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A CHARACTERIZATION OF CYLINDRICAL HELIX STRIP

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Abstract. In this paper, we investigate cylindrical helix strips. We give a new definition and a characterization of cylindrical helix strip. We use some charecterizations of general helix and the Terquem theorem (one of the Joachimsthal Theorems for constant distances between two surfaces).

1. Introduction

In 3-dimensional Euclidean Space, a regular curve is described by its curvatures k_1 and k_2 and also a strip is descibed by its curvatures k_n, k_g and t_r. The relations between the curvatures of a strip and the curvatures of the curve can be seen in many differential books and papers. We know that a regular curve is called a general helix if its first and second curvatures k_1 and k_2 are not constant, but $\frac{k_1}{k_2}$ is constant ([2], [7]). Also if a helix lie on a cylinder, it is called a cylindrical helix and a cylindrical helix has the strip at $\alpha(s)$. The cylindrical helix strips provide being a helix condition and cylindrical helix condition at the point $\alpha(s)$ of the strip by using the curvatures of helix k_1 and k_2.

2. Preliminaries

2.1. The Theory of the Curves.

Definition 2.1. If $\alpha : I \subset \mathbb{R} \to E^n$ is a smooth transformation, then α is called a curve (from the class of C^∞). Here I is an open interval of $\mathbb{R}$ ([11]).

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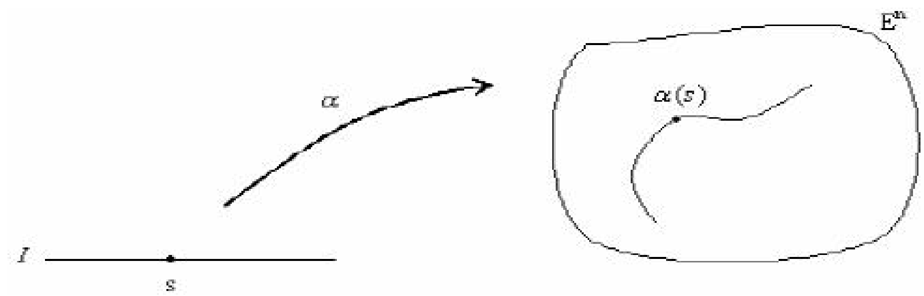

Figure1 The curve in E^n

Definition 2.2. Let the curve $\alpha \subset E^n$ be a regular curve coordinate neigbourhood and $\{V_1(s), V_2(s), ..., V_r(s)\}$be the Frenet frame at the point $\alpha(s)$ that correspond for every $s \in I$. Accordingly,

$$\begin{aligned} k_i : \quad & I \rightarrow R \\ & s \rightarrow k_i(s) = \langle V_i'(s), V_{i+1}(s) \rangle . \end{aligned}$$

We know that the function k_i is called $i-th$ curvature function of the curve and the real number $k_i(s)$ is called $i-th$ curvature of the curve for each $s \in I$ ([2]). The relation between the derivatives of the Frenet vectors among α and the curvatures are given with a theorem as follows:

Definition 2.3. Let $M \subset E^n$ be the curve with neigbouring (I, α). Let $s \in I$ be arc parameter. If $k_i(s)$ and $\{V_1(s), V_2(s), ..., V_r(s)\}$ be the $i-th$ curvature and the Frenet r-frame at the point $\alpha(s)$, then

$$\left\{ \begin{array}{rrcl} \textbf{i}. & V_1'(s) & = & k_1(s)V_2(s) \\ \textbf{ii}. & V_i'(s) & = & -k_{i-1}(s)V_{i-1}(s) \quad + \quad k_i(s)V_{i+1}(s), \quad \ldots \quad 1\langle i \langle r, \\ \textbf{iii}. & V_r'(s) & = & -k_{r-1}(s)V_{r-1}(s) \end{array} \right.$$

([2]).

The equations that about the covariant derivatives of the Frenet r-frame $\{V_1(s), V_2(s), ..., V_r(s)\}$ the Frenet vectors $V_i(s)$ along the curve can be written as

$$\begin{bmatrix} V_1'(s) \\ V_2'(s) \\ V_3'(s) \\ \vdots \\ V_{r-2}'(s) \\ V_{r-1}'(s) \\ V_r'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{r-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{r-2} & 0 & k_{r-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{r-1} & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \\ \vdots \\ V_{r-2}(s) \\ V_{r-1}(s) \\ V_r(s) \end{bmatrix}$$

These formulas are called Frenet Formulas ([2]).

In special case if we take $n = 3$ above the last matrix equations, we obtain following matrix the equation

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \text{ or } \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

.

The first curvature of the curve $k_1(s)$ is called only curvature and the second curvature of the curve $k_2(s)$ is known as torsion ([2]).
If the Frenet vectors are shown as $V_1 = t$, $V_2 = n$, $V_3 = b$ in E^3 , and the curvatures of the curve are shown as $k_1 = \kappa$ and $k_2 = \tau$,

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

or the equations are as follows,

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n. \end{aligned}$$

2.2. **The Strip Theory.**

Definition 2.4. Let M and α be a surface in $E^3$and a curve in $M \subset E^3$. We define a surface element of M is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve α is called a strip or curve-surface pair and is shown as (α, M).

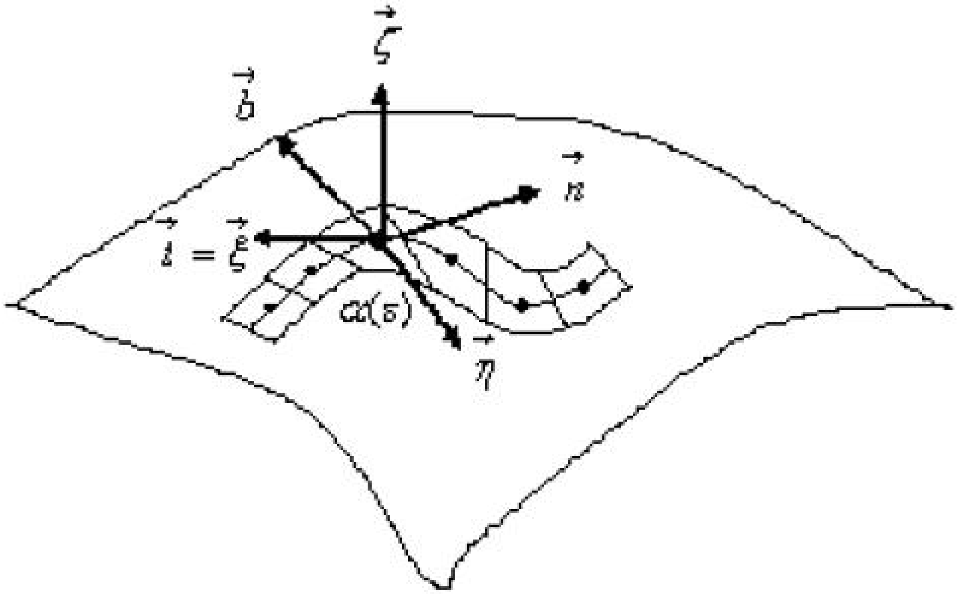

Figure 2 A Strip in E^3 (Hacısalihoğlu1982)

2.3. **Vector Fields of a Strip in E^3.**

Definition 2.5. We know the Frenet vectors fields of a curve α in $M \subset E^3$ are $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$. $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$is called Frenet Frame or Frenet Trehold. Also Frenet vectors of the curve is shown as $\left\{\overrightarrow{V_1}, \overrightarrow{V_2}, \overrightarrow{V_3}\right\}$. In here $\overrightarrow{V_1} = \overrightarrow{t}$, $\overrightarrow{V_2} = \overrightarrow{n}$, $\overrightarrow{V_3} = \overrightarrow{b}$. Let $\overrightarrow{t}$ be the tangent vector field of the curve α, $\overrightarrow{n}$ be the normal vector field of the curve α and $\overrightarrow{b}$ be the binormal vector field of the curve α.

$$\alpha : \quad I \subset M \to E^3$$
$$s \to \alpha(s).$$

If $\alpha : I \to E^3$ is a curve in E^3 with $\|\alpha'(s)\| = 1$, then α is called unit velocity. Let $s \in I$ be the arc length parameter of α. In E^3 for a curve α with unit velocity, $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ Frenet vector fields are calculated as follows ([2])

$$\begin{aligned} \overrightarrow{t} &= \alpha'(s), \\ \overrightarrow{n} &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\ \overrightarrow{b} &= \overrightarrow{t} \times \overrightarrow{n}. \end{aligned}$$

Strip vector fields of a strip which belong to the curve α are $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$. These vector fields are;
Strip tangent vector field is $\overrightarrow{t} = \overrightarrow{\xi}$,
Strip normal vector field is $\overrightarrow{\zeta} = \overrightarrow{N}$,
Strip binormal vector field is $\overrightarrow{\eta} = \overrightarrow{\zeta} \Lambda \overrightarrow{\xi}$ ([6]).

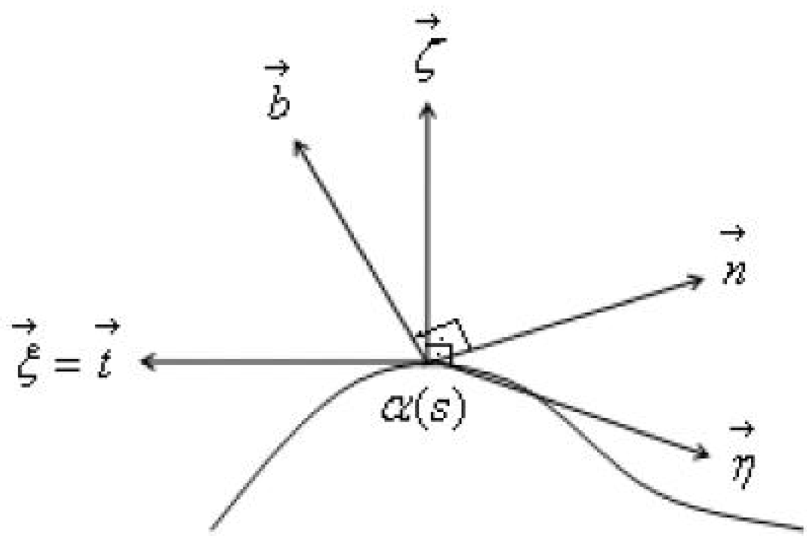

Figure 3 *Strip and curve vector fields in* E^3

Let α be a curve in $M \subset E^3$. If $\alpha\'(s) = \overrightarrow{t}$ $(\overrightarrow{t} = \overrightarrow{\xi})$ and $\overrightarrow{\zeta}$ is a unit strip vector field of a surface M at the point $\alpha(s)$, than we have $\overrightarrow{\eta}\ |_{\alpha(s)} = \overrightarrow{\zeta}\ |_{\alpha(s)}\ \Lambda \overrightarrow{\xi}\ |_{\alpha(s)}$ ([6]). That is $\overrightarrow{\eta}\ |_{\alpha(s)}$is perpendicular $\overrightarrow{\zeta}\ |_{\alpha(s)}$and also $\overrightarrow{\xi}\ |_{\alpha(s)}$ .So we obtain $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ orthonormal vector fields system is called strip three-bundle ([6]).

2.4. **Curvatures of a Strip.** Let $k_n = -b$, $k_g = c$, $t_r = a$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip ([6]).

Let $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ be the strip vector fields on α. Then we have

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

or,

$$\begin{array}{rcl} \xi' & = & c\eta - b\zeta \\ \eta' & = & -c\xi + a\zeta \\ \zeta' & = & b\xi - a\eta \end{array} . \qquad (1)$$

2.5. **Some Relations between Frenet Vector Fields of a Curve and Strip Vector Fields of a Strip.** Let $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$, $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$ and φ be the unit strip vector fields, the unit Frenet vector fields and the angle between $\overrightarrow{\eta}$ and $\overrightarrow{n}$ on α.

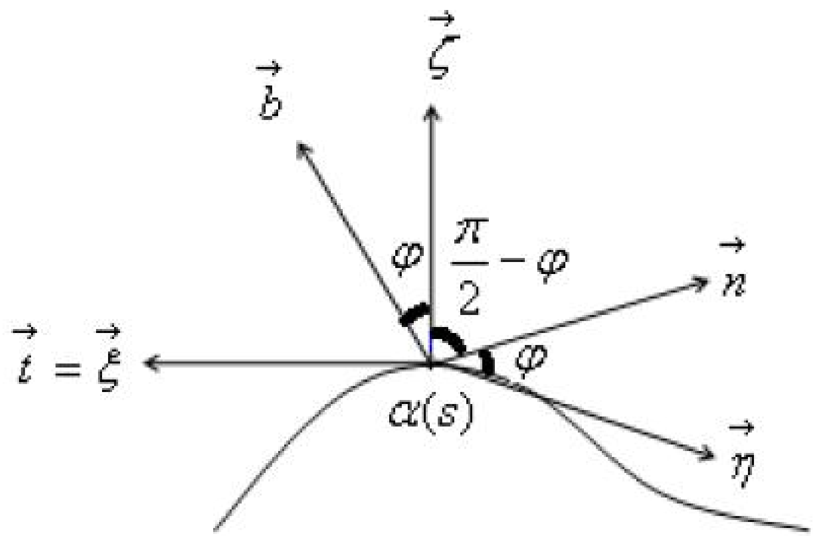

Figure 4 *Strip and curve vector fields and the angle* φ between $\overrightarrow{\eta}$ and $\overrightarrow{n}$ *in* E^3

We can see that $\overrightarrow{\eta}, \overrightarrow{\zeta}, \overrightarrow{n}, \overrightarrow{b}$ vectors are in the same surface from the Figure 4. then we obtain the following equations

$$\begin{aligned}\left\langle\overrightarrow{t}, \overrightarrow{\zeta}\right\rangle &= 0\\ \left\langle\overrightarrow{t}, \overrightarrow{n}\right\rangle &= 0\\ \left\langle\overrightarrow{t}, \overrightarrow{b}\right\rangle &= 0\\ \left\langle\overrightarrow{t}, \overrightarrow{\eta}\right\rangle &= 0.\end{aligned}$$

2.5.1. *The Equations of the Strip Vector Fields in type of the Frenet vector Fields.* Let $\left\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\right\}$, $\left\{\overrightarrow{\xi}, \overrightarrow{\eta}, \overrightarrow{\zeta}\right\}$ and φ be the Frenet Vector fields, strip vector fields and the angle between $\overrightarrow{\eta}$ and $\overrightarrow{n}$. We can write the following equations by the Figure 4.

$$\begin{aligned}\overrightarrow{\xi} &= \overrightarrow{t}\\ \overrightarrow{\eta} &= \cos\varphi\ \overrightarrow{n} - \sin\varphi\ \overrightarrow{b}\\ \overrightarrow{\zeta} &= \sin\varphi\ \overrightarrow{n} + \cos\varphi\ \overrightarrow{b}\end{aligned}$$

or in matrix form

$$\begin{bmatrix} \overrightarrow{\xi} \\ \overrightarrow{\eta} \\ \overrightarrow{\zeta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix}.$$

2.5.2. *The Equations of the Frenet Vector Fields in type of the Strip Vector Fields.* By the help of the Figure 4 we can write

$$\begin{aligned} \overrightarrow{t} &= \overrightarrow{\xi} \\ \overrightarrow{n} &= \cos\varphi \ \overrightarrow{\eta} + \sin\varphi \ \overrightarrow{\zeta} \\ \overrightarrow{b} &= -\sin\varphi \ \overrightarrow{\eta} + \cos\varphi \ \overrightarrow{\zeta} \end{aligned}$$

or in matrix form

$$\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \overrightarrow{\xi} \\ \overrightarrow{\eta} \\ \overrightarrow{\zeta} \end{bmatrix}.$$

2.5.3. *Some Relations between a, b, c invariants (Curvatures of a Strip) and κ, τ invariants (Curvatures of a Curve).* We know that a curve α has two curvatures κ and τ. A curve has a strip and a strip has three curvatures k_n, k_g and t_r.

$$\begin{aligned} k_n &= -b \\ k_g &= c \\ t_r &= a \end{aligned}$$

([4], [6]).From the derivative equations we can write

$$\acute{\xi} = c\eta - b\zeta.$$

If we substitude $\overrightarrow{\xi} = \overrightarrow{t}$ in last equation, we obtain

$$\acute{\xi} = \kappa n$$

and

$$\begin{aligned} b &= -\kappa \sin\varphi \\ c &= \kappa \cos\varphi \end{aligned}$$

([4], [8]). From last two equations we obtain,

$$\kappa^2 = b^2 + c^2.$$

This equation is a relation between the curvature κ of a curve α and normal curvature and geodesic curvature of a strip ([6], [10]).

By using similar operations, we obtain a new equation as follows

$$\tau = -a + \frac{\acute{b}c - b\acute{c}}{b^2 + c^2}$$

([6], [10]). This equation is a relation between τ (torsion or second curvature of α) and a, b, c curvatures of a strip that belongs to the curve α.

And also we can write

$$a = \varphi^{'} + \tau.$$

The special case: ıf φ =constant, then $\varphi^{'} = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the strip is equal to torsion of the curve.

Definition 2.6. Let α be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve α is equal to zero, then the curve-surface pair (α, M) is called a curvature strip ([6]).

3. General Helix

Definition 3.1. Let α be a curve in E^3 and V_1 be the first Frenet vector field of $\alpha.U \in \chi(E^3)$ be a constant unit vector field. If

$$\langle V_1, U\rangle = \cos\varphi \quad \text{(constant)}$$

α, φ and Sp$\{U\}$ is called an general helix, the slope angle and the slope axis ([1], [2]).

Definition 3.2. A regular curve is called a general helix if its first and second curvatures κ, τ are not constant but $\frac{\varkappa}{\tau}$ is constant ([1], [7]).

Definition 3.3. A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio $\frac{\varkappa}{\tau}$ is constant ([8]) .

Definition 3.4. A helix is a curve in 3-dimensional space. The following parametrisation in Cartesian coordinates defines a helix ([12]).

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \\ z(t) &= t. \end{aligned}$$

As the parameter t increases, the point $(x(t), y(t), z(t))$ traces a right-handed helix of pitch 2π and radius 1 about the z-axis, in a right-handed coordinate system.In cylindrical coordinates (r, θ, h), the same helix is parametrised by

$$\begin{aligned} r(t) &= 1 \\ \theta(t) &= t \\ h(t) &= t. \end{aligned}$$

Definition 3.5. If the curve α is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. The ratio $\frac{\tau}{\kappa}$ is called first Harmonic Curvature of the curve and is denoted by H_1 or H.

Theorem 3.6. *A regular curve $\alpha \subset E^3$ is a general helix if and only if $H(s) = \frac{k_1}{k_2}$ =constant for $\forall s \in I$ ([2]).*

Proof. ($\Rightarrow$) Let α be a general helix. The slope axis of the curve α is shown as $Sp\{U\}$. Note that

$$\left\langle \alpha'(s), U \right\rangle = \cos\varphi = \text{constant}.$$

If the Frenet trehold is $\{V_1(s), V_2(s), V_3(s)\}$ at the point $\alpha(s)$, then we have

$$\langle V_1(s), U\rangle = \cos\varphi.$$

If we take derivative of the both sides of the last equation, then we have

$$\langle k_1(s)V_2(s), U\rangle = 0 \Rightarrow \langle V_2(s), U\rangle = 0.$$

Hence

$$U \in Sp\{V_1(s), V_3(s)\}.$$

Therefore

$$U = \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s).$$

U is the linear combination of $V_1(s)$ and $V_3(s)$. By differentiating the equation $\langle V_2(s), U\rangle = 0$, we obtain

$$\begin{aligned} \langle -k_1(s)V_1(s) + k_2(s)V_3(s), U\rangle &= 0 \\ -k_1(s)\langle V_1(s), U\rangle + k_2(s)\langle V_3(s), U\rangle &= 0 \\ -k_1(s)\cos\varphi + k_2(s)\sin\varphi &= 0. \end{aligned}$$

By using the last equation, we see that

$$H = \text{constant}.$$

($\Leftarrow$**)** Let $H(s)$ be constant for $\forall s \in I$, and $\lambda = \tan\varphi$, then we obtain

$$U = \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s).$$

1) If U is a constant vector,then we have

$$D_{\overset{\bullet}{\alpha}} U = (k_1(s)\cos\varphi - \sin\varphi\ k_2(s))V_2(s).$$

By substituting $H(s) = \tan\varphi$ is in the last equation, we see that

$$k_1(s)\cos\varphi - k_2(s)\sin\varphi = 0,$$

and so

$$U = \text{constant}.$$

2) If α is an inclined curve with slope axis $Sp\{U\}$. Since

$$\begin{aligned} \left\langle \alpha'(s), U \right\rangle &= \langle V_1(s), \cos\varphi\ V_1(s) + \sin\varphi\ V_3(s)\rangle \\ &= \cos\varphi\ \langle V_1(s), V_1(s)\rangle + \sin\varphi\ \langle V_1(s), V_3(s)\rangle \end{aligned}$$

we obtain

$$\left\langle \alpha'(s), U \right\rangle = \cos\varphi = \text{constant}.$$

□

Definition 3.7. Let α be a helix that lie on the cylinder. A helix which lies on the cylinder is called cylindrical helix.

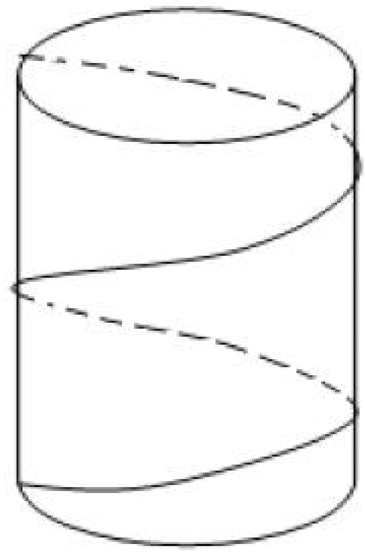

Figure 5 Cylindrical helix

Definition 3.8. Let M be a cylinder in E^3, and α be a helix on M. We define a surface element of M as the part of a tangent plane at the neighbourhood of a point of the cylindrical helix. The locus of the surface element along the cylindrical helix is called a helix strip.

Definition 3.9. Let M be a cylinder in E^3, and α be a helix on M. The part of the tangent plane on the cylindrical helix is called the surface element of the cylinder. The locus of the surface element along the cylindrical helix is called a strip of cylindrical helix.

Theorem 3.10. *Suppose that $\kappa\rangle 0$. Then α is a strip of cylindrical helix if and only if the ratio $\frac{\varkappa}{\tau}$ is constant.*

Proof. Let θ be the angle between the tangent vector field ξ and slope vector u of a strip of cylindrical helix. Since $\xi.u = \cos\theta$ is constant, we have

$$0 = (\xi.u)\acute{} = \acute{\xi}u = \kappa\zeta.u$$

Because $\varkappa\rangle 0$ and $\zeta.u = 0$, we see that u is perpendicular to ζ and so

$$u = \cos\theta.\xi + \sin\theta.\eta.$$

By differentiating the last equation,

$$(\kappa\cos\theta - \tau\sin\theta)\zeta = 0$$

or

$$\tan\theta = \frac{\kappa}{\tau}.$$

Since $\tan\theta$ is constant, $\frac{\kappa}{\tau}$ is also constant ([9]). □

Theorem 3.11. *(Terquem Theorem) Let M_1, M_2 be two different surfaces in E^3.Let α and β be nonplanar curve in M_1 and a curve on M_2.*
i. *The points of the curves α and β corresponds to each other 1:1 on a plane ε which rolls on the M_1 ve M_2, such that the distance is constant between corresponding points.*
ii. *(α, M_1) is a curvature strip.*
iii.(β, M_2) *is a curvature strip* ([6]).

Claim**:** Two of the three lemmas gives third ([6]).

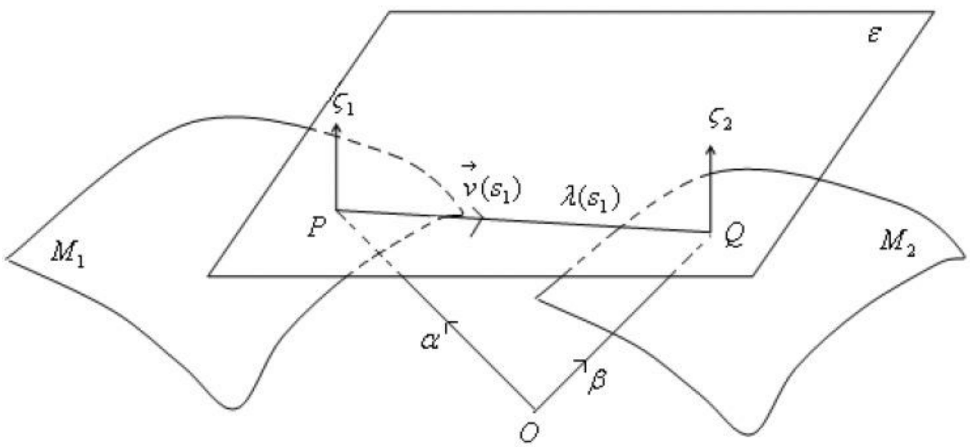

Figure 6

By applying the similar way in the proof of Theorem II.3.11 in [6] to the strip of cylindric helix strip, we give the following theorem.

Theorem 3.12. *Let L and M be be a cylindrical helix and a surface in E^3. Suppose that L and M have common tangent plane along β and cylindrical helix α. If the curve-surface pair (β, M) is a curvature strip, then the curve β is a a helix strip.*

Proof.

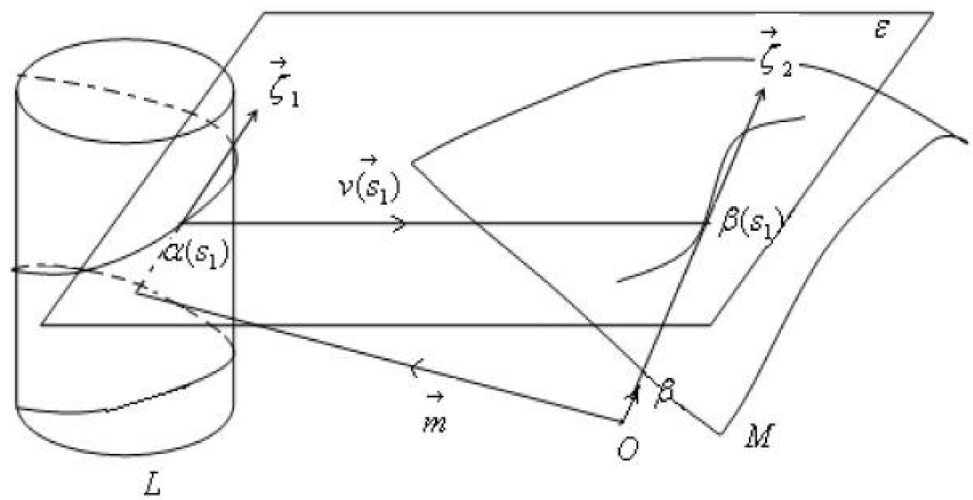

Figure 7 The cylinder L and the surface M.

If the curve α is a helix on L, then it provides $\frac{\kappa_1}{\tau_1}$ is constant. We have to show that β is a helix strip on M, that is, $\frac{\kappa_2}{\tau_2}$ =constant.

By the Figure 7, we have

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1)\overrightarrow{v}(s_1) \tag{2}$$

where

$$\alpha(s_1) = \overrightarrow{m} + r\ \zeta_1(s_1). \tag{3}$$

By differentiating both side of (3), we see that

$$\overrightarrow{\xi}_1 = \frac{d\alpha}{ds_1} = r\frac{d\overrightarrow{\zeta}_1}{ds_1}.$$

By (1),

$$\overrightarrow{\xi}_1 = r(b_1\overrightarrow{\xi}_1 - a_1\overrightarrow{\eta_1}),$$

we obtain $a_1 = 0$ and $b_1 = 1$.

Let r be the distance between gravity center of the cylinder and $\alpha(s_1)$. We denote

$r = 1$. If $\overrightarrow{m}$ is a position vector of the gravity center of cylinder, then $\overrightarrow{m}$ must be a constant vector.

Since $a_1 = 0$, (α, L) is a curvature strip. By the strips (α, L) and (β, M) are curvature strips and by Theorem 17, we see that λ is non-zero constant. Let $\overrightarrow{v}(s_1)$ be a vector in $\mathrm{Sp}\left\{\overrightarrow{\xi}_1, \overrightarrow{\eta_1}\right\}$, and let φ be the angle between $\overrightarrow{\xi}_1$ and $\overrightarrow{v}(s_1)$. Then we write

$$\overrightarrow{v}(s_1) = \cos\varphi \overrightarrow{\xi}_1 + \sin\varphi \overrightarrow{\eta_1}.$$

By substituting (3) and (4) in (2), and differentiating both sides, we obtain (5).

$$\frac{d\beta}{ds_1} = \frac{d\overrightarrow{m}}{ds_1} + \frac{d\overrightarrow{\zeta_1}}{ds_1} + \frac{d\lambda}{ds_1}(\cos\varphi\ \overrightarrow{\xi}_1 + \sin\varphi\ \overrightarrow{\eta_1}) + \lambda(s_1)\frac{d(\cos\varphi\ \overrightarrow{\xi}_1 + \sin\varphi\ \overrightarrow{\eta_1})}{ds_1}. \quad (5)$$

Since the vector $\overrightarrow{m}$ and λ are constant, we obtain the following equation

$$\frac{d\beta}{ds_1} = \frac{d\overrightarrow{\zeta_1}}{ds_1} + \lambda(s_1)\frac{d(\cos\varphi\ \overrightarrow{\xi}_1 + \sin\varphi\ \overrightarrow{\eta_1})}{ds_1}$$

or

$$\frac{d\beta}{ds_1} = \frac{d\overrightarrow{\zeta_1}}{ds_1} + \lambda(s_1)(-\frac{d\varphi}{ds_1}\sin\varphi\ \overrightarrow{\xi}_1 + \cos\varphi\ \frac{d\overrightarrow{\xi}_1}{ds_1}) + \frac{d\varphi}{ds_1}\cos\varphi\ \overrightarrow{\eta_1} + \sin\varphi\ \frac{d\overrightarrow{\eta_1}}{ds_1}).$$

By (1), we see that

$$\frac{d\beta}{ds_1} = \left[1 - \lambda(\frac{d\varphi}{ds_1} + c_1)\sin\varphi\right]\overrightarrow{\xi}_1 + \lambda(\frac{d\varphi}{ds_1} + c_1)\cos\varphi\overrightarrow{\eta_1} - \lambda\cos\varphi\overrightarrow{\zeta_1}. \quad (6)$$

Since the cylindric helix and the surface M have the same tangent plane along the curves α and β, we can write

$$\left\langle \frac{d\beta}{ds_1}, \overrightarrow{\zeta_1} \right\rangle = 0.$$

By subsitituting (6) in the last equation, we obtain $\cos\varphi = 0$. By using that equation in (6), we have

$$\frac{d\beta}{ds_1} = (1 \mp \lambda c_1)\overrightarrow{\xi}_1 \quad (7)$$

If we calculate the second and third derivatives of the curve β, then we get

$$\begin{aligned}
\frac{d^2\beta}{ds_1^2} &= \mp\lambda c_1^{'}\overrightarrow{\xi}_1 + (1 \mp \lambda c_1)c_1\overrightarrow{\eta_1} - (1 \mp \lambda c_1)\overrightarrow{\zeta_1} \\
\frac{d^3\beta}{ds_1^3} &= \left[\mp\lambda c_1^{''} - (1 \mp \lambda c_1)c_1^2 - (1 \mp \lambda c_1)\right]\overrightarrow{\xi}_1 + \left[\mp\lambda c_1 c_1^{'} \mp \mp\lambda c_1 c_1^{'} + (1 \mp \lambda c_1)c_1^{'}\right]\overrightarrow{\eta}_1 \\
&\quad + (\mp\lambda c_1^{'} \mp \lambda c_1^{'})\overrightarrow{\zeta_1}.
\end{aligned}$$

Since the same result is obtained by using the other form of (7), we use the form $\frac{d\beta}{ds_1} = (1 - \lambda c_1)\overrightarrow{\xi}_1$ of (7) in the rest of our proof. By differentiating both sides of

(7), we obtain

$$\begin{aligned}\frac{d\beta}{ds_1} &= (1-\lambda c_1)\overrightarrow{\xi}_1 \\ \frac{d^2\beta}{ds_1^2} &= -\lambda c_1^{'}\overrightarrow{\xi}_1 + (1-\lambda c_1)c_1\overrightarrow{\eta_1} - (1-\lambda c_1)\overrightarrow{\zeta_1} \\ \frac{d^3\beta}{ds_1^3} &= \left[-\lambda c_1^{''} - (1-\lambda c_1)c_1^2 - (1-\lambda c_1)\right]\overrightarrow{\xi}_1 + \left[-3\lambda c_1 c_1^{'} + c_1^{'}\right]\overrightarrow{\eta}_1 + 2\lambda c_1^{'}\overrightarrow{\zeta_1}.\end{aligned}$$

By applying Gram-Schmidt to the $\left\{\beta^{'},\beta^{''},\beta^{'''}\right\}$, then we have

$$\begin{aligned}F_1 &= (1-\lambda c_1)\overrightarrow{\xi}_1 \\ F_2 &= (1-\lambda c_1)c_1\overrightarrow{\eta_1} - (1-\lambda c_1)\overrightarrow{\zeta_1} \\ F_3 &= \frac{(1-\lambda c_1)c_1^{'}}{c_1^2+1}\overrightarrow{\eta_1} + \frac{(1-\lambda c_1)c_1^{'}c_1}{c_1^2+1}\overrightarrow{\zeta_1}.\end{aligned}$$

By [6], we have

$$\kappa_1^2 = b_1^2 + c_1^2, \quad b_1 = 1 \tag{8}$$

and

$$\tau_1^2 = -a_1 + \frac{b_1^{'}c_{1-}b_1c_1^{'}}{b_1^2+c_1^2}, \quad a_1 = 0 \tag{9}$$

By (8) and (9), we see that

$$\tau_1 = \frac{-c_1^{'}}{\kappa_1^2}. \tag{10}$$

By using (10) in F_3, we obtain

$$F_3 = -(1-\lambda c_1)\tau_1\overrightarrow{\eta_1} - (1-\lambda c_1)c_1\tau_1\overrightarrow{\zeta_1}.$$

If we calculate κ_2 and τ_2, then we have

$$\kappa_2 = \frac{\kappa_1}{|1-\lambda c_1|}$$

and

$$\tau_2 = \frac{\tau_1}{|1-\lambda c_1|}$$

Dividing by κ_2 to τ_2 , we obtain

$$\frac{\kappa_2}{\tau_2} = \frac{\kappa_1}{\tau_1}. \tag{11}$$

We obtain the proof of theorem from last equation. □

ÖZET: Bu çalışmada silindirik helis şeridleri incelendi. Yeni bir tanım ve bir karekterizayon verildi. Genel helis ve Terquem teoremlerinin (herhangi iki yüzey arasındaki uzaklığın sabit olmasıyla ilgili Joachimsthal teoremlerinden biri) karekterizasyonlarından yararlanıldı.

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