,\ if & g(\vec{q},\ \vec{q}) < 0, \\ lightlike,\ if & g(\vec{q},\ \vec{q}) = 0,\ and\ \vec{q} \neq \vec{0}. \end{array} \tag{2.3}$$

3. Point-line displacement with equiform transformations of $\mathbb{R}^3_1$

3.1. **Point-line operator of $\mathbb{R}^3_1$.** A point-line is represented by an oriented (directed) line and an incident point on this line. Moreover, an oriented (directed) line can be represented with a unit dual vector or signed Plücker coordinates. Thus, we can say the point-line representation can be built up as a dual split vector or signed Plücker coordinates.

Let L be an oriented (directed) line and P be a reference point in Minkowski three-space $\mathbb{R}^3_1$. If we take N as the foot of the perpendicular from P to the directed line L and E is an incident on this directed line L, then the distance h from N to E depends on the location of E and the oriented (directed) line L, (see Fig. 1).

The oriented (directed) line L passing through points E and N can be represented by a unit dual split vector.

Let $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$ be a unit dual split vector satisfying $g(\vec{a},\ \vec{a}) = 1$ (resp. $g(\vec{a},\ \vec{a}) = -1$) and $g(\vec{a}, \overrightarrow{a}_0) = 0$ where the split vector $\overrightarrow{a}$ denotes the unit vector along the oriented line, and the split vector $\overrightarrow{a}_0$ is the moment vector of the oriented line with respect to the origin of reference frame $O - xyz$.

A point-line can be represented by multiplication of a dual number $exp(\varepsilon h) = 1 + \varepsilon h$, and $\vec{A}$, namely

$$\begin{aligned} \hat{A} &= exp(\varepsilon h)\vec{A} \\ &= \left\|\hat{A}\right\| \vec{A} \\ &= \vec{a} + \varepsilon \overrightarrow{a}'_0, \end{aligned} \tag{3.1}$$

where $\overrightarrow{a}'_0 = \overrightarrow{a}_0 + h\vec{a}$ and $\hat{A}$ is a dual split vector with dual length $exp(\varepsilon h)$.

When we have the point-line coordinates, the incident offset h, the directed line, and the incident point can be determined easily.

3.1.1. Case. If $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$ is a unit spacelike dual split vector, then we have

$$\vec{A} = \vec{a} + \varepsilon \left(\overrightarrow{a}'_0 - h\vec{a}\right), \tag{3.2}$$

and

$$h = g(\vec{a}, \overrightarrow{a}'_0). \tag{3.3}$$

Here, the value of h changes related to the reference point. Without losing generality, if we assume that the reference point is the origin of the coordinate system, we can write the position vector of the incident E as

$$\vec{r}_E = \overrightarrow{PN} + \overrightarrow{NE},$$

where $\overrightarrow{a}_0 = \overrightarrow{PN} \times \vec{a}$ and $\overrightarrow{NE} = h\vec{a}$. Therefore, from Theorem 1 and $\overrightarrow{a}'_0 = \overrightarrow{a}_0 + h\vec{a}$, the position vector $\vec{r}_E$ of the incident E is

$$\begin{aligned} \vec{r}_E &= -\vec{a} \times \overrightarrow{a}_0 + h\vec{a} \\ &= -\vec{a} \times \overrightarrow{a}'_0 + g(\vec{a}, \overrightarrow{a}'_0)\vec{a}, \end{aligned}$$

where $\times$ is the Lorentzian cross-product.

3.1.2. Case. If $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$ is a unit timelike dual split vector, then we have

$$\vec{A} = \vec{a} + \varepsilon \left(\overrightarrow{a}'_0 - h\vec{a}\right), \tag{3.4}$$

and

$$h = -g(\vec{a}, \overrightarrow{a}'_0). \tag{3.5}$$

Here, the value of h changes related to the reference point. Without losing generality, if we assume that the reference point is the origin of the coordinate system, we can write the position vector of the incident E as

$$\vec{r}_E = \overrightarrow{PN} + \overrightarrow{NE},$$

where $\overrightarrow{a}_0 = \overrightarrow{PN} \times \vec{a}$ and $\overrightarrow{NE} = h\vec{a}$. Therefore, from Theorem 1 and $\overrightarrow{a}'_0 = \overrightarrow{a}_0 + h\vec{a}$, the position vector $\vec{r}_E$ of the incident E is

$$\begin{aligned} \vec{r}_E &= \vec{a} \times \overrightarrow{a}_0 + h\vec{a} \\ &= \vec{a} \times \overrightarrow{a}'_0 - g(\vec{a}, \overrightarrow{a}'_0)\vec{a}, \end{aligned}$$

where $\times$ is the Lorentzian cross-product ([4])

3.2. **Equiform transformations of $\mathbb{R}^3_1$.** This section describes equiform transformations, which means affine transformations whose linear part is composed from an orthogonal transformation and a homothetical transformation in Minkowski three-space $\mathbb{R}^3_1$. Such an equiform transformation maps points $x \in \mathbb{R}^3_1$ according to

$$\begin{aligned} &\varphi : \mathbb{R}^3_1 \longrightarrow \mathbb{R}^3_1 \\ &\quad x \longrightarrow \varphi(x) = y(t) = \alpha(t)R(t)x + b(t), \end{aligned} \tag{3.6}$$

where $R \in O_1(3)$, $b \in \mathbb{R}^3_1$ and α is a homothetic scale. R, α and b are differentiable functions of class C^∞ of a parameter t.
The velocity $\dot{y}(t)$ has the form

$$v(y) = \dot{R}R^T y + \frac{\dot{\alpha}}{\alpha} y - \dot{R}R^T b - \frac{\dot{\alpha}}{\alpha} b + \dot{b}, \tag{3.7}$$

where $\dot{y}(t) = \frac{dy}{dt}$.

Since R is orthogonal, the matrix $\dot{R}R^T := C^\times$ is skew-symmetric in Lorentz sense and the product $C^\times x$ can be written in the form $c \times x$ in Minkowski three-space $\mathbb{R}^3_1$:

$$v(y) = c \times y + \gamma y + \bar{c}, \tag{3.8}$$

where $\gamma = \frac{\dot{\alpha}}{\alpha}$ and $\bar{c} = \dot{R}R^T b - \frac{\dot{\alpha}}{\alpha} b + \dot{b}$.
Any triple $(c, \bar{c}, \gamma) \in \mathbb{R}^7$ defines a uniform equiform motion in Minkowski three-space $\mathbb{R}^3_1$, uniquely.

3.3. **Plücker coordinates of line elements of $\mathbb{R}^3_1$.** Let L be an oriented (directed) line in Minkowski three-space $\mathbb{R}^3_1$ passing through a point $\vec{x}$. In order to assign coordinates to the line element $(L, \vec{x})$, we use the familiar definition of Plücker coordinates. The triple $(\vec{a}, \overrightarrow{a}_0, h) \in \mathbb{R}^7$ is called the Plücker coordinates of the line element $(L, \vec{x})$ in $\mathbb{R}^3_1$, if $\vec{a} \neq \vec{0}$ spacelike (resp. timelike) is parallel to L, then $\overrightarrow{a}_0 = \vec{x} \times \vec{a}$, $h = g(\vec{x}, \vec{a})$ (resp. $h = -g(\vec{x}, \vec{a})$). It is easy to show that
3.3.1. Case. If $\vec{A}$ is a unit spacelike dual split vector, then

$$\vec{x} = N(\vec{a}, \overrightarrow{a}_0) + h\vec{a}, \tag{3.9}$$

where $N(\vec{a}, \overrightarrow{a}_0) = -\vec{a} \times \overrightarrow{a}_0$.
The point $N(\vec{a}, \overrightarrow{a}_0)$ is the foot point of the origin on the line L. We know that Plücker coordinates satisfy $g(\vec{a}, \overrightarrow{a}_0) = 0$, and $\vec{a} \neq \vec{0}$ occurs as coordinates of lines in $\mathbb{R}^3_1$. Therefore, from (3.9) we obtain the equation

$$\vec{x} = -\vec{a} \times \overrightarrow{a}_0 + h\vec{a}, \tag{3.10}$$

where $h = g(\vec{x}, \vec{a})$ and $\vec{a}$ is a unit parallel spacelike vector to the line L.

If the corresponding line has an orientation, then a line element becomes oriented. The equiform transformation (3.6) transforms the line element $(\vec{a}, \overrightarrow{a}_0, h_1)$ into $(\vec{u}, \vec{u}_0, h_2)$ with $\vec{x}' = \alpha R\vec{x} + \vec{b}$, $\vec{u} = R\vec{a}$, $\vec{u}_0 = \vec{x}' \times \vec{u}$, $h_2 = g(\vec{x}', \vec{u})$. In block matrix form, this transformation, as in the Euclidean case [7], reads

$$\begin{bmatrix} \vec{u} \\ \vec{u}_0 \\ h_2 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ R^{\times}R & \alpha R & 0 \\ \vec{b}^T R & 0^T & \alpha \end{bmatrix} \begin{bmatrix} \vec{a} \\ \overrightarrow{a}_0 \\ h_1 \end{bmatrix}, \tag{3.11}$$

where $R \in O_1(3)$, $b \in \mathbb{R}^3_1$, α is a homothetic scale R, $R^{\times}\vec{x} = \vec{b} \times \vec{x}$, $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$, $g(\vec{a}, \vec{a}) = 1$, $g(\vec{a}, \overrightarrow{a}_0) = 0$ and $\vec{U} = \overrightarrow{u} + \varepsilon \overrightarrow{u}_0$, $g(\overrightarrow{u}, \overrightarrow{u}) = 1$, $g(\overrightarrow{u}, \overrightarrow{u}_0) = 0$.

3.3.2. Case. If $\vec{A}$ is a unit timelike dual split vector, then

$$\vec{x} = N(\vec{a}, \overrightarrow{a}_0) + h\vec{a}, \tag{3.12}$$

where $N(\vec{a}, \overrightarrow{a}_0) = \vec{a} \times \overrightarrow{a}_0$.

The point $N(\vec{a}, \overrightarrow{a}_0)$ is the foot point of the origin on the line L. We know that Plücker coordinates satisfy $g(\vec{a}, \overrightarrow{a}_0) = 0$, and $\vec{a} \neq \vec{0}$ occurs as coordinates of lines in $\mathbb{R}^3_1$. Therefore, from (3.12) we obtain the equation

$$\vec{x} = \vec{a} \times \overrightarrow{a}_0 + h\vec{a}, \tag{3.13}$$

where $h = -g(\vec{x}, \vec{a})$ and $\vec{a}$ is a unit parallel timelike vector to the line L.

If the corresponding line has an orientation, then a line element becomes oriented. The equiform transformation (3.6) transforms the line element $(\vec{a}, \overrightarrow{a}_0, h_1)$ into $(\vec{u}, \vec{u}_0, h_2)$ with $\vec{x}' = \alpha R\vec{x} + \vec{b}$, $\vec{u} = R\vec{a}$, $\vec{u}_0 = \vec{x}' \times \vec{u}$, $h_2 = -g(\vec{x}', \vec{u})$. In block matrix form, this transformation, as in the Euclidean case [7], reads

$$\begin{bmatrix} \vec{u} \\ \vec{u}_0 \\ h_2 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ R^{\times}R & \alpha R & 0 \\ -\vec{b}^T R & 0^T & \alpha \end{bmatrix} \begin{bmatrix} \vec{a} \\ \overrightarrow{a}_0 \\ h_1 \end{bmatrix}, \tag{3.14}$$

where $R \in O_1(3)$, $b \in \mathbb{R}^3_1$, α is a homothetic scale R, $R^{\times}\vec{x} = \vec{b} \times \vec{x}$, $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$, $g(\vec{a}, \vec{a}) = -1$, $g(\vec{a}, \overrightarrow{a}_0) = 0$ and $\vec{U} = \overrightarrow{u} + \varepsilon \overrightarrow{u}_0$, $g(\overrightarrow{u}, \overrightarrow{u}) = -1$, $g(\overrightarrow{u}, \overrightarrow{u}_0) = 0$.

Using the correspondence between line elements and point-lines we observe the following:

Conclusion 1. *Let $\hat{A} = \left\|\hat{A}\right\| \vec{A}$ and $\hat{U} = \left\|\hat{U}\right\| \vec{U}$ be two point-lines. When the reference point is chosen as the origin of the coordinate system for a point-line, the transformations* (3.11) *(resp.* (3.14)*) transform the point-line $\hat{A}$ to the point-line $\hat{U}$ if $\vec{A}$ is a unit spacelike dual split vector (resp. if $\vec{A}$ is a unit timelike dual split vector).*

We can obtain the oriented (directed) line elements in the equation (3.11) and (3.14) by using dual split quaternions. Moreover, as in the Euclidean case [8], we can transform a point-line to another point-line by using dual split quaternions with the following theorem.

Theorem 3.1. *A dual split quaternion Q transforms a given point-line to another given point-line and is defined by*

$$Q = \frac{1}{\left\|\hat{A}\right\|^2} \left(g\left(\hat{A}, \hat{U}\right) - \left(\hat{A} \times \hat{U}\right) \right), \tag{3.15}$$

where $\hat{A}$ and $\hat{U}$ denote two point-lines, $\times$ is Lorentzian cross product and the Q is called the point-line operator which acts on point-lines, represented by split vectors, via split quaternion multiplication.

Proof. Let $\hat{A}$ and $\hat{U}$ be two point-lines defined by $\hat{A} = \left\|\hat{A}\right\| \vec{A}$ and $\hat{U} = \left\|\hat{U}\right\| \vec{U}$. Here, from (3.1) $\vec{A}$ and $\vec{U}$ are unit spacelike (resp. timelike) dual split vectors, dual length $\left\|\hat{A}\right\| = \exp\varepsilon(h_1)$ of $\hat{A}$ and dual length $\left\|\hat{U}\right\| = \exp\varepsilon(h_2)$ of $\hat{U}$.
Case 1. If $\vec{A}$ is a unit spacelike dual split vector and if we apply quaternion multiplication to the Eq. (3.15) with $\hat{A}$ from right-side, then we have

$$Q\hat{A} = \frac{1}{\left\|\hat{A}\right\|^2}\left[g\left(\hat{A},\hat{U}\right)\hat{A} - (\hat{A}\times\hat{U})\times\hat{A}\right]$$

and from Theorem 1 we have

$$Q\hat{A} = \frac{1}{\left\|\hat{A}\right\|^2}\left[g\left(\hat{A},\hat{U}\right)\hat{A} + g\left(\hat{A},\hat{A}\right)\hat{U} - g\left(\hat{A},\hat{U}\right)\hat{A}\right]$$

and from $\frac{g\left(\hat{A},\ \hat{A}\right)}{\left\|\hat{A}\right\|^2} = 1$

$$Q\hat{A} = \hat{U}.$$

Also, since

$$\begin{aligned}\hat{A} &= \left\|\hat{A}\right\|\vec{A},\\ \hat{U} &= \left\|\hat{U}\right\|\vec{U},\end{aligned}$$

Eq. (3.15) can be modified

$$Q = \frac{\left\|\hat{U}\right\|}{\left\|\hat{A}\right\|}\left(g\left(\vec{A},\ \vec{U}\right) - \left(\vec{A}\times\vec{U}\right)\right)$$

and from Eq. (3.1) since $\left\|\hat{A}\right\| = \exp\varepsilon(h_1)$ and $\left\|\hat{U}\right\| = \exp\varepsilon(h_2)$, the last equation can be rewritten as

$$Q = \{\exp\left[\varepsilon(h_2 - h_1)\right]\} Q_0,$$

where $\frac{\left\|\hat{U}\right\|}{\left\|\hat{A}\right\|} = \exp\left[\varepsilon(h_2 - h_1)\right]$ is the dual length of Q and $Q_0 = g\left(\vec{A},\ \vec{U}\right) - \left(\vec{A}\times\vec{U}\right)$.
Because $g\left(\vec{A},\ \vec{U}\right)$ is the scalar part of Q_0 and $-\left(\vec{A}\times\vec{U}\right)$ is the vector part of Q_0, Q is a dual split quaternion.
Case 2. If $\vec{A}$ is a unit timelike dual split vector and if we apply quaternion multiplication to the Eq. (3.15) with $\hat{A}$ from right-side, then we have

$$Q\hat{A} = \frac{1}{\left\|\hat{A}\right\|^2}\left[g\left(\hat{A},\hat{U}\right)\hat{A} - (\hat{A}\times\hat{U})\times\hat{A}\right]$$

and from Theorem 1 we have

$$Q\hat{A} = \frac{1}{\left\|\hat{A}\right\|^2}\left[g\left(\hat{A},\hat{U}\right)\hat{A} + g\left(\hat{A},\hat{A}\right)\hat{U} - g\left(\hat{A},\hat{U}\right)\hat{A}\right]$$

as $Q\hat{A} = -\hat{U}$ is obtained by use of $g\left(\hat{A},\ \hat{A}\right) = -1$. Moreover, due to

$$\begin{aligned}\hat{A} &= \left\|\hat{A}\right\|\vec{A},\\ \hat{U} &= \left\|\hat{U}\right\|\vec{U}.\end{aligned}$$

Eq. (3.15) can be rewritten as

$$Q = \frac{\left\|\hat{U}\right\|}{\left\|\hat{A}\right\|}\left(g\left(\vec{A},\ \vec{U}\right) - \left(\vec{A}\times\vec{U}\right)\right)$$

and from Eq. (3.1). Use of $\left\|\hat{A}\right\| = \exp\varepsilon(h_1)$ and $\left\|\hat{U}\right\| = \exp\varepsilon(h_2)$ leads

$$Q = \left\{\exp\left[\varepsilon(h_2 - h_1)\right]\right\}Q_0,$$

where $\frac{\left\|\hat{U}\right\|}{\left\|\hat{A}\right\|} = \exp\left[\varepsilon(h_2 - h_1)\right]$ is dual length of Q and $Q_0 = g\left(\vec{A},\ \vec{U}\right) - \left(\vec{A}\times\vec{U}\right)$. Because $g\left(\vec{A},\ \vec{U}\right)$ is the scalar part of Q_0 and $-\left(\vec{A}\times\vec{U}\right)$ is the vector part of Q_0, then Q is a dual split quaternion. $\square$

4. Conclusion

In this study, we constructed two block matrices to represent the equiform transformation mapping a given point-line to another given one.

We also prove that dual split quaternions can be used to map a given point-line to another given one. Since it is compact, free of redundancies and easier to compute compared to the matrices given in Eq. (3.11) and Eq. (3.14), this approach has some advantages.

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References

[1] Aydogmus,O., Kula, L. and Yayli, Y., On point-line displacement in Minkowski 3-space, Differential Geometry -Dynamical Systems, 10 (2008), 32-43.
[2] Clifford, W. K., Preliminary skecth of biquaternions, Proceedings of London Math. Soc. 4 (1873), 361-395.
[3] Gundogan, H. and Kecilioglu, O., Lorentzian Matrix Multiplication and the Motions on Lorentzian Plane, Glasnik Matematicki, 41(2006), No. 61, 329-334.
[4] Koc Ozturk, E.B., Spiral vector fields and applications, Ankara University, Ph.D. Thesis, 2012.
[5] Kula, L., Yayli, Y. Dual Split Quaternions and Screw Motions in Minkowski 3- Space, Iranian Journal of Science and Technology (Trans A). Vol. 30, Number A3 (2006), 245-258.
[6] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press 1983.
[7] Odehnal, B., Pottmann, H. and Wallner, J., Equiform kinematics and the geometry of line elements, Beiträge zur Algebra und Geometrie, 47(2006), No. 2, 567-582.
[8] Zhang, Y. and Ting, K.L., On point-line geometry and displacement, Mech. Mach. Theory 39 (2004), 1033-1050.

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