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PROPERTIES OF Δ^*-CLOSED MAPS IN TOPOLOGICAL SPACES

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Abstract. This paper is concerned with the introduction of a new notion of closed maps namely, Δ^*-closed maps using Δ^*-closed sets and the analysis of their significant properties in topological spaces. Also the nature of Δ^*-closed maps under composition mappings and their applications are explored in this paper.

1. Introduction and Notation

The concept of closed maps plays a vital role in the development of the nature of topological spaces. The notion of δ-closed functions was introduced by T.Noiri [7] in the year 1978. The idea of generalised closed functions was initiated and investigated by S.R. Malghan [2] in 1982. Julian Dontchev [1] introduced δg-closed maps in 1996. Since then several types of closed functions were studied by many authors. In the year 2013, R.Sudha [8] described δg^*-closed maps. In this article a new class of closed maps called, Δ^*-closed maps via Δ^*-closed sets are established and their significant characterizations, behaviour under composition mappings and applications are exhibited. In this paper (X, τ) and (Y, τ) denote non empty topological spaces with no separation axioms are imposed on them if it is not stated specifically.

Remark: In 2014, a new class of closed sets namely, Δ^*-closed sets [6] were introduced and initially denoted by $\boldsymbol{\delta}(\boldsymbol{\delta}\mathbf{g})^*$-closed sets by the authors.

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2. Preliminaries

Definition 1. *A subset A of a topological space (X,τ) is called a Δ^***-closed set** [3] if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$, U is δg-open in (X,τ). The class of all Δ^*-closed sets of (X,τ) is denoted by $\Delta^*C(X,\tau)$.*
*The complement of a Δ^***-closed set** is called a Δ^***-open set**.*

Definition 2. *A space (X,τ) is said to be a ${}_{\Delta^*}T_\delta$**-space** [4] if every Δ^*-closed subset of (X,τ) is δ-closed in (X,τ).*

Definition 3. *A space (X,τ) is said to be a ${}_{\Delta^*}T_{\delta g^*}$**-space** [4] if every Δ^*-closed subset of (X,τ) is δg^*-closed in (X,τ).*

Definition 4. *A space (X,τ) is said to be a ${}_{\delta g^*}T_\delta$**-space** [8] if every δg^*-closed subset of (X,τ) is δ-closed in (X,τ).*

Definition 5. *A mapping $f:(X,\tau)\to(Y,\sigma)$ is said to be Δ^***-continuous** [5] if the inverse image of every closed set in (Y,σ) is Δ^*-closed in (X,τ).*

Definition 6. *A map $f:(X,\tau)\to(Y,\sigma)$ is called a Δ^***-irresolute map** [6] if f(v) is a Δ^*-open set in (X,τ) for every Δ^*-open set V in (Y,σ).*

Definition 7. *A map $f:(X,\tau)\to(Y,\sigma)$ is called a δg**-irresolute map** [1] if $f(v)$ is a δg-open set in (X,τ) for every δg-open set in (Y,σ).*

Definition 8. *A mapping $f:(X,\tau)\to(Y,\sigma)$ is called a δ**-closed map** [7] if the image each closed set in (X,τ) is a δ-closed set in (Y,σ).*

Definition 9. *A map $f:(X,\tau)\to(Y,\sigma)$ is called a δg^***-closed map** [1] if the image of each closed set in (X,τ) is a δg^*-closed set in (Y,σ).*

3. Properties of Δ^*-Closed Maps

Definition 10. *A map $f:(X,\tau)\to(Y,\sigma)$ is called a Δ^***-closed map** if the image of each closed set in (X,τ) is a Δ^*-closed set in (Y,σ).*

Proposition 1. *Every δ-closed map is a Δ^*-closed map but not conversely.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a δ-closed map. Let V be a closed set in (X, τ). Then its image $f(V)$ is δ-closed in (Y, σ). Since every δ-closed set is Δ^*-closed [3], $f(V)$ is Δ^*-closed in (Y, σ). Hence f is a Δ^*-closed map.

□

Counter example 1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = a, f(b) = c, f(c) = b$ where $X = \{a, b, c\} = Y$, $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}\}$. Then f is a Δ^*-closed map but not a δ-closed map as the image of the closed set {b,c} in (X, τ) is not a δ-closed set in (Y, σ).*

Remark 1. *The following counter examples show that the Δ^*-closed map is independent from a δg-closed map.*

Counter example 2. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = c, f(b) = b, f(c) = a$. Then f is a Δ^*-closed map but not a δg-closed map as the image of the closed set {b,c} in (X, τ), $f[\{b, c\}] = \{a, b\}$ is not δg-closed in (Y, σ).*

Counter example 3. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = c, f(b) = c, f(c) = a$. Then f is a δg-closed map but not a Δ^*-closed map since for the closed set {b,c} in (X, τ), $f[\{b, c\}] = \{a, b\}$ is not Δ^*-closed in (Y, σ).*

Remark 2. *The Δ^*-closed map and Δ^*-continuity are independent as shown by the following examples.*

Counter example 4. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = c, f(b) = a, f(c) = c$. Then f is Δ^*-continuous but not a Δ^*-closed map since for the closed set {b,c} in (X, τ), $f[\{b, c\}] = \{a, c\}$ is not Δ^*-closed in (Y, σ).*

Counter example 5. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = a, f(b) = c, f(c) = b$. Then f is a Δ^*-closed map but not Δ^*-continuous since for the closed set {c} in (Y, σ), $f^{-1}\{c\}$={b} is not Δ^*-closed in (X, τ).*

Theorem 11. *A map $f : (X, \tau) \to (Y, \sigma)$ is Δ^*-closed if and only if for each subset U of (Y, σ) and for each open set V of (X, τ) containing $f^{-1}(U)$ there exists a Δ^*-open set G of (Y, σ) such that $U \subseteq G$ and $f^{-1}(G) \subseteq V$.*

Proof. **(Necessity):** Suppose that $f:(X,\tau)\to(Y,\sigma)$ is a Δ^*-closed map and U be a subset of (Y,σ). Let V be an open subset of (X,τ) containing $f^{-1}(U)$. Then $(X-V)$ is closed in (X,τ). Since f is Δ^*-closed, $f(X-V)$ is Δ^*-closed in (Y,σ). Hence $Y-f(X-V)$ is a Δ^*-open set in (Y,σ). Take $G=Y-f(X-V)$. Then G is Δ^*-open in (Y,σ) containing U such that $f^{-1}(G)\subseteq V$.
(Sufficiency): Let H be a closed subset of (X,τ). Then $f^{-1}[Y-f(H)]\subseteq(X-H)$ and X - H is open. By hypothesis there is a Δ^*-open set G of (Y,σ) such that $Y-f(H)\subseteq G$ and $f^{-1}(G)\subseteq X-H$. Therefore $H\subseteq X-f^{-1}(G)$. Hence $Y-G\subseteq f(H)\subseteq f[X-f^{-1}(G)]\subseteq Y-G$ which implies that $f(H)=Y-H$ and $f(H)$ is Δ^*-closed in (Y,σ). Thus f is a Δ^*-closed map. □

Theorem 12. *A bijection mapping $f:(X,\tau)\to(Y,\sigma)$ is a Δ^*-closed map if and only if $f(U)$ is a Δ^*-open set in (Y,σ) for every open set U in (X,τ).*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a Δ^*-closed map and U be any open set in (X,τ). Then U^c is a closed set in (X,τ). Therefore by the hypothesis,$f(U^c)$ is Δ^*-closed in (Y,σ). Since f is bijective, $f(U^c)=[f(U)]^c$ is Δ^*-closed in (Y,σ). Hence $f(U)$ is Δ^*-open in (Y,σ). Conversely, let U be a closed subset of (X,τ). Then U^c is a open set in (X,τ). By the hypothesis, $f(U)^c$ is Δ^*-open in (Y,σ). Since f is a bijection map, $f(U)^c=[f(U)]^c$. Thus $f(U)$ is Δ^*-closed in (Y,σ). Hence f is a Δ^*-closed map.

□

Remark 3. *In the above proposition, bijection condition on f is necessary which is proved in the following example.*

Example 1. *Let $X=\{a,b,c\}=Y$ with $\tau=\{\phi,X,\{a\}\}$ and $\sigma=\{\phi,Y,\{a\},\{a,b\}\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be a map such that $f(a)=b,f(b)=a,f(c)=a$. Then for the only open set $\{a\}$, $f\{a\}$ is Δ^*-open but not Δ^*-closed as for the closed set $\{b,c\}$ in (X,τ), $f(\{b,c\})=\{a\}$ is not Δ^*-closed in (Y,σ).*

Proposition 2. *If $f:(X,\tau)\to(Y,\sigma)$ is δg-irresolute and Δ^*-closed map then $f(A)$ is Δ^*-closed subset of (Y,σ) where A is a Δ^*-closed subset of (X,τ).*

Proof. Let U be a δg-open set in (Y,σ) such that $f(A)\subseteq U$. Since f is δg-irresolute, $f^{-1}(U)$ is a δg-open set containing A. That is $A\subseteq f^{-1}(U)$. Hence $\delta cl(A)\subseteq f^{-1}(U)$. since every δ-closed set is closed [7], $\delta cl(A)$ is closed. Since f is a Δ^*-closed map, $f(\delta cl(A))$ is Δ^*-closed contained in the δg-open set U which implies that $\delta cl[f(\delta cl(A)]\subseteq U$ and hence $\delta cl[f(A)]\subseteq U$. Thus $f(A)$ is a Δ^*-closed subset of (Y,σ).

□

4. Composition of Δ^*-Closed Maps

Proposition 3. *The composition mapping $(g \circ f) : (X, \tau) \to (Z, \eta)$ of a closed map $f : (X, \tau) \to (Y, \sigma)$ and a Δ^*-closed map $g : (Y, \sigma) \to (Z, \eta)$ is a Δ^*-closed map.*

Proof. The image $f(U)$ of any closed subset U of X under the closed map $f : (X, \tau) \to (Y, \sigma)$ is closed in (Y, σ). Since $g : (Y, \sigma) \to (Z, \eta)$ is a Δ^*-closed map, $g[f(U)]$ is Δ^*-closed in (Z, η) and hence $(g \circ f)$ is a Δ^*-closed map.

□

Remark 4. *The following example shows that the composition of a Δ^*-closed map and a closed map is need not be a Δ^*-closed map.*

Counter example 6. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c\}\}$. Consider the Δ^*-closed map $f : (X, \tau) \to (Y, \sigma)$ defined as $f(a) = a, f(b) = a, f(c) = c$ and a closed map $g : (Y, \sigma) \to (Z, \eta)$ defined as $g(a) = c, g(b) = b, g(c) = a$. Then the composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ is not a Δ^*-closed map as the image of the closed set $\{b, c\}$ in (X, τ) is not Δ^*-closed in (Z, η).*

Theorem 13. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two maps.*
i) If $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-closed map and g is a Δ^*-irresolute injective map then f is a Δ^*-closed map.
ii) If $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-irresolute map and g is a Δ^*-closed injective map then f is a Δ^*-continuous map.

Proof. **i)** Let U be any closed set in (X, τ). Since $(g \circ f)$ is Δ^*-closed, $(g \circ f)(U)$ is Δ^*closed in (Z, η). Therefore $g[f(U)]$ is Δ^*-closed in (Z, η). Since g is Δ^*-irresolute, $g^{-1}[g(f(U)]$ is Δ^*-closed in (Y, σ). Since g is injective, $g^{-1}[g(f(U)] = f(U)$ is Δ^*-closed in (Y, σ). Hence f is a Δ^*-closed map.
ii) Let V be a closed set in (Y, σ). Since g is Δ^*-closed, $g(V)$ is Δ^*-closed in (Z, η). Since $(g \circ f)$ is Δ^*-irresolute, $(g \circ f)^{-1}[g(V)]$ is Δ^*-closed in (X, τ). Therefore $f^{-1}((g^{-1}[g(V)])$ is Δ^*-closed in (X, τ). Since g is injective, $g^{-1}[g(V)] = V$ and hence $g^{-1}(V)$ is Δ^*-closed (X, τ). Thus f is a Δ^*-continuous map.

□

Proposition 4. *The composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ of the Δ^*-closed maps $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ is a Δ^*-closed map if (Y, σ) is a ${}_{\Delta^*}T_{\delta g^*}$-space.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a Δ^*-closed map. Then $f(A)$ is a Δ^*-closed in (Y, σ) and hence δ-closed in (Y, σ) as (Y, σ) is a ${}_{\Delta^*}T_{\delta g^*}$-space. Since every δ-closed set is closed [7], $f(A)$ becomes closed in (Y, σ). Thus $g[f(A)]=(g \circ f)(A)$ is Δ^*-closed in (Z, η) as g is a Δ^*-closed map. Hence the composition mapping $(g \circ f)$ is a Δ^*-closed map.

□

Proposition 5. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be Δ^*-closed maps. If (Y, σ) is a ${}_{\Delta^*}T_{\delta g^*}$-space and ${}_{\delta g^*}T_{\delta}$-space then their composition $(g \circ f) : (X, \tau) \to (Z, \eta)$ is a Δ^*-closed map.*

Proof. Let A be a closed set in (X, τ). Then $f(A)$ is Δ^*-closed in (Y, σ). Since (Y, σ) is a ${}_{\Delta^*}T_{\delta g^*}$-space and ${}_{\delta g^*}T_{\delta}$-space, $f(A)$ is δg^*-closed and hence it is δ-closed in (Y, σ). Since every δ-closed set is closed [7], $f(A)$ is closed in (Y, σ). Since g is a Δ^*-closed map, $g[f(A)] = (g \circ f)(A)$ is Δ^*-closed in (Z, η). Hence the composition map $(g \circ f)$ is Δ^*-closed.

□

Remark 5. *The composition of two Δ^*-closed maps need not be a Δ^*-closed map as seen from the following examples.*

Counter example 7. *Let $X = \{a, b, c\} = Y$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$, $\sigma = \{\phi, Y, \{a, b\}\}$ and $\eta = \{\phi, Z, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $f(a) = a, f(b) = a, f(c) = c$. Let $g : (Y, \sigma) \to (Z, \eta)$ be a map such that $g(a) = c, g(b) = b, g(c) = a$. Then both f and g are Δ^*-closed maps. But their composition map $(g \circ f) : (X, \tau) \to (Z, \eta)$ is not a Δ^*-closed map since for the closed set $\{b, c\}$ in (X, τ), $(g \circ f)[\{b, c\}] = \{a, c\}$ is not Δ^*-closed in (Z, η).*

5. Conclusion

The properties of newly defined Δ^*-closed maps are analysed in this paper. Also it is shown that Composition two Δ^*-closed maps is not a Δ^*-closed map. In continuation of this work we have extended this concept to Δ^*-Homeomorphisms in topological spaces.

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