

A NOVEL APPROACH TO DEFINITION OF FUZZY FUNCTIONS

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ABSTRACT. Purpose of the paper is to suggest a new approach in the fuzzy function concept. This approach not only answers the needs of application but also obeys the mathematical rules. Corresponding fuzzy function is defined, continuity of it is investigated and how to use the function concept, which is defined in applied problems, is presented in this paper.

1. INTRODUCTION

Fuzzy logic and fuzzy set theory were created by Zadeh at 1965 [1]. Although it is very popular today and it takes attention of lots of researchers now, fuzzy logic did not take so much attention at the first years of its discovery. Developing technology shows how useful fuzzy logic is in lots of applications. Since the number of engineers who are interested in researches about fuzzy sets is increasing rapidly, mathematical base of it needed to be developed. There are often large data in applications and finding the connection between data is one of the most common problems that an engineer faces with. Recently If Then Rule bases models such as Sugeno-Yasukava [2], Takagi-Sugeno [3], Tanaka [4] are used. Fuzzy rule bases are determined by either of fuzzy clustering methods, such as Fuzzy C-Means (FCM) [5] or by experts in order to get the membership descriptions of the input fuzzy sets that form the left right-hand side and the output fuzzy sets that form right-hand sides. This approach was firstly proposed by Zadeh [1] and applied by Mamdani and Assilian [6]. Lately engineers try to define the functions between data in order to use functions instead of fuzzy rules. We can mention Türkşen [7] and his studies about this subject. Actually mathematical description of fuzzy function was defined by Demirci [8] and Sasaki [9], but these definitions were so abstract that they did not serve the purpose of engineers. Suggesting a new concept of the fuzzy function is our main purpose in this paper. This concept answers the needs of application

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and obeys the mathematical rules. In accordance with this purpose we defined fuzzy function investigate its continuity and display how to use the function concept.

2. FUZZY COORDINATES

The most popular kind of fuzzy numbers is triangular fuzzy numbers. These numbers are denoted as $\Delta = (a, b, c)$ and their membership functions are defined as follows

$$\mu(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{x-c}{b-c}, & b \leq x \leq c \end{cases}$$

where $b \neq a$, $b \neq c$.

Remark : A crisp number a may be regarded as the triangular fuzzy number (a, a, a) .

For a given possibility $\alpha \in [0, 1]$, the left and the right boundaries of α cuts of a triangular fuzzy number $\Delta = (a, b, c)$ are given by $\Delta_L(\alpha) = a + \alpha(b - a)$ and $\Delta_R(\alpha) = c + \alpha(b - c)$, respectively.

Definition 2.1. Let $\Delta = (a, b, c)$ be a triangular fuzzy number. The number $\Delta_{cr} = b$ is called to be the crisp part of Δ .

In the definitions below, $\Delta_1 = (a, b, c)$ and $\Delta_2 = (d, e, f)$ are triangular fuzzy numbers.

Definition 2.2. $\Delta_1 = \Delta_2$ if and only if $a = d$, $c = f$, $b = e$.

Definition 2.3. $\Delta_1 + \Delta_2 = (a + d, b + e, c + f)$.

Definition 2.4. Multiplication of a triangular fuzzy number with a real number k is defined as follows:

$$k\Delta_1 = \begin{cases} (ka, kb, kc), & k \geq 0 \\ (kc, kb, ka), & k < 0 \end{cases}$$

Definition 2.5. $\Delta_1 - \Delta_2 = \Delta_1 + (-1)\Delta_2$

Definition 2.6. Let D and F be two fuzzy sets. Let $\mu_D(x)$ and $\mu_F(x)$ be the membership functions of D and F , respectively. The set

$$G = \{(x, y) \mid x \in D, y \in F, \mu_G(x, y) = \min(\mu_D(x), \mu_F(y))\}$$

is called a Cartesian product of fuzzy sets D and F .

The Cartesian product can be denoted as $G = D \times F$. Let fuzzy triangular numbers $\Delta_x = (a_x, b_x, c_x)$ and $\Delta_y = (a_y, b_y, c_y)$ be given. The region which is bounded with the rectangle $A_1A_2A_3A_4$ with vertices $A_1(a_x, a_y)$, $A_2(a_x, c_y)$, $A_3(c_x, c_y)$, $A_4(c_x, a_y)$ in XY Cartesian coordinate system, denotes the Cartesian product of fuzzy sets defined by the triangular numbers $\Delta_x = (a_x, b_x, c_x)$ and $\Delta_y = (a_y, b_y, c_y)$. Point $B(b_x, b_y)$ in this rectangle, corresponds to the point with grade 1. We call the point $B(b_x, b_y)$ as the crisp center of rectangle $A_1A_2A_3A_4$ and denote this rectangle by $S(B)$. In XY coordinate system there exist unique

rectangle $S(B)$ for each pair of fuzzy triangular numbers $(\Delta_x = (a_x, b_x, c_x), \Delta_y = (a_y, b_y, c_y))$ and conversely each rectangle $S(B)$ defines unique pair $(\Delta_x = (a_x, b_x, c_x), \Delta_y = (a_y, b_y, c_y))$ of fuzzy triangular numbers. That is why we can call the rectangle as the fuzzy rectangle (point) with coordinates (Δ_x, Δ_y) and denote by $\tilde{S}(\Delta_x, \Delta_y)$. We will call Δ_x as the aphis of \tilde{S} and Δ_y as the ordinate.

3. FUZZY FUNCTIONS

Let us denote the set of fuzzy triangular numbers by E^1 . We call every function, defined in set $\tilde{A} \subseteq E^1$ to $\tilde{B} \subseteq E^1$, a fuzzy function and denote by $\tilde{y} = \tilde{f}(\tilde{x})$.

Example 1. $\tilde{f}(\tilde{x}) = 2\tilde{x} + \tilde{b}$ is a fuzzy function in our sense. Here $\tilde{b} = (1, 2, 3)$ is a given triangular number.

In general case if k is given reel number and \tilde{b} is given triangular number then

$$\tilde{f}(\tilde{x}) = k\tilde{x} + \tilde{b}$$

is a fuzzy function.

In coordinate system for each pair (\tilde{x}, \tilde{y}) the geometrical place of the rectangles $\tilde{S}(\tilde{x}, \tilde{y})$ are defined as the graph of \tilde{f} fuzzy function, where $\tilde{y} = \tilde{f}(\tilde{x})$ and the geometrical place of the crisp centers of the rectangles $\tilde{S}(\tilde{x}, \tilde{y})$ are called as the main trajectory of fuzzy function. We note that main trajectory may not be graph of a function in classical sense. We can show that by the following example.

Example 2. Let $\tilde{A} \{(1, 3, 4), (2, 3, 5)\}$ and $\tilde{B} \{(1, 3, 5), (2, 4, 6)\}$ fuzzy triangular numbers be given. We define the function $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$ as following:

$\tilde{f}((1, 3, 4)) = (1, 3, 5); \tilde{f}((2, 3, 5)) = (2, 4, 6)$. In this case the graph of \tilde{f} will be like that:

Fig.1

Main trajectory $\{(3, 3), (3, 4)\}$ does not define a function.

Definition 3.1. If all the conditions $a_1 \neq a_2, b_1 \neq b_2, c_1 \neq c_2$ hold for triangular numbers $\Delta_1 = (a_1, b_1, c_1)$ and $\Delta_2 = (a_2, b_2, c_2)$, then Δ_1 and Δ_2 are defined as strictly different triangular numbers.

Let set \tilde{A} be defined as strictly different triangular number set. Now main trajectory will also be a function graph for each fuzzy function defined in set \tilde{A} . We define the crisp function d in the set $E^1 \times E^1$ as below:

Definition 3.2. For triangular numbers $\Delta_1 = (a_1, b_1, c_1)$ and $\Delta_2 = (a_2, b_2, c_2)$,

$$d(\Delta_1, \Delta_2) = |a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2|. \quad (3.1)$$

Proposition 1. *Function $d(\Delta_1, \Delta_2)$ is a metric.*

Proof. It is clear that $d(\Delta_1, \Delta_2) \geq 0$ and $d(\Delta_1, \Delta_2) = 0$ hold if and only if for $\Delta_1 = \Delta_2$. We can easily see that $d(\Delta_1, \Delta_2) = d(\Delta_2, \Delta_1)$.

For the numbers $\Delta_1 = (a_1, b_1, c_1)$, $\Delta_2 = (a_2, b_2, c_2)$ and $\Delta_3 = (a_3, b_3, c_3)$

$$\begin{aligned} d(\Delta_1, \Delta_3) &= |a_1 - a_3| + |b_1 - b_3| + |c_1 - c_3| \\ &\leq |a_1 - a_2| + |a_2 - a_3| + |b_1 - b_2| + |b_2 - b_3| + |c_1 - c_2| + |c_2 - c_3| \quad \square \\ &= d(\Delta_1, \Delta_2) + d(\Delta_2, \Delta_3) \end{aligned}$$

Definition 3.3. Let \tilde{A} be set of strictly different triangular numbers and we have the following notations:

$$\begin{aligned} a_1 &= \inf \left\{ a \mid (a, b, c) = \Delta \in \tilde{A} \right\}, \quad a_2 = \sup \left\{ a \mid (a, b, c) = \Delta \in \tilde{A} \right\}, \\ b_1 &= \inf \left\{ b \mid (a, b, c) = \Delta \in \tilde{A} \right\}, \quad b_2 = \sup \left\{ b \mid (a, b, c) = \Delta \in \tilde{A} \right\}, \\ c_1 &= \inf \left\{ c \mid (a, b, c) = \Delta \in \tilde{A} \right\}, \quad c_2 = \sup \left\{ c \mid (a, b, c) = \Delta \in \tilde{A} \right\} \end{aligned}$$

If for any $a \in (a_1, a_2)$, $b \in (b_1, b_2)$ and $c \in (c_1, c_2)$ a triangular number $\Delta = (a, b, c) \in \tilde{A}$, then \tilde{A} is called a dense fuzzy set.

Definition 3.4. Let fuzzy triangular number sets \tilde{A} and \tilde{B} be given. Let \tilde{A} be defined as dense set and fuzzy function \tilde{f} be defined in $\tilde{A} \rightarrow \tilde{B}$. Let $\Delta_1 \in \tilde{A}$. If for each $\varepsilon > 0$ there exist such $\delta > 0$ that for all triangular numbers Δ holding the condition $d(\Delta, \Delta_1) < \delta$, the inequality $d(\tilde{f}(\Delta), \tilde{f}(\Delta_1)) < \varepsilon$ holds, then function \tilde{f} is called continuous for Δ_1 .

If a function \tilde{f} is continuous for all $\Delta_1 \in \tilde{A}$, then it is called continuous function in the set \tilde{A} .

Proposition 2. *Main trajectory of the continuous function \tilde{f} in set \tilde{A} is also continuous.*

Proof. Let function \tilde{f} be continuous for $\Delta_1 \in \tilde{A}$. By the definition of continuity, for all $\varepsilon > 0$, there exists an $\delta > 0$ such that for all triangular number Δ holding the condition

$$d(\Delta, \Delta_1) < \delta \tag{3.2}$$

the inequality

$$d(\tilde{f}(\Delta), \tilde{f}(\Delta_1)) < \varepsilon \tag{3.3}$$

holds. Let $\Delta_1 = (a_1, b_1, c_1)$, $\Delta = (a, b, c)$ and the function corresponds to main trajectory of fuzzy function \tilde{f} be g . By the definition 3.2 we have the inequality

$$|b - b_1| < d(\Delta, \Delta_1) \tag{3.4}$$

and by the definition of the main trajectory can be written also the inequality

$$|g(b) - g(b_1)| < d(f(\Delta), \tilde{f}(\Delta_1)) \quad (3.5)$$

For any $\varepsilon > 0$, we can choose $\delta > 0$ which holds the condition (3.2). Therefore from (3.2), (3.3), (3.4) and (3.5) for any b holding the condition

$$|b - b_1| < d(\Delta, \Delta_1) < \delta,$$

we have $|g(b) - g(b_1)| < d(f(\Delta), \tilde{f}(\Delta_1)) < \varepsilon$. As result the function g is continuous for b_1 . \square

According to proposition 2 we can explain a graph of continuous function \tilde{f} as below. There are rectangles which are moving continuously on the continuous main trajectory.

Fig.2

For example let $\tilde{f}((4, 5, 6)) = (4, 6, 8)$ for a function \tilde{f} , main trajectory of which is graphed below.

Fig.3

As we can see from the Fig. 3, for with grade 0,5 the function \tilde{f} receives all the values from the interval $[4, 8]$ with different grades. For example we have,

$$\begin{aligned} \tilde{f}((4, 5; 0, 5)) &= 4; 0 \\ \tilde{f}((4, 5; 0, 5)) &= 5; 0, 5 \\ \tilde{f}((4, 5; 0, 5)) &= 6; 1 \\ \tilde{f}((4, 5; 0, 5)) &= 7; 0, 5 \\ \tilde{f}((4, 5; 0, 5)) &= 8; 0 \end{aligned}$$

4. CONCLUSIONS

Thus, in this paper a new approach for fuzzy function is introduced. This presentation differs from if... then... rule basis, is convenient for applications, and also obeys the mathematical rules. Examples of such function are presented.

ÖZET: Bu çalışmada bulanık fonksiyon kavramı için yeni bir yaklaşım geliştirilmiştir. Bu yaklaşım uygulama problemlerinin gereksinimlerini karşılamakta, hem de klasik matematik kurallarına uyumaktadır. Makalede farklı bir bulanık fonksiyon tanımı yapılmış,

bu fonksiyonun sürekliliği araştırılmış ve uygulama problemlerinde nasıl kullanılacağı gösterilmiştir.

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