CONSTRUCTIVE MATHEMATICAL ANALYSIS
2 (2019), No. 3, pp. 130-143
http://dergipark.gov.tr/cma
ISSN 2651-2939

A Sequence of Kantorovich-Type Operators on Mobile Intervals

MIRELLA CAPPELLETTI MONTANO AND VITA LEONESSA*

ABSTRACT. In this paper, we introduce and study a new sequence of positive linear operators, acting on both spaces of continuous functions as well as spaces of integrable functions on $[0,1]$. We state some qualitative properties of this sequence and we prove that it is an approximation process both in $C([0,1])$ and in $L^p([0,1])$, also providing some estimates of the rate of convergence. Moreover, we determine an asymptotic formula and, as an application, we prove that certain iterates of the operators converge, both in $C([0,1])$ and, in some cases, in $L^p([0,1])$, to a limit semigroup. Finally, we show that our operators, under suitable hypotheses, perform better than other existing ones in the literature.

Keywords: Kantorovich-type operators, Positive approximation processes, Rate of convergence, Asymptotic formula, Generalized convexity.

2010 Mathematics Subject Classification: 41A36, 41A25, 47D06.

1. INTRODUCTION

In [13], the author proposed a modification of the classical Bernstein operators B_n on $[0,1]$ that, instead of fixing constants and the function x, fixes the constants and x^2, obtaining, in such a way, an order of approximation at least as good as the order of approximation of the operators B_n in the interval $[0,1/3[$. More precisely, those operators are defined by setting, for every continuous function on $[0,1]$, $\widetilde{B}_n(f) = B_n(f) \circ r_n$, where, for every $x \in [0,1]$,

$$r_n(x) = \begin{cases} x^2 & \text{if } n = 1\,, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{nx^2}{n-1} + \dfrac{1}{4(n-1)^2}} & \text{if } n \geq 2\,. \end{cases}$$

Subsequently, other modifications of the classical Bernstein operators, as well as of many other well-known operators, that fix suitable functions were introduced (see [2] and the references quoted therein). Here we limit ourselves to mention that, for example, in [9], the authors considered a family of sequences of operators $(B_{n,\alpha})_{n\geq 1}$, $\alpha \geq 0$, that preserve the constants and the function $x^2 + \alpha x$. A further extension was presented in [12]; in that paper, Gonska, Raşa and Piţul considered the operators $V_n^\tau(f) = B_n(f) \circ \tau_n$ $(f \in C([0,1]))$, where $\tau_n = (B_n(\tau))^{-1} \circ \tau$ and τ is a strictly increasing function on $[0,1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$. In particular, the operators V_n^τ preserve the constants and the function τ.
In [10], instead, the authors introduced a modification of Bernstein operators fixing constants and a strictly increasing function τ in the following way: considering a strictly increasing function τ which is infinitely many times continuously differentiable on [0, 1] and such that $\tau(0) = 0$

Received: 28 May 2019; Accepted: 29 July 2019; Published Online: 30 July 2019
*Corresponding author: Vita Leonessa; vita.leonessa@unibas.it
DOI: 10.33205/cma.571078

and $\tau(1) = 1$, they introduced the operators

$$B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1, f \in C([0,1])).$$

The authors studied shape preserving and approximation properties of the operators B_n^τ, and compared them, under suitable assumptions, with the B_n's and the $V_n^t au's$. General sequences of positive linear operators fixing τ and τ^2 have been recently studied in [1].
In this paper, motivated by works [7], [4] and [5], we present a Kantorovich-type modification of the operators B_n^τ. In particular, in [7], among other things, the authors introduced a sequence of positive linear operators $(C_n)_{n\geq 1}$ that generalize the classical Kantorovich operators on $[0,1]$ and present the advantage to reconstruct any integrable function on $[0,1]$ by means of its mean value on a finite numbers of subintervals of $[0,1]$ that do not need to be a partition of $[0,1]$.
Accordingly, in this work, for any integrable function f on $[0,1]$ we shall study the operators

$$C_n^\tau(f) = C_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1),$$

where τ is a strictly increasing function that is infinitely many times continuously differentiable on [0, 1] and such that $\tau(0) = 0$ and $\tau(1) = 1$.
The paper is organized as follows; after giving some preliminaries, we discuss some qualitative properties of the operators C_n^τ; in particular, we prove that they preserve some generalized convexity. We also prove that the sequence $(C_n^\tau)_{n\geq 1}$ is an approximation process for spaces of continuous as well as integrable functions and we evaluate the rate of convergence in both cases by means of suitable moduli of smoothness. As a byproduct, we obtain a simultaneous approximation result for the operators B_n^τ.
By using some results of [5], we prove that the operators C_n^τ satisfy an asymptotic formula with respect to a second order elliptic differential operator and, as an application, that suitable iterates of the C_n^τ's can be employed in order to constructively approximate strongly continuous semigroups in the function spaces considered in the paper.
Finally, as a further consequence of the above mentioned asymptotic formula, we compare the sequence $(C_n^\tau)_{n\geq 1}$ and the sequence $(C_n)_{n\geq 1}$, showing that, under suitable conditions, the former perform better.

2. Preliminaries

From now on, we denote by $C([0,1])$ the space of all real-valued continuous functions on the interval $[0,1]$. As usual, $C([0,1])$ will be equipped with the uniform norm $\|\cdot\|_\infty$.
For every $i \geq 1$, the symbol e_i stands for the functions $e_i(x) := x^i$ for all $x \in [0,1]$; moreover $\mathbf{1}$ will indicate the constant function on $[0,1]$ of constant value 1. If $X \subset \mathbb{R}$, we denote by $\mathbf{1}_X$ the characteristic function of X, defined by setting, for every $x \in \mathbb{R}$,

$$\mathbf{1}_X(x) := \begin{cases} 1 & \text{if } x \in X; \\ 0 & \text{if } x \notin X. \end{cases}$$

Moreover, for every $k \in \mathbb{N}$, we denote by $C^k([0,1])$ the space consisting of all real-valued functions which are continuously differentiable up to order k on $[0,1]$. In particular, if $f \in C^k([0,1])$, for every $i = 0,\ldots,k$, $D^{(i)}(f)$ is the derivative of order i of f. For simplicity, if $i = 1,2$, we might also use the usual symbols f' and f''. Further, $C^\infty([0,1])$ is the space of all real-valued functions which are infinitely many times continuously differentiable on $[0,1]$.
Finally, for every $p \in [1,+\infty[$, we denote by $L^p([0,1])$ the space of all (the equivalence classes of) Borel measurable real-valued functions on $[0,1]$ whose p^{th} power is integrable with respect

to the Borel-Lebesgue measure λ_1 on [0, 1]. The space $L^p([0,1])$ is endowed with the norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p} \qquad (f \in L^p([0,1])).$$

In what follows we recall the definition of certain operators acting on the space $L^1([0,1])$ which represent a generalization of the classical Kantorovich operators on $[0,1]$. They were studied in [7, Examples 1.2, 1] and subsequently extended to the multidimensional setting in [4, 5].
Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers such that, for every $n \geq 1$, $0 \leq a_n < b_n \leq 1$. Then, consider the positive linear operator $C_n : L^1([0,1]) \longrightarrow C([0,1])$ defined by setting, for any $f \in L^1([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$C_n(f)(x) = \sum_{k=0}^{n} \left(\frac{n+1}{b_n - a_n}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(t)\,dt\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{2.1}$$

Since $C_n(\mathbf{1}) = \mathbf{1}$, the restriction to $C([0,1])$ of each C_n is continuous and we have $||C_n|| = 1$ for any $n \geq 1$, where $\|\cdot\|$ denotes the usual operator norm on $C([0,1])$.
We notice that if, in particular, $a_n = 0$ and $b_n = 1$ for any $n \geq 1$, the operators C_n turn into the classical Kantorovich operators on $[0,1]$.
For every $n \geq 1$,

$$C_n(e_1) = \frac{n}{n+1}\,e_1 + \frac{a_n + b_n}{2(n+1)}\,\mathbf{1}, \tag{2.2}$$

$$C_n(e_2) = \frac{1}{(n+1)^2}\left(n^2 e_2 + n e_1(1-e_1) + n(a_n+b_n)e_1 + \frac{b_n^2 + a_n b_n + a_n^2}{3}\,\mathbf{1}\right). \tag{2.3}$$

We also point out that (see [7, Formula (4.2)]), the operators C_n are closely related to the classical Bernstein operators on $[0,1]$.
In fact, if one denotes by B_n the n-th Bernstein operator on $C([0,1])$, for every $f \in L^1([0,1])$, considering the function

$$F_n(f)(x) := \frac{n+1}{b_n - a_n}\int_{\frac{nx+a_n}{n+1}}^{\frac{nx+b_n}{n+1}} f(t)\,dt = \int_0^1 f\left(\frac{(b_n-a_n)t + a_n + nx}{n+1}\right)dt \tag{2.4}$$

$(x \in [0,1], n \geq 1)$, it turns out that

$$C_n(f) = B_n(F_n(f)) \tag{2.5}$$

$(f \in L^1([0,1]), n \geq 1)$.
As quoted in the Introduction, in [10] the authors introduced a modification of Bernstein operators that fixes suitable functions.
More precisely, consider a function $\tau \in C^\infty([0,1])$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for every $x \in [0,1]$.
The operators introduced in [10] are defined by

$$B_n^\tau(f) := B_n(f \circ \tau^{-1}) \circ \tau \qquad (n \geq 1, f \in C([0,1])). \tag{2.6}$$

Namely, for every $f \in C([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$B_n^\tau(f)(x) := \sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right). \tag{2.7}$$

After the above preliminaries, we pass to introduce a new sequence of positive linear operators acting on integrable functions on [0, 1], which is a combination of (2.1) and (2.6). More precisely, for any $f \in L^1([0,1])$ and $n \geq 1$, we set

$$C_n^\tau(f) := C_n(f \circ \tau^{-1}) \circ \tau; \tag{2.8}$$

hence, for every $f \in L^1([0,1])$, $n \geq 1$ and $x \in [0,1]$,

$$C_n^\tau(f)(x) = \sum_{k=0}^{n} \left(\frac{n+1}{b_n - a_n} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} \left(f \circ \tau^{-1}\right)(t)\, dt \right) \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k},$$

where we have used the fact that, thanks to the change of variable theorem, $f \circ \tau^{-1} \in L^1([0,1])$ provided $f \in L^1([0,1])$.
Note that, if $\tau = e_1$, the operators C_n^τ turn into the operators C_n defined by (2.1), and hence in the classical Kantorovich operators whenever $a_n = 0$ and $b_n = 1$ for every $n \geq 1$.
The operators C_n^τ can be viewed as integral modification of Kantorovich-type of the operators B_n^τ with mobile intervals.

3. Shape preserving properties of the C_n^τ's

This section is devoted to show some qualitative properties of the operators C_n^τ. To this end, we first remark that, taking (2.4), (2.5) and (2.8) into account, the following formula holds true:

$$C_n^\tau(f) = B_n(F_n(f \circ \tau^{-1})) \circ \tau \tag{3.9}$$

$(f \in L^1([0,1]), n \geq 1)$.
Hence, one can recover some properties of the operators C_n^τ by means of the relevant ones held by the B_n's.
First off, as $F_n(f)$ is increasing whenever f is (continuous and) increasing, the B_n's map (continuous) increasing functions into increasing functions (see, e.g., [3, Remark p. 461]), and τ is increasing, we have that the operators C_n^τ map (continuous) increasing functions into increasing functions.
The C_n^τ's preserve also a particular form of convexity.
We recall (see [17]) that a function $f \in C([0,1])$ is said to be convex with respect to τ if, for every $0 \leq x_0 < x_1 < x_2 \leq 1$, one has

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \geq 0.$$

In particular, it can be proven that a function f is convex with respect to τ if and only if $f \circ \tau^{-1}$ is convex.
In [7, Proof of Th. 4.3]) it has been shown that the operators C_n map (continuous) convex functions into (continuous) convex functions; hence, thanks to (2.8), the operators C_n^τ map (continuous) convex functions with respect to τ into (continuous) convex functions with respect to τ.
Moreover, we investigate the monotonicity of the sequence $(C_n^\tau)_{n\geq 1}$ on convex functions with respect to τ.

Proposition 3.1. *If $f \in C([0,1])$ is convex with respect to τ and increasing (resp., decreasing), then, for every $n \geq 1$,*

$$f \leq C_n^\tau(f) \quad \text{on} \quad \left[0, \tau^{-1}\left(\frac{a_n + b_n}{2}\right)\right], \tag{3.10}$$

(resp.,

$$f \le C_n^\tau(f) \quad \textit{on} \ \left[\tau^{-1}\left(\frac{a_n+b_n}{2}\right), 1\right]). \tag{3.11}$$

Moreover, if $(a_n)_{n\ge1}$ *and* $(b_n)_{n\ge1}$ *are constant sequences and* $f \in C([0,1])$ *is convex with respect to* τ*, then*

$$C_{n+1}^\tau(f) \le \frac{n+1}{n+2} C_n^\tau(f) + \frac{1}{n+2} B_{n+1}^\tau(f), \tag{3.12}$$

B_n^τ *being defined by* (2.7).

Proof. In [7, Proposiotion 4.5] it has been proven that, if g is convex and increasing, then $g \le C_n(g)$ on $\left[0, \frac{a_n+b_n}{2}\right]$. Hence because f is convex with respect to τ and increasing, $f\circ\tau^{-1}$ is convex and increasing, so that

$$f\circ\tau^{-1} \le C_n(f\circ\tau^{-1}) \quad \text{on} \ \left[0, \frac{a_n+b_n}{2}\right],$$

and from this we get (3.10). Reasoning in the same way, one can establish (3.11).
Moreover, fix $f \in C([0,1])$ convex function with respect to τ. In [7, Theorem 4.4] it was established that, if $g \in C([0,1])$ is convex, then, for all $n \ge 1$, $C_{n+1}(g) \le \frac{n+1}{n+2} C_n(g) + \frac{1}{n+2} B_{n+1}(g)$, so that, by applying this result to $f\circ\tau^{-1}$, we get (3.12). □

Besides the convexity with respect to τ, the operators C_n^τ preserve another type of convexity. More precisely, given $\varphi \in C^\infty([0,1])$ such that $\varphi'(x) \neq 0$ for all $x \in [0,1]$ and $\varphi(0) = 0$, and $k \in \mathbb{N}$, a function $f \in C^k([0,1])$ is said to be φ-convex of order k if, for every $x \in [0,1]$,

$$D_\varphi^{(k)}(f)(x) := D^{(k)}(f\circ\varphi^{-1})(\varphi(x)) \ge 0.$$

For more details about φ-convex functions of order k see [14].
Since in our case $\tau : [0,1] \to [0,1]$ is a bijection and a positive function, it is easy to show that a function $f \in C^k([0,1])$ is τ-convex of order k if and only if

$$D_\tau^{(k)}(f) := D^{(k)}(f\circ\tau^{-1}) \ge 0.$$

In other words, f is τ-convex of order k iff $f\circ\tau^{-1}$ is k-convex. Here we recall that a function $g \in C^k([0,1])$ is said to be k-convex if $D^{(k)}(g) \ge 0$.
By using the fundamental theorem of calculus, F_n maps k-convex functions into k-convex functions and the same happens for the B_n's (see, for example, [6, Prop. A.2.5]). Then, thanks to (3.9) we have that the C_n^τ's map τ-convex functions of order k into τ-convex functions of order k.
We point out that the operators C_n^τ do not preserve the convexity. In order to construct an example, we use the following alternative representation for the operators C_n^τ: for every $n \ge 1$ and $f \in L^1([0,1])$,

$$C_n^\tau(f) = B_n^\tau(G_n^\tau(f\circ\tau^{-1})),$$

where

$$G_n^\tau(f)(x) := \frac{n+1}{b_n - a_n} \int_{\frac{n\tau(x)+a_n}{n+1}}^{\frac{n\tau(x)+b_n}{n+1}} f(t)\,dt\,.$$

Then, choosing $a_n = 0$, $b_n = 1$ for all $n \ge 1$, $\tau = (e_1+e_2)/2$ and $f = e_1$,

$$C_n^\tau(e_1) = \frac{n}{n+1} B_n^\tau(e_1) + \frac{1}{2(n+1)}.$$

Recalling that in this case $B_n^\tau(e_1)$ is not convex for lower n (see [10]), we get that the same happens for $C_n^\tau(e_1)$.
Now we pass to show that each C_n^τ preserves the class of Hölder continuous functions. Given $M > 0$ and $0 \le \alpha \le 1$, we shall write $f \in \mathrm{Lip}_M \alpha$ if

$$|f(x) - f(y)| \le M|x-y|^\alpha \quad \text{for every } 0 \le x, y \le 1.$$

In particular, if $\alpha = 1$, we get the space of all Lipschitz functions of Lipschitz constant M.
First observe that, from hypotheses on τ, both τ and τ^{-1} are Lipschitz functions. Precisely, $\tau \in \mathrm{Lip}_L 1$ with $L := \|\tau'\|_\infty$ and $\tau^{-1} \in \mathrm{Lip}_N 1$ with $N := (\min_{[0,1]} \tau')^{-1}$. Therefore, by recalling that $C_n(\mathrm{Lip}_M 1) \subset \mathrm{Lip}_{CM} 1$ with $C := \max\{1, |f(0)| + |f(1)|\}$ (see [7, Th. 4.1 and Example n. 1]), from (2.8) it follows that

$$C_n^\tau(\mathrm{Lip}_M 1) \subset \mathrm{Lip}_{CLMN} 1 \quad \text{for every } n \ge 1\,. \tag{3.13}$$

On account of [3, Cor. 6.1.20], since $\|C_n^\tau\| = 1$ and property (3.13) holds, for every $n \ge 1$, $f \in C([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \le 1$,

$$\omega(C_n^\tau(f), \delta) \le (1 + C)\omega(f, \delta) \quad \text{and} \quad C_n^\tau(\mathrm{Lip}_M \alpha) \subset \mathrm{Lip}_{(CLN)^\alpha M} \alpha\,.$$

Finally, for every $k \in \mathbb{N}$, denote by $\mathbb{P}_{\tau,k}$ the linear subspace generated by the set $\{\tau^i : i = 0, \ldots, k\}$. $\mathbb{P}_{\tau,k}$ is said to be the space of the τ-polynomials of degree k. Since both the B_n's and the F_n's map polynomials of degree k into polynomials of degree k, taking (3.9) into account, we have that

$$C_n^\tau(\mathbb{P}_{\tau,k}) \subset \mathbb{P}_{\tau,k} \qquad (k \in \mathbb{N}, n \ge 1).$$

4. Approximation properties of the C_n^τ's

In this section, we prove that $(C_n^\tau)_{n\ge1}$ is a positive approximation process both in $C([0,1])$ and in $L^p([0,1])$, $1 \le p < +\infty$, and we provide some estimates of the rate of convergence, by means of suitable moduli of smoothness. As a byproduct of the uniform convergence, we obtain a property of the operators B_n^τ introduced in [10], which seems to be new.
We begin by stating the following result.

Theorem 4.1. *For every $f \in C([0,1])$, we have that*

$$\lim_{n\to\infty} C_n^\tau(f) = f \tag{4.14}$$

uniformly on $[0,1]$.

Proof. From (2.2) and (2.3) it easily follows that

$$C_n^\tau(\tau) = \frac{n}{n+1}\tau + \frac{a_n + b_n}{2(n+1)}\mathbf{1}, \tag{4.15}$$

$$C_n^\tau(\tau^2) = \frac{1}{(n+1)^2}\left(n^2\tau^2 + n\tau(1-\tau) + n(a_n+b_n)\tau + \frac{b_n^2 + a_n b_n + a_n^2}{3}\mathbf{1}\right); \tag{4.16}$$

since $C_n^\tau(\mathbf{1}) = \mathbf{1}$ and $\{\mathbf{1}, \tau, \tau^2\}$ is an extended Tchebychev system on [0,1], (4.14) comes directly by an application of Korovkin Theorem (see [3, Example 5, p. 246]). □

In order to get a quantitative version of the above uniform convergence, we use a result due to Paltanea (see [15]) which involves the usual modulus of continuity of the first and second order, denoted, respectively, by $\omega(f,\delta)$ and $\omega_2(f,\delta)$. To this end, we need some further preliminaries.
For $x \in [0,1]$, we denote by $e_{\tau,i}^x$ the function

$$e_{\tau,i}^x(t) = (\tau(t) - \tau(x))^i \quad (i = 0, 1, 2, \ldots)\,.$$

When $\tau = e_1$ we shall simply write $\psi_x^i(t) = (t-x)^i$.
In particular, for any $n \geq 1$ and $x \in [0,1]$) (see (4.15) and (4.16)),

$$C_n^\tau(e_{\tau,2}^x)(x) = \frac{1-n}{(n+1)^2}\tau^2(x) + \frac{n-a_n-b_n}{(n+1)^2}\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3(n+1)^2}. \tag{4.17}$$

Moreover, we recall the following result (see [11, Formula (8)]): there exists a constant $K > 0$ such that

$$K\psi_x^2(t) \leq \tau'(x)e_{\tau,2}^x(t) \quad \text{for every } x, t \in [0,1]\,. \tag{4.18}$$

Obviously, $K = 1$ if $\tau = e_1$.

Proposition 4.2. *Consider $n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$, and $\delta > 0$. Then*

$$|C_n^\tau(f)(x) - f(x)| \quad \leq \omega(f, \delta_n^\tau(x)) + \frac{3}{2}\omega_2(f, \delta_n^\tau(x))\,, \tag{4.19}$$

where

$$\delta_n^\tau(x) = \frac{\sqrt{\tau'(x)}}{(n+1)\sqrt{K}}\sqrt{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}}.$$

Moreover,

$$\|C_n^\tau(f) - f\|_\infty \leq \omega\left(f, \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}\right) + \frac{3}{2}\omega_2\left(f, \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}\right). \tag{4.20}$$

Proof. Let $n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$ and $\delta > 0$. Paltanea's estimate ([15, Theorem 2.2.1]; see, also, [6, Theorem 1.6.2]) runs as follows:

$$\begin{aligned} |C_n^\tau(f)(x) - f(x)| &\leq |f(x)||C_n^\tau(\mathbf{1})(x) - 1| \\ &\quad + \delta^{-1}|C_n^\tau(\psi_x)(x)|\omega(f,\delta) + \left(C_n^\tau(\mathbf{1})(x) + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x)\right)\omega_2(f,\delta) \\ &= \delta^{-1}|C_n^\tau(\psi_x)(x)|\omega(f,\delta) + (1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x))\omega_2(f,\delta)\,. \end{aligned}$$

Cauchy-Schwarz inequality yields

$$|C_n^\tau(\psi_x)| \leq \sqrt{C_n^\tau(\psi_x^2)}\,,$$

so that

$$|C_n^\tau(f)(x) - f(x)| \leq \delta^{-1}\sqrt{C_n^\tau(\psi_x^2)(x)}\omega(f,\delta) + (1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x))\omega_2(f,\delta)\,.$$

From (4.18), (4.17) and the positivity of C_n^τ's, we have

$$\begin{aligned} &KC_n^\tau(\psi_x^2)(x) \leq \tau'(x)C_n^\tau(e_{\tau,2}^x) \\ &= \frac{\tau'(x)}{(n+1)^2}\left\{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}\right\}. \end{aligned}$$

Therefore,

$$C_n^\tau(\psi_x^2) \leq \frac{\tau'(x)}{K(n+1)^2}\left\{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2+a_nb_n+a_n^2}{3}\right\} \tag{4.21}$$

and, for $\delta = \delta_n^\tau(x)$, we get (4.19). Estimate (4.20) follows by noting that

$$\delta_n^\tau(x) \leq \frac{\|\tau'\|_\infty^{1/2}}{\sqrt{K}\sqrt{n+1}}$$

since $0 \leq \tau(x) \leq 1$. $\square$

As a byproduct of Theorem 4.1, we present a simultaneous approximation result for the operators B_n^τ given by (2.7). As far as we know, this property is new.

Theorem 4.2. *Suppose that* $a_n = 0$ *and* $b_n = 1$ *for every* $n \geq 1$. *Then, for every* $f \in C^1([0,1])$,

$$B_{n+1}^\tau(f)' = \tau' C_n^\tau\left(f'/\tau'\right). \tag{4.22}$$

Moreover,

$$\lim_{n\to\infty} B_n^\tau(f)' = f' \quad \textit{uniformly on } [0,1]. \tag{4.23}$$

Proof. Let $x \in [0,1]$, $f \in C^1([0,1])$, and $n \geq 1$. From (2.7) if follows that

$$\begin{aligned}
B_{n+1}^\tau(f)'(x) &= \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \\
&\times (n+1)\left(\left(f\circ\tau^{-1}\right)\left(\frac{k+1}{n+1}\right) - \left(f\circ\tau^{-1}\right)\left(\frac{k}{n+1}\right)\right) \\
&= \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (f\circ\tau^{-1})'(t)\,dt\right) \\
&= \tau'(x) C_n^\tau\left(\frac{f'}{\tau'}\right)(x),
\end{aligned}$$

and this completes the proof of (4.22). Formula (4.23) immediately follows from (4.22) and Theorem 4.1, because τ' is bounded. $\square$

Now we prove that the sequence $(C_n^\tau)_{n\geq 1}$ is a positive approximation process also in $L^p([0,1])$ for any $p \in [1,+\infty[$.

Theorem 4.3. *Assume that*

$$\sup_{n\geq 1} \frac{1}{b_n - a_n} = M \in \mathbb{R}.$$

Then, for every $p \in [1,+\infty[$ *and* $f \in L^p([0,1])$,

$$\lim_{n\to\infty} C_n^\tau(f) = f \quad \textit{in} \quad L^p([0,1]). \tag{4.24}$$

Proof. By Theorem 4.1, for every $f \in C([0,1])$, $\lim_{n\to\infty} C_n(f) = f$ in L^p-norm, as well. Since $C([0,1])$ is dense in $L^p([0,1])$, in order to prove the statement it is sufficient to show, thanks to Banach-Steinhaus theorem, that the sequence of operators $C_n^\tau : L^p([0,1]) \to L^p([0,1])$ $(n \geq 1)$ is equicontinuous, i.e.,

$$\sup_{n\geq 1} \|C_n^\tau\|_{L^p,L^p} < +\infty.$$

To this end, for every $n \geq 1$, $f \in L^p([0,1])$ and $x \in [0,1]$, we preliminary notice that, since the function $|t|^p$ $(t \in \mathbb{R})$ is convex,

$$|C_n^\tau(f)(x)|^p \leq \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left[\frac{(n+1)}{(b_n - a_n)} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} \left(f\circ\tau^{-1}\right)(t)\,dt\right]^p.$$

By applying Jensen's inequality (see, e.g., [8, Theorem 3.9]) to the probability measure $\dfrac{n+1}{b_n-a_n}\,\mathbf{1}_{[\frac{k+a_n}{n+1},\frac{k+b_n}{n+1}]}\lambda_1$ on $[0,1]$, we get

$$\begin{aligned}|C_n^\tau(f)(x)|^p &\le \sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\frac{(n+1)}{(b_n-a_n)}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}}\left|\left(f\circ\tau^{-1}\right)(t)\right|^p\,dt\\ &=\sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\frac{(n+1)}{(b_n-a_n)}\int_{\tau^{-1}\left(\frac{k+a_n}{n+1}\right)}^{\tau^{-1}\left(\frac{k+b_n}{n+1}\right)}|f(y)\tau'(y)|^p\,dy\\ &\le \|\tau'\|_\infty^p\frac{(n+1)}{(b_n-a_n)}\sum_{k=0}^{n}\binom{n}{k}\tau(x)^k(1-\tau(x))^{n-k}\int_{\tau^{-1}\left(\frac{k+a_n}{n+1}\right)}^{\tau^{-1}\left(\frac{k+b_n}{n+1}\right)}|f(y)|^p\,dy.\end{aligned}$$

We point out that

$$\int_0^1\tau(x)^k(1-\tau(x))^{n-k}\,dx=\int_0^1\frac{t^k(1-t)^{n-k}}{\tau'(\tau^{-1}(t))}\,dt\le\frac{1}{\min_{y\in[0,1]}\tau'(y)}\frac{1}{\binom{n}{k}(n+1)}.$$

Hence, by integrating with respect to x, we obtain

$$\|C_n^\tau(f)\|_p^p\le MN\|f\|_p^p,$$

where

$$N:=\frac{\|\tau'\|_\infty^p}{\min_{y\in[0,1]}\tau'(y)};$$

hence $\|C_n^\tau\|_{L^p,L^p}\le(MN)^{1/p}<+\infty$. □

An estimate of the convergence in (4.24) can be obtained by using a result due to Swetits and Wood [16, Theorem 1] which involves the second-order integral modulus of smoothness defined, for $f\in L^p([0,1])$, $1\le p<+\infty$, as

$$\omega_{2,p}(f,\delta):=\sup_{0<t\le\delta}\|f(\cdot+t)-2f(\cdot)+f(\cdot-t)\|_p\quad(\delta>0).$$

We define

(4.25)

$$\beta_{n,p,\tau}:=\frac{1}{(n+1)\sqrt{K}}\times\left\|\sqrt{\tau'}\left\{(n-1)\tau(1-\tau)+(1-a_n-b_n)\tau+\frac{b_n^2+a_nb_n+a_n^2}{3}\mathbf{1}\right\}^{1/2}\right\|_p^{1/2}$$

and

$$\begin{aligned}\gamma_{n,p,\tau}&:=\frac{1}{(n+1)^{2p/(2p+1)}K^{p/(2p+1)}}\\ &\times\left\|\tau'\left\{(n-1)\tau(1-\tau)+(1-a_n-b_n)\tau+\frac{b_n^2+a_nb_n+a_n^2}{3}\mathbf{1}\right\}\right\|_p^{p/(2p+1)},\end{aligned}\tag{4.26}$$

where K is the strictly positive constant in (4.18).
Then we can state the following result.

Proposition 4.3. *Under the hypotheses of Theorem 4.3, for every $p\in[1,+\infty[$ there exists $C_p>0$ such that, for every $f\in L^p([0,1])$ and for n sufficiently large,*

$$\|C_n^\tau(f)-f\|_p\le C_p(\alpha_{n,p,\tau}^2\|f\|_p+\omega_{2,p}(f,\alpha_{n,p,\tau}))$$

where $\alpha_{n,p,\tau}=\max\{\beta_{n,p,\tau},\gamma_{n,p,\tau}\}$.

Proof. First we introduce the following auxiliary functions:

$$F_n^\tau(x) := C_n^\tau(\psi_x)(x), \;\; G_n^\tau(x) := C_n^\tau(\psi_x^2)(x) \quad (x \in [0,1], n \geq 1).$$

Hence, the result in [16] applied to the uniformly bounded sequence $(C_n^\tau)_{n\geq 1}$ yields that there exists a constant $C_p > 0$ such that

$$\|C_n^\tau(f) - f\|_p \leq C_p(\mu_{n,p}^2 \|f\|_p + \omega_{2,p}(f, \mu_{n,p})),$$

where the sequence $\mu_{n,p} \to 0$ as $n \to \infty$ and it is defined as follows:

$$\begin{aligned}\mu_{n,p} &:= \max\left\{\|C_n^\tau(\mathbf{1}) - \mathbf{1}\|_p^{1/2}, \|F_n^\tau\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\} \\ &= \max\left\{\|F_n^\tau\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\}.\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$|F_n^\tau|^p \leq (\sqrt{G_n^\tau})^p,$$

so

$$\mu_{n,p} \leq \max\left\{\|\sqrt{G_n^\tau}\|_p^{1/2}, \|G_n^\tau\|_p^{p/(2p+1)}\right\}.$$

From (4.21) it follows that $\|\sqrt{G_n^\tau}\|_p^{1/2} \leq \beta_{n,p,\tau}$ and $\|G_n^\tau\|_p^{p/(2p+1)} \leq \gamma_{n,p,\tau}$ (see (4.25) and (4.26)). Moreover,

$$\begin{aligned}\gamma_{n,p,\tau} &\leq \frac{\|\tau'\|_\infty^{p/(2p+1)}}{(n+1)^{2p/(2p+1)} K^{p/(2p+1)}}(n+1)^{p/(2p+1)} \\ &= \frac{\|\tau'\|_\infty^{p/(2p+1)}}{(n+1)^{p/(2p+1)} K^{p/(2p+1)}} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Similarly,

$$\beta_{n,p,\tau} \leq \frac{\|\sqrt{\tau'}\|_\infty^{1/2}}{(n+1)\sqrt{K}}(n+1)^{1/4} = \frac{\|\sqrt{\tau'}\|_\infty^{1/2}}{(n+1)^{3/4}\sqrt{K}} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, setting $\alpha_{n,p,\tau} = \max\{\beta_{n,p,\tau}, \gamma_{n,p,\tau}\}$, we have that $\alpha_{n,p,\tau} \to 0$ as $n \to \infty$ and this completes the proof. □

5. Asymptotic formula for the C_n^τ's

In this section we establish an asymptotic formula for the operators C_n^τ, which, in addition, allows us to derive other properties of them. To this end, from now assume that

(5.27) $$\text{there exists } l := \lim_{n\to\infty}(a_n + b_n) \in \mathbb{R}$$

and consider the differential operator $(V_l, C^2([0,1]))$ defined by setting

$$V_l(u)(x) := \frac{1}{2}x(1-x)u''(x) + \left(\frac{l}{2} - x\right)u'(x),$$

$(u \in C^2([0,1]), x \in [0,1])$.

Theorem 5.4. *Assume that (5.27) holds true. Then, for each $f \in C([0,1])$, twice differentiable at a certain $x \in]0,1[$,*

$$\begin{aligned}\lim_{n\to\infty} n(C_n^\tau(f)(x) - f(x)) &= \frac{\tau(x)(1-\tau(x))}{2} D_\tau^2(f)(x) + \left(\frac{l}{2} - \tau(x)\right) D_\tau(f)(x) \\ &= \frac{\tau(x)(1-\tau(x))}{2\tau'(x)^2} f''(x) + \frac{1}{\tau'(x)}\left(\frac{l}{2} - \tau(x) - \frac{\tau(x)(1-\tau(x))}{2\tau'(x)^2}\tau''(x)\right) f'(x).\end{aligned} \tag{5.28}$$

Moreover, for every $u \in C^2([0,1])$

$$\lim_{n\to\infty} n(C_n^\tau(u) - u) = V_l(u \circ \tau^{-1}) \circ \tau \tag{5.29}$$

uniformly in $[0,1]$.

Proof. In [5, Theorem 3.1] it was proven that

$$\lim_{n\to\infty} n(C_n(u) - u) = V_l(u),$$

for every $u \in C^2([0,1])$ uniformly on $[0,1]$, but it is easy to prove that the same limit relationship holds pointwise for each $f \in C([0,1])$, twice differentiable at a certain $x \in]0,1[$. From this, formulas (5.28) and (5.29) easily follow. □

5.1. **An application to iterates of the operators C_n^τ.** In this subsection we show how iterates of operators C_n^τ can be employed in order to approximate constructively certain semigroups of operators. For unexplained terminology concerning Semigroup Theory and its connection with Approximation Theory, we refer, e.g., to [6, Chapter 2].
We begin by recalling that, as shown in [5, Theorem 3.2] the operator $(V_l, C^2([0,1]))$ is closable and its closure generates a Markov semigroup $(T_l(t))_{t\geq 0}$ on $C([0,1])$ such that, if $t \geq 0$ and if $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, then

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = T_l(t)(f) \qquad \text{uniformly on } [0,1]$$

for every $f \in C([0,1])$, where $C_n^{\rho_n}$ denotes the iterate of C_n of order ρ_n.
Moreover (see [5, Theorem 3.4, Remark 3.5,1]), if either $a_n = 0$ and $b_n = 1$ for every $n \geq 1$, or the following properties hold true

(i) $0 < b_n - a_n < 1$ for every $n \geq 1$;
(ii) there exist $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$;
(iii) $M_1 := \sup_{n\geq 1} n(1 - (b_n - a_n)) < +\infty$,

for every $p \geq 1$, $(T_l(t))_{t\geq 0}$ extends to a positive C_0-semigroup $(\widetilde{T}(t))_{t\geq 0}$ on $L^p([0,1])$ such that, if $(\rho_n)_{n\geq 1}$ is a sequence of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$, then for every $f \in L^p([0,1])$,

$$\lim_{n\to\infty} C_n^{\rho_n}(f) = \widetilde{T}(t)(f) \qquad \text{in } L^p([0,1]).$$

We remark that, for every $f \in C([0,1])$ and $k \geq 1$,

$$(C_n^\tau)^k(f) = C_n^k(f \circ \tau^{-1}) \circ \tau.$$

From this we get the following result.

Theorem 5.5. *Under assumption (5.27), for every $f \in C([0,1])$, $t \geq 0$ and for every sequence $(\rho_n)_{n\geq 1}$ of positive integers such that $\lim_{n\to\infty} \rho_n/n = t$,*

$$\lim_{n\to\infty} (C_n^\tau)^{\rho_n}(f) = T_l(t)(f \circ \tau^{-1}) \circ \tau \qquad \textit{uniformly on } [0,1].$$

Moreover, assume that either $a_n = 0$ *and* $b_n = 1$ *for every* $n \geq 1$*, or the following properties hold true*

(i) $0 < b_n - a_n < 1$ *for every* $n \geq 1$;

(ii) *there exist* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} b_n = 1$;

(iii) $M_1 := \sup_{n\geq 1} n(1 - (b_n - a_n)) < +\infty$.

Then, if $t \geq 0$ *and if* $(\rho_n)_{n\geq 1}$ *is a sequence of positive integers such that* $\lim_{n\to\infty} \rho_n/n = t$*, then for every* $f \in L^p([0,1])$,

$$\lim_{n\to\infty} (C_n^\tau)^{\rho_n}(f) = \widetilde{T}(t)(f \circ \tau^{-1}) \circ \tau \qquad \text{in } L^p([0,1]).$$

5.2. **Comparing the operators C_n^τ and C_n.** The asymptotic formula (5.28) can be also used to prove that, under suitable conditions, the operators C_n^τ perform better than the operators C_n in approximating certain functions. In fact, arguing as in the proof of [10, Theorem 9], we are able to show the following result.

Theorem 5.6. *Let* $f \in C^2([0,1])$ *and assume that there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$ *and* $x \in]0,1[$,

$$f(x) \leq C_n^\tau(f)(x) \leq C_n(f)(x)\,.$$

Then, for $x \in]0,1[$,

$$\begin{aligned} f''(x) &\geq \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - l)}{\tau(x)(1-\tau(x))} f'(x) \\ &\geq \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x) + \frac{\tau'(x)^2(2x-l)}{\tau(x)(1-\tau(x))} f'(x)\,. \end{aligned} \tag{5.30}$$

In particular, $f'' \geq 0$ *in* $]0, l/2[$ *(resp., in* $]l/2, 1[$*) whenever* f *is decreasing in* $]0, l/2[$ *(resp.,* f *is increasing in* $]l/2, 1[$*).*

Conversely, assume that at a given point $x_0 \in]0,1[$*, (5.30) holds with strict inequalities. Then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$,

$$f(x_0) < C_n^\tau(f)(x_0) < C_n(f)(x_0)\,.$$

Example 5.1. *Take*

$$\tau = \frac{e_2 + \alpha e_1}{1+\alpha} \qquad (\alpha > 0)$$

and suppose that $f \in C^2([0,1])$ *is increasing and strictly convex.*

Moreover, assume that the sequences $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *are such that* $l = \lim_{n\to\infty}(a_n + b_n) = 2$*. We show that there exist* $x_\alpha \in]0,1[$ *and* $n_0 \in \mathbb{N}$ *such that, for each* $x \in]x_\alpha, 1]$ *and* $n \geq n_0$,

$$f(x) < C_n^\tau(f)(x) < C_n(f)(x)\,.$$

On account of Theorem 5.6, it is sufficient to prove that there exists $x_\alpha \in]0,1[$ *such that, for* $x \in]x_\alpha, 1[$,

$$\begin{aligned} f''(x) &> \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - 2)}{\tau(x)(1-\tau(x))} f'(x) \\ &> \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right) f''(x) + \frac{\tau'(x)^2(2x-2)}{\tau(x)(1-\tau(x))} f'(x)\,. \end{aligned} \tag{5.31}$$

The first inequality in (5.31) is satisfied for $\alpha > 2f'(1)/M$*, where* $M = \min_{[0,1]} f''(x)$*. Indeed, for this choice,*

$$f''(x) > \frac{2}{2x+\alpha} f'(x) > \frac{2}{2x+\alpha} f'(x) - 2\frac{2x+\alpha}{x^2+\alpha x} f'(x), \qquad x \in]0,1[\,.$$

The second inequality in (5.31) *is obviously fulfilled for those* x *for which*

$$\frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))} \geq 1 \tag{5.32}$$

and

$$\frac{\tau''(x)}{\tau'(x)} > 2\frac{\tau'(x)^2}{\tau(x)(1-\tau(x))}\left(x-1-\frac{\tau(x)-1}{\tau'(x)}\right). \tag{5.33}$$

From one hand (5.32) *is verified for* $x \in]y_\alpha, 1]$ *where*

$$y_\alpha := \frac{1-2\alpha+\sqrt{4\alpha^2+8\alpha+1}}{6}$$

(see [10, Corollary 11, (iii)]*). On the other hand* (5.33) *is equivalent to solve (with respect to* x*) the following inequality:*

$$g_\alpha(x) := (x^2+\alpha x)(1+x+\alpha) - (2x+\alpha)^2(1-x) > 0\,.$$

By observing that $g_\alpha(0) < 0$*,* $g_\alpha(1) > 0$*, and evaluating the critical points of* g_α *and their position within the interval [0, 1] depending on* $\alpha > 0$*, we can conclude that, for every* $\alpha > 0$*, there exists* $z_\alpha \in]0,1[$ *such that* $g_\alpha(z_\alpha) = 0$ *and* $g_\alpha(x) > 0$ *for every* $z_\alpha < x \leq 1$*. By setting* $x_\alpha = \max\{y_\alpha, z_\alpha\}$ $(\alpha > 2f'(1)/M)$*, we get the claim.*
We point out that, in the case $\alpha = 0$*,* $\tau = e_2$ *and the corresponding operators* C_n^τ *are a Kantorovich-type modification on mobile intervals of the operators in* [10, p. 159]*. On the other hand,* $\tau_\infty = \lim_{\alpha\to+\infty}\tau = e_1$ *uniformly w.r.t.* $x \in [0,1]$*, so that* $C_n^{\tau_\infty} = C_n$ *for any* $n \geq 1$*.*

Aknowledgements.
The authors acknowledge the support of "INdAM GNAMPA Project 2019 - Approssimazione di semigruppi tramite operatori lineari e applicazioni".
The authors wish also to thank the anonymous referees for their careful reading of the manuscript and their comments and suggestions.

References

[1] T. Acar, A. Aral, I. Raşa, *Positive linear operators preserving* τ *and* τ^2, Constr. Math. Anal. **2** (3) (2019), 98–102.
[2] T. Acar, M. Cappelletti Montano, P. Garrancho, V. Leonessa, *On sequences of J. P. King-type operators*, J. Funct. Spaces, 2019, Article ID 2329060.
[3] F. Altomare, M. Campiti, Korovkin-type approximation theory and its applications, de Gruyter Studies in Mathematics **17**, Walter de Gruyter & Co., Berlin, 1994.
[4] F. Altomare, M. Cappelletti Montano, V. Leonessa, *On a generalization of Kantorovich operators on simplices and hypercubes*, Adv. Pure Appl. Math. **1**(3) (2010), 359-385.
[5] F. Altomare, M. Cappelletti Montano, V. Leonessa, *Iterates of multidimensional Kantorovich-type operator and their associated positive* C_0*-semigroups*, Studia Universitatis Babes-Bolyai. Mathematica **56**(2) (2011), 236-251.
[6] F. Altomare, M. Cappelletti Montano, V. Leonessa, I. Raşa, Markov Operators, Positive Semigroups and Approximation Processes, *de Gruyter Studies in Mathematics* **61**, *Walter de Gruyter GmbH, Berlin/Boston*, 2014.
[7] F. Altomare, V. Leonessa, *On a sequence of positive linear operators associated with a continuous selection of Borel measures*, Mediterr. J. Math. **3** (2006), 363-382.
[8] H. Bauer, Probability Theory, de Gruyter Studies in Mathematics **23**, Walter de Gruyter & Co., Berlin, 1996.
[9] D. Cardenas-Morales, P. Garrancho, F.J. Muños-Delgado, *Shape preserving approximation by Bernstein-type operators which fix polinomials*, Appl. Math. Comput. **182** (2006) 1615–1622.
[10] D. Cardenas-Morales, P. Garrancho, I. Raşa, *Bernstein-type operators which preserve polynomials*, Comput. Math. Appl. **62**(1) (2011), 158–163.
[11] G. Freud, *On approximation by positive linear methods I, II*, Stud. Scin. Math. Hungar. **2** (1967) 63–66, **3** (1968), 365–370.
[12] H. Gonska, P. Piţul, I. Raşa, *General King-type operators*, Results Math. **53**(3-4) (2009) 279–286.
[13] J.P. King, *Positive linear operators which preserve* x^2, Acta Math. Hungar. **99**(3) (2003) 203–208.

[14] A.-J. López-Moreno, F.-J. Muñoz-Delgado, *Asymptotic expression of derivatives of Bernstein type operators*, Suppl. Cir. Mat. Palermo Ser. II **68** (2002), 615-624.
[15] R. Paltanea, Approximation theory using positive linear operators, *Birkhäuser, Boston*, (2004).
[16] J. J. Swetits, B. Wood, *Quantitative estimates for L_p approximation with positive linear operators*, J. Approx. Theory **38** (1983), 81–89.
[17] Z. Ziegler, *Linear approximation and generalized convexity*, J. Approx. Theory **1** (1968), 420–433.

UNIVERSITY OF BARI
DEPARTMENT OF MATHEMATICS
VIA E. ORABONA, 4, BARI, ITALY
Email address: mirella.cappellettimontano@uniba.it

UNIVERSITY OF BASILICATA
DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE AND ECONOMICS
VIALE DELL'ATENEO LUCANO 10, POTENZA, ITALY
Email address: vita.leonessa@unibas.it