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OPTION PRICING WITH PADÉ APPROXIMATIONS

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Abstract. In this paper, Padé approximations are applied Black-Scholes model which reduces to heat equation. This paper shows various Padé approximaitons to obtain an effective and accurate solution to the Black-Scholes equation for a European put/call option pricing problem. At the end of the paper, results of closed-form solution of Black-Scholes problem , solution of Crank-Nicolson approach and the solution of $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$ Padé approximations are given at a table.

1. Introduction

In recent years, studies of solutions of Black-Scholes partial differential equations have increased. Although in 1970's Merton [1,2] and Black and Scholes [3] has formulated Black-Scholes model according to stochastic differential equations, nowadays the model has been solved both stochastic and numerical solutions. Especially, in books of Seydel [6], Ugur [7] and Brandimarte [5] the results of examples solved by applying finite differences. In this paper, we will give a new approach for solving the Black-Scholes model to reduced heat equation. Firstly, as implementing the finite difference algorithms to the diffusion equation, the equation will be transformed the system of ordinary differential equation. Then the system will be solved with Padé approximations ($(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$) and the results obtained will be compared with results of Crank-Nicolson solution, Closed-Form solution and Matlab solution.

2. Padé Approximations

Black-Scholes equation for European option V(S,t):

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\delta)S\frac{\partial V}{\partial S}-rV=0 \tag{2.1}$$

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is parabolic partial differential equation in domain
$D_V = \{(S,t) : S > 0, 0 \leq t \leq T\}$. Black-Scholes equation with appropriate variable transformation is equating to heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{2.2}$$

Thus, domain of Black-Scholes equation is change with domain
$D_u = \{(x,\tau) : -\infty < x < \infty, 0 \leq \tau \leq \frac{\sigma^2}{2}\}$ [6]. If x derivative of eq.(2.2) is changed with following finite difference formula
$\frac{1}{h^2}\{u(x-h,\tau) - 2u(x,\tau) + u(x+h,\tau)\} + O(h^2)$, then eq.(2.2) can be written as

$$\frac{du(\tau)}{d\tau} = \frac{1}{h^2}\{u(x-h,\tau) - 2u(x,\tau) + u(x+h,\tau)\} + O(h^2) \tag{2.3}$$

Heat equation can be reduced ordinary differential equation system as form:

$$\frac{d}{dt}\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-2} \\ V_{N-1} \end{pmatrix} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-2} \\ V_{N-1} \end{pmatrix} + \frac{1}{h^2}\begin{pmatrix} V_0 \\ 0 \\ \vdots \\ 0 \\ V_N \end{pmatrix} \tag{2.4}$$

That is, above matrix form can be shown as

$$\frac{\mathbf{dV}(\tau)}{d\tau} = \mathbf{AV}(\tau) + \mathbf{b} \tag{2.5}$$

where $V(\tau) = [V_1, V_2, \ldots, V_{N-1}]^T$ is approximation of u, b is a column vector which has zeros and known boundary values and

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}$$

is $(N-1)$ order matrix. Solution of ordinary differential equation $\frac{dV}{d\tau} = AV + b$ such that $V(0) = [g_1, g_2, \ldots, g_{N-1}]^T = g$ initial condition is

$$V(\tau) = -A^{-1}b + exp(\tau A)(g + A^{-1}b) \tag{2.6}$$

.

In step $(\tau + k)$, eq.(2.6) can be written as

$$V(\tau + k) = -A^{-1}b + exp(kA)(V(\tau + A^{-1}b)$$

[4].

In this paper, we have made approximation to $exp(kA)$ with Padé approximations.

$(1,1)$ Padé approximation as matrix form:

$$(I-\frac{1}{2}kA)V(\tau+k)=(I+\frac{1}{2}kA)V(\tau)+kb$$

$(1,2)$ Padé approximation as matrix form:

$$(I-\frac{1}{3}kA)V(\tau+k)=(I+\frac{2}{3}kA+\frac{1}{6}k^2A^2)V(\tau)+(I+\frac{1}{6}kA)b$$

$(2,0)$ Padé approximation as matrix form:

$$(I-kA+\frac{1}{2}k^2A^2)(V(\tau+k)=V(\tau)+(kb-\frac{1}{2}k^2Ab)$$

$(2,1)$ Padé approximation as matrix form:

$$(I-\frac{2}{3}kA+\frac{1}{6}k^2A^2)V(\tau+k)=(I+\frac{1}{3}kA)V(\tau)+(I-\frac{1}{6}kA)kb$$

$(2,2)$ Padé approximation as matrix form:

$$(I-\frac{1}{2}kA+\frac{1}{12}k^2A^2)V(\tau+k)=(I+\frac{1}{2}kA+\frac{1}{12}k^2A^2)V(\tau)+kb$$

.

Padé approximations given above form following systems of linear equations:

$$CV^{(j+1)}=BV^{(j)}+b^{(j)} \tag{2.7}$$

The matrix C can be written as $LU-$decomposition $C=LU$, where L is a lower and U is an upper triangular matrix. The solution to the system of linear equations (2.7) can be written,

$$V^{(j+1)}=U^{-1}L^{-1}(BV^j+b^j)$$

.

For each Padé approximations, the solution of Black-Scholes Model reduced to heat equation is given. The results can be seen at Table 1. Also, in this table the results of Crank-Nicolson solution of Black-Scholes Model reduced to heat equation is illustrated. For put option and call option, it has been taken $S0=10$, $K=10$, $r=0.25$, $sigma=0.6$, $div=0.2$ and maturity time $T=1$. Values at Table 1 and Table 2 has been found respectively put and call options.

S	C-N	(1,1)	(1,2)	(2,0)
10	1.688723	1.688723	-85914525509.448166	1.688514
20	0.332834	0.332834	-19140576.582597	0.333149
30	0.084809	0.084809	0.248896	0.085082
40	0.024170	0.024170	0.024057	0.024190
50	0.009358	0.009358	0.009330	0.009283
60	0.003313	0.003313	0.003278	0.003216
70	0.001435	0.001435	0.001407	0.001351
80	0.000592	0.000592	0.000572	0.000530
90	0.000320	0.000320	0.000305	0.000273
100	0.000169	0.000169	0.000159	0.000135

S	(2,1)	(2,2)	Closed-Form Sol.	Matlab Sol.
10	1.688828	1.688824	1.593673	1.690363
20	0.332945	0.332948	0.284594	0.34044
30	0.084902	0.084902	0.066802	0.08533
40	0.024174	0.024173	0.017639	0.02559
50	0.009330	0.009329	0.006431	0.008799
60	0.003280	0.003279	0.002123	0.003366
70	0.001407	0.001407	0.000866	0.001401
80	0.000572	0.000572	0.000333	0.000625
90	0.000304	0.000305	0.000171	0.000295
100	0.000158	0.000158	0.000085	0.000147

Table 1.

S	C-N	(1,1)	(1,2)	(2,0)
10	2.088070	2.088070	-85914525509.0493	2.087858
20	8.983799	8.983799	19140567.883981	9.032449
30	16.892984	16.892984	17.055754	16.893249
40	25.437537	25.437537	25.439717	25.437547
50	32.773602	32.773602	32.776947	32.773515
60	41.745939	41.745939	41.751329	41.745827
70	49.759815	49.759814	49.766811	49.759713
80	55.811386	55.811386	55.822353	55.811297
90	66.103337	66.103334	66.099704	66.103267
100	73.874374	73.874366	73.874562	73.874313

S	(2,1)	(2,2)	Closed-Form Sol.	Matlab Sol.
10	2.088174	2.088170	1.992973	2.0897
20	9.032248	9.032251	8.983799	8.9270
30	16.893074	16.893074	16.874825	16.8592
40	25.437538	25.437536	25.430801	24.9868
50	32.773571	32.773569	32.770423	33.1573
60	41.745900	41.745900	41.744440	41.3392
70	49.759780	49.759780	49.758884	49.5245
80	55.811356	55.811356	55.810648	57.7111
90	66.103312	66.103312	66.102724	65.8981
100	73.874349	73.874349	73.873777	74.0852

Table 2.

3. **Conclusion**

In this study Black-Scholes equation for European put/ call options model solved by using Padé approximations which applied to heat equation which is the classical reduced form of Black-Scholes model. Tables 1 and 2 show various estimations of Padé approximations along with Crank-Nicholson (C-N), closed form solutions and Matlab solution. The maturity times shown in Tables 1 and 2 illustrate how the closed form (exact) solutions as well as the approximate solutions obtained via Crank-Nicholson solution, various Padé approximations and Matlab solution behave. Although the discretizations used for spatial variables are uniform, the discretization for asset price is non-uniform, this generate the slight differences on the maturity times. Although, these slight differences insignificant, I think that it's due to the transformation $S = e^x$. Hence, a shortcoming of these transformations perhaps that the resulting grid is not uniform for the asset price but it is inevitable in our case. Of course, one can get a uniform grid in the asset price by choosing constant step size in the asset price. However, the construction of non-uniform grids for the finite difference methods for the heat equation may not be as easy as the one over a uniform grid. Another difficulty is that after solving the equation numerically, a back transformation must be used to interpret the solution in terms of option values which is also a tricky task which we will confront, but this is our prospect study in the future.

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Özet: Bu makalede Padé yaklaşımları ısı denklemine indirgenen Black-Scholes modeline uygulanıyor. Makale Avrupa put ve call opsiyon problemi için Black-Scholes denkleminin etkili ve doğru çözümünü elde etmek için çeşitli Padéyaklaşımlarını gösteriyor. Makalenin sonunda tablo halinde Black-Scholes probleminin kapalı-çözümü, Crank-Nicolson çözümü ve $(1,1)$, $(1,2)$,$(2,0)$, $(2,1)$, $(2,2)$ Padé yaklaşımlarının çözümleri verilmektedir.

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