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REGULARIZED TRACE OF STURM-LIOUVILLE EQUATION WITH SINGILARITY ON A BOUNDED SEGMENT

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Abstract. Let $\mu_1 \le \mu_2 \le ... \le \mu_n \le ...$ and $\lambda_1 \le \lambda_2 \le ... \le \lambda_n \le ...$ be the eigenvalues of the operators L_0 and L which are formed by differential expressions

$$\ell_o[y] = -y'' + \frac{v^2 - 1/4}{x^2} y, \quad \ell[y] = -y'' + \frac{v^2 - 1/4}{x^2} y + q(x)y, \quad \nu \ge 1/2$$

respectively and with the same boundary conditions Eq.(2.4)

$$y(0) = y(1) = 0$$

We prove that under some conditions on $q(x)$, the following formula for traces

$$\sum_{n=1}^{\infty} (\lambda_n - \mu_n) = -\frac{2\nu q(0) + q(1)}{4}$$

with Eq.(2.4) holds.

1. Introduction

In literature there are numerous papers devoted to the calculation of regularized trace of scalar differential operators which is the generalization of concept of matrix trace. First work in this direction belongs to I.M.Gelfand and B.M Levitan [1], where the formula for the sum of diffrences of eigenvalues of two regular Sturm-Liouville operators on the $[0, \pi]$, was obtained. This work has numerous continuations. In [2-16] the regularized traces are calculated in several cases. In [2] the formula was obtained for the sum of diffrences of eigenvalues of two singular self-adjoint Sturm-Liouville operators which differ by finite potentials. In [4-8] by using zeta function and teta function V.A.Sadovnichiy has obtained formulae for regularized traces for wide class of differential operators. In [7] the author considered the following operator

$$Ly = -y^{''} + \frac{\nu^2 - {}^1\!/_4}{x^2} y + q(x)y = \lambda^2 y$$

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on $(0, \pi]$, for the case when the potential $q(x)$ is finite on the neighborhood of zero and is a sufficiently smooth function. In this case the series $\sum\limits_{n=1}^{\infty}(\lambda_n - (n + \frac{\nu}{2} - \frac{1}{2})^2)$ is calculated.

The aim of this article is to calculate the regularized trace with the peculiarity on the bounded segment. Namely we consider the following problem.

Let L_0 and L be operators acting in $L_2(0,1)$and formed by differential expressions

$$\begin{aligned} \ell_o[y] &= -y'' + \tfrac{v^2-1/4}{x^2}y \\ \ell[y] &= -y'' + \tfrac{v^2-1/4}{x^2}y + q(x)y, \quad \nu \geq 1/2 \end{aligned}$$

respectively and with same boundary conditions Eq.(2.4)

$$y(0) = y(1) = 0,$$

where potential $q(x)$ is a bounded function and satisfies the following conditions:

a) $q(x)$ satisfies the Hölder condion with order $\alpha > 0$ on the neigbour of zero, i.e. there is $\varsigma > 0$such that for any $x \in [0, \varsigma]$ the following inequality holds

$$|q(x) - q(0)| < \text{const. } x^{\alpha}$$

b) $\int\limits_0^1 q(x)dx = 0$

Our aim is to find the regularized trace of these operators.

2. Main results

Let $\mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq ...$ and $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$ be the eigenvalues of the operators $L_0$and L, respectively. Then the following theorem holds

Theorem. *Let the function q(x) satisfy conditions a) and b).*

Then $\sum\limits_{n=1}^{\infty}(\lambda_n - \mu_n) = -\frac{2\nu q(0)+q(2.4)}{4}$

It is shown in [11] that $\sum\limits_{n=1}^{\infty}(\lambda_n - \mu_n - (q\ell_n, \ell_n)) = 0$, where ℓ_n are eigenfunctions of L$_0$. If $\sum\limits_{n=1}^{\infty}(q\ell_n, \ell_n)$ is convergent then $\sum\limits_{n=1}^{\infty}(\lambda_n - \mu_n) = \sum\limits_{n=1}^{\infty}(q\ell_n, \ell_n)$.

Eigenfunctions of the operator L_0 have the form [2]

$\ell_n = \sqrt{2x}\frac{Jv(j_n x)}{J_{v+1}(j_n)}$ where $J_v(x)-$is the Bessel function, $j_1 \leq j_2 \leq j_3 \leq ...$ are positive roots of $J_v(z)$.

$$\sum_{n=1}^{\infty}(q\ell_n,\ell_n)=\lim_{N\to\infty}\sum_{n=1}^{N}(q\ell_n,\ell_n) \tag{2.1}$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}\int_0^1 2x\frac{J_v^2(j_nx)}{J_{v+1}^2(j_n)}q(x)dx \tag{2.2}$$

$$=\lim_{N\to\infty}\int_0^1\left(\sum_{n=1}^{N}\frac{2xJ_v^2(j_nx)}{J_{v+1}^2(j_n)}\right)q(x)dx \tag{2.3}$$

To find this limit we will study the asymptotical behavior of the function

$\mathrm{T}_N(x)=\sum\limits_{n=1}^{N}2x\frac{J_v^2(j_nx)}{J_{v+1}^2(j_n)}$ with and $\varepsilon>0$

For later use we note the following lemma

Lemma. *For the function $T_N(x)$,*

$$T_N(x)=\frac{A_N}{\pi}-\frac{\cos(2xA_N-v\pi)}{2\sin\pi x}+\frac{\psi(A_Nx)}{x}$$

where $A_N=(N+\frac{v}{2}+\frac{1}{4})\pi$, *when* $0<x<1$,

$\psi(A_Nx)\to 0$ *as* $N\to\infty$.

Proof of Lemma. To get formula for $T_N(x)$ we express the m^{th} term of the sum $T_N(x)$ in the form of residue at point j_m of some function of complex variable z, which has poles at the points $j_1,j_2,...,j_N$.

Consider the following complex function.

$$\frac{zx\{J_v^2(xz)-J_{v-1}(xz)J_{v+1}(xz)\}}{J_v^2(z)}$$

First, prove that this function has a residue $2x\frac{J_v^2(j_mx)}{J_{v+1}^2(j_m)}$ at $z = j_m$.

If $z=j_m+\theta$, where θ is small, then

$$J_\nu(z)=\theta J_\nu^{'}(j_m)+\frac{1}{2}\theta^2J_\nu^{''}(j_m)+...$$

and hence

$$zJ_v^2(z)=\theta^2j_mJ_\nu'^2(j_m)+\theta^3\left(j_m\right)\left\{j_mJ_\nu''(j_m)+J_\nu'(j_m)\right\}+...$$

Using Bessel differential equation it is easy to prove that the coefficient of θ^3 on the right hand side of this expression is zero.

Therefore the residue of function

$$\frac{g(z)}{zJ_v^2(z)}=\frac{z^2x\left\{J_v^2(xz)-J_{v-1}(xz)J_{v+1}(xz)\right\}}{zJ_v^2(z)} \tag{2.4}$$

at j_m is

$$\frac{g'(j_m)}{j_m J'2}(j_m) = 2x\frac{J_v^2(j_m x)}{J_{v+1}^2(j_m)}$$

As a contour of integration we take the rectangle with vertex at $\pm iB$, $A_N \pm iB$. Here $B \to \infty$ and $j_N < A_N < j_{N+1}$. For A_N we take the value $\left(N + \frac{v}{2} + \frac{1}{4}\right)\pi$, in the case when N is sufficiently large this value is between j_N and j_{N+1}.

It is easy to prove that the function (2.4) is an odd function of z, therefore the integral on the left sides of rectangle is zero. If $z = u + iw$, then for large $|w|$ and for $u \geq 0$ integrand will have an order $O(e^{-2(1-x)|w|})$ and consequently for given value of A_N, integrals on upper and lower sides converge to zero as $B \to \infty$, when $0 < x < 1$.

Thus we obtain

$$T_N(x) = \frac{1}{2\pi i}\int_{A_N - iB}^{A_N + iB} \frac{zx\{J_v^z(xz) - J_{v+1}(xz)J_{v-1}(xz)\}}{J_v^2(z)}dz$$

When $j_N^{-1+\varepsilon} \leq x < 1$, where $0 < \varepsilon < 1/2, \quad |xz| \to \infty$, the Bessel functions in the integrant can be replaced by the corresponding asymptotes with large arguments

$$J_v^2(xz) - J_{v-1}(xz)J_{v+1}(xz) = \frac{2}{\pi xz}\left[1 - \frac{1}{2xz}\cos(2xz - v\pi)\right]\left(1 + 0\left(\frac{1}{(xz)^2}\right)\right)$$

$$J_v^2(z) = \frac{2}{\pi z}\left[\frac{1}{2} + \frac{\sin(2z - v\pi)}{2} + \frac{\cos(2z - v\pi)(2v-1)(2v+1)}{8z}\right]\left(1 + 0\left(\frac{1}{z^2}\right)\right)$$

Then if $N \to \infty$

$$
\begin{aligned}
T_N(x) &= \frac{1}{2\pi i} \lim_{B\to\infty} \int\limits_{A_N-iB}^{A_N+iB} \frac{xz\{J_v^2(xz) - J_{v+1}(xz)J_{v-1}(xz)\}}{J_v^2(xz)} dz \\
&= \frac{1}{2\pi i} \lim_{B\to\infty} \int\limits_{A_N-iB}^{A_N+iB} \frac{z[1 - \frac{1}{2xz}\cos(2xz - v\pi)(1 + 0\left(\frac{1}{(xz)^2}\right) dz}{} \\
&\left(\frac{1}{2} + \frac{\sin(2z - v\pi)}{2} + \frac{\cos(2z - v\pi)(2v-1)(2v+1)}{8z}\left[1 + 0\left(\frac{1}{z^2}\right)\right]\right) \\
&\approx \frac{1}{\pi i} \int\limits_{A_N-i\infty}^{A_N+i\infty} \frac{z\,[\,1 - \frac{1}{2xz}\cos(2xz - v\pi)\,]}{1 + \sin(2z - v\pi)} dz \\
&= \frac{1}{\pi i} \int\limits_{A_N-i\infty}^{A_N+i\infty} [\frac{z}{1 + \sin(2z - v\pi)} - \frac{1}{2x}\frac{\cos(2xz - v\pi)}{1 + \sin(2z - v\pi)}] dz \\
&= \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \frac{(A_N + iw)dv}{1 + \cos(2iw)} - \frac{1}{2x\pi} \int\limits_{-\infty}^{\infty} \frac{\cos(2xA_N + 2ixw - \nu\pi)}{1 + \cos(2iv)} dw \\
&= \frac{A_N}{\pi} - \frac{\cos(2xA_N - v\pi)}{2x\pi} \int\limits_{-\infty}^{\infty} \frac{\cosh(2xw)}{1 + \cosh(2w)} dw \\
&= \frac{A_N}{\pi} - \frac{\cos(2xA_N - v\pi)}{2\sin \pi x}
\end{aligned}
$$

That is we obtain

$$
T_N(x) = \frac{A_N}{\pi} - \frac{\cos(2xA_N - v\pi)}{2\sin \pi x} + \frac{\psi(A_N x)}{x},
$$

where $A_N = (N + \frac{v}{2} + \frac{1}{4})\pi$ and $\psi(A_N x) = O(\lim\limits_{B+\infty} \int\limits_{A_N-i\infty}^{A_N+i\infty} \frac{\sin(2xz-v\pi)}{xz(1+\sin(2z-v\pi))} dz)$
when $0 < x < 1$, $\quad \psi(A_N x) \to 0$ as $N \to \infty$.
Really, if we put $z = A_N + iw$, the integral

$$
\int\limits_{A_N-iB}^{A_N+iB} \frac{\sin(2z - v\pi)}{xz(1 + \sin(2z - v\pi))} dz \tag{2.5}
$$

can be rewritten as

$$2A_N \sin(2xA_N - v\pi) \int_0^\infty \frac{\cosh(2xw)}{x(A_N^2 + w^2)(1 + \cosh(2w))} dw$$
$$+2\cos(2xA_N - v\pi) \int_0^\infty \frac{w \sinh(2xw)dw}{x(A_N^2 + w^2)(1 + \cosh(2w))}$$

When $j_W^{-1+\varepsilon} \leq x < 1$, absolute value of (2.5) doesn't exceed

$$\frac{1}{xA_N} \int_0^\infty \frac{\cosh(2xv)}{1 + \cosh(2v)} dv + \frac{2}{xA_N^2} \int_0^\infty \frac{v \sinh(2xv)}{1 + \cosh(2v)} dv$$

This shows, that $\psi(A_N x) \to 0$ as $N \to \infty$.

If $x = 1$

$$\lim_{B\to\infty} \int_{A_N - iB}^{A_N + iB} \frac{\sin(2z - v\pi)}{z(1 + \sin(2z - v\pi))} dz = \lim_{B\to\infty} \int_{-B}^{B} \frac{\cosh(2w)dw}{(A_N + iw)(1 + \cosh(2w))}$$

$$= \lim_{B\to\infty} \left[2A_N \int_0^B \frac{\cosh(2w)}{(A_N^2 + w^2)(1 + \cosh(2w))} dw - i \int_{-B}^{B} \frac{w \cosh(2w)dw}{(A_N^2 + w^2)(1 + \cosh(2w))} \right]$$

$$= \lim_{B\to\infty} 2A_N \int_0^B \frac{\cosh(2w)}{(A_N^2 + w^2)(1 + \cosh(2w))} dw \quad .$$

Absolute value of this integral doesn't exceed $2A_N \int_0^\infty \frac{dw}{A_N^2 + w^2} = \pi$.

We obtain that $\psi(A_N x) \to 0$ as $N \to \infty$ when $j_N^{-1+\varepsilon} < x < 1$ and $|\psi(A_N x)| < c$ when $x = 1$.

Proof of Theorem. Now using the asymtotes of the function $T_N(x) = \sum_{n=1}^{N} 2x \frac{J_v^2(j_n x)}{J_{v+1}^2(j_n)}$, we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}(\lambda_n-\mu_n) &= \lim_{N\to\infty}\sum_{n=1}^{N}\int_0^1 \tfrac{2xJ_v^2(j_nx)}{J_{v+1}^2(j_n)}q(x)dx = \lim_{N\to\infty}\int_0^1 T_N(x)q(x)dx \\
&= \lim_{N\to\infty}\left[\int_0^1 T_N(x)(q(x)-q(0))dx+q(0)N\right] \\
&= \lim_{N\to\infty}\int_0^{A_N^{-1+\varepsilon}} T_N(x)(q(x)-q(0))dx \\
&+ \int_{A_N^{-1+\varepsilon}}^1 \left(\tfrac{A_N}{\pi}-\tfrac{\cos(2A_Nx-\nu\pi)}{2\sin\pi x}-\tfrac{\psi(A_Nx)}{x}\right)(q(x)-q(0)dx+q(0)N\,] \\
&= \lim_{N\to\infty}\left[\int_0^{A_N^{-1+\varepsilon}} T_N(x)(q(x)-q(0))dx\right. \\
&- \int_0^{A_N^{-1+\varepsilon}} (\tfrac{A_N}{\pi}-\tfrac{\cos(2A_Nx-\nu\pi)}{2\sin\pi x})(q(x)-q(0))dx \\
&+\int_0^1(\tfrac{A_N}{\pi}-\tfrac{\cos(2A_Nx-\nu\pi)}{2\sin\pi x})(q(x)-q(0))dx+q(0)N \\
&\left.+\int_{A_N^{-1+\varepsilon}}^1 \tfrac{\psi(A_Nx)}{x}(q(x)-q(0))dx\right] \\
&= \lim_{N\to\infty}\left[\int_0^{A_N^{-1+\varepsilon}} T_N(x)(q(x)-q(0))dx+\right.
\end{aligned}
$$

when $0<\varepsilon<\frac{\alpha}{\alpha+1}$.

Using the inequality $|\sqrt{x}J_v(x)|<const$, for any x [17], it is easy to prove that $T_N(x)\leq$const N (3)

Take into account (3) and condition a), one can easily see that

$$
\begin{aligned}
\lim_{N\to\infty}\left|\int_0^{A_N^{-1+\varepsilon}} T_N(x)(q(x)-q(0)dx\right| &\leq \lim_{N\to\infty}\int_0^{A_N^{-1+\varepsilon}} T_N(x)\left|q(x)-q(0)\right|dx \\
&\leq \lim_{N\to\infty} const\int_0^{A_N} Nx^{\alpha}dx \\
&= const\lim_{N\to\infty} NA_N^{-1-\alpha+\varepsilon(\alpha+1)} \\
&= 0
\end{aligned}
$$

for $0<\varepsilon<\frac{\alpha}{\alpha+1}$.

Moreover if is easy to prove that

$$
\lim_{N\to\infty}\int_0^{A_N^{-1+\varepsilon}}\left(\frac{A_N}{\pi}-\frac{\cos(2A_Nx-v\pi)}{2\sin\pi x}\right)(q(x)-q(0))dx=0 \qquad (2.6)
$$

$$\lim_{N\to\infty}\int\limits_{A_N^{-1+\varepsilon}}^{1}\frac{\psi(A_Nx)}{x}(q(x)-q(0)dx=0 \tag{2.7}$$

Taking into account (3) – (2.7) and condition b), we obtain

$$\lim_{N\to\infty}\left[\int\limits_0^1\left(\frac{A_N}{\pi}-\frac{\cos(2A_Nx-v\pi)}{2\sin\pi x}\right)(q(x)-q(0))dx+q(0)N\right]$$

$$=\lim_{N\to\infty}\frac{A_N}{\pi}\int\limits_0^1 q(x)dx--\frac{A_N}{\pi}q(0)+Nq(0)$$

$$-\frac{1}{4}\int\limits_0^1\frac{\cos(2N+\frac{1}{2})}{\cos\frac{\pi}{2}x}\frac{\cos(v\pi(x-1))}{\sin\frac{\pi}{2}x}(q(x)-q(0))dx$$

$$=-\left(\frac{v}{2}+\frac{1}{4}\right)q(0)-\frac{q(1)-q(0)}{4}=-\frac{2vq(0)+q(1)}{4}$$

That is,

$$\sum_{n=1}^{\infty}(\lambda_n-\mu_n)=-\frac{2vq(0)+q(1)}{4} \quad .$$

This completes the proof of the theorem.

ÖZET:

$\mu_1\le\mu_2\le...\le\mu_n\le...$ ve $\lambda_1\le\lambda_2\le... \quad \le\lambda_n\le$ ...sayıları

$\ell_o[y]=-y''+\frac{v^2-1/4}{x^2}y,\quad \ell[y]=-y''+\frac{v^2-1/4}{x^2}y+q(x)y,\quad \nu\ge 1/2$

diferansiyel ifadeleri ve $y(0)=y(1)=0$ sınır şartları ile tanımlanmış sırasıyla L_0 ve L_1 operatörlerinin özdeğerleri olsun. Makalede $q(x)$ fonkisyonu bazı şartları sağladığında

$\sum\limits_{n=1}^{\infty}(\lambda_n-\mu_n)=-\frac{2\nu q(0)+q(1)}{4}$ iz formülü ıspatlanmıştır.

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