Commun.Fac.Sci.Univ.Ank.Series A1
Volume 62, Number 1, Pages 1–10 (2013)
ISSN 1303–5991

ON THE KOLMOGOROV-PETROVSKII-PISKUNOV EQUATION

ARZU ÖĞÜN ÜNAL

Abstract. We prove existence and uniqueness of the solutions of Kolmogorov-Petrovskii-Piskunov (KPP) equation. We study asymptotic stability and instability of the equilibrium solution $u(x,t) \equiv 0$ of KPP equation with subject to the traveling wave solutions. We show that KPP equation has not got any periodic traveling wave solution. Also, we obtain some exact traveling wave solutions of KPP equation by the first integral method.

1. Introduction

In this paper, we are interested in the equation of Kolmogorov-Petrovskii-Piskunov

$$u_t - u_{xx} + \mu u + \nu u^2 + \delta u^3 = 0, \quad x \in \mathbb{R}, \quad t \in [0,\infty) \tag{1}$$

with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

KPP equation first appeared in the genetics model for the spread of an advantageous gene through a population [12]. Later, it has been applied to a number of physics, biological and chemical models. KPP equation contains various well known nonlinear equations in mathematical physics; In the case of $\mu = -1,\ \nu = 0,\ \delta = 1$, it reduces to the Newell-Whitehead equation, for $\mu = a,\ \nu = -(a+1),\ \delta = 1$, it is called FitzHugh-Nagumo equation and for $\mu = -1,\ \nu = 1,\ \delta = 0$, it is a special case of Fisher equation $u_t - u_{xx} = u - u^2$.

The reason for our interest in the KPP equation is that there exist solutions to the KPP equation whose qualitative behavior resembles the traveling wave solutions. In recent years, various techniques such as Bäcklund transformation method [10, 15, 17], tanh method [11], Adomian method [2], $\frac{G'}{G}$-expansion method [8], numerical methods [5] and as well a direct algebraic method [13] have been used to obtain some exact traveling wave solutions of Eq. (1). Yet as we know, the first integral method has not been applied to Eq. (1) for the same purpose. This method first

Received by the editors Nov. 22, 2012, Accepted: Feb. 22, 2013.
2000 *Mathematics Subject Classification.* Primary 35B35, 35B40, 35B10, 35C07 .
Key words and phrases. Existence and uniqueness of solutions, asymptotic stability, instability, periodicity, traveling wave solutions, first integral method.

introduced by Feng to solve the Burgers Korteweg-de Vries equation [9] and after that it was applied to various types of nonlinear equations [1, 3, 7, 14, 18, 20].

Our aim is firstly to study the asymptotic stability and instability of zero solution of KPP equation with subject to all traveling wave solutions by means of qualitative theory of ordinary dfferential equations, secondly to explore the periodic traveling wave solution of KPP equation and thirdly to find some exact traveling wave solutions of KPP equation by using the first integral method. But, for all these, it is necessary to guarantee the existence and uniqueness of solutions of IVP (1)-(2). So, this paper is designed as follow:
In Section 2, the existence and uniqueness solutions of (1)-(2) is proved. In Section 3, asymptotic stability and instability of zero solution $u(x,t) \equiv 0$ of KPP equation are studied. The stability regions of zero solution are sketched. Also, a negative result is given for the periodicity. In Section 4, some exact traveling wave solutions of KPP equation are obtained by the first integral method. In the final section, we showed that if our conditions are satisfied, then a traveling wave solution that we obtained can approach to zero.

2. Existence and Uniqueness of Solutions

Let us consider the initial value problem (IVP)

$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}, \quad x \in \Omega, \quad t \in (0,\infty), \tag{3}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{4}$$

where $\Omega \subset \mathbb{R}$ and D is a diffusion coefficient. Equation (3) is known as a reaction-diffusion equation which includes the KPP equation. We first give the following well known result about existence and uniqueness for the solution of (3)-(4). [4, 6, 16]

Theorem 1. *Consider the IVP (3)-(4) problem. Suppose that $u_0(x)$ is continuous for $x \in \bar{\Omega}$ or $x \in \mathbb{R}$. In addition, suppose there exists constants a and b such that $a \leq u_0(x) \leq b$ for $x \in \bar{\Omega}$, $f(a) \geq 0$, $f(b) \leq 0$, and f is uniformly Lipschitz continuous, that is, there exists a constant c such that,*

$$|f(y) - f(z)| \leq c\,|y - z| \tag{5}$$

for all values $y, z \in [a,b]$. Then the Cauchy problem (3)-(4) has a unique bounded solution $u(x,t)$ for $x \in \Omega$ or $x \in \mathbb{R}$ and $t \in (0,\infty)$. In addition, the solution $u(x,t) \in [a,b]$.

Now, it is easy to prove that there exists a unique bounded solution of the IVP (1)-(2).

Theorem 2. *Suppose that $u_0(x)$ is continuous and $0 \leq u_0(x) \leq \beta$ for $x \in \mathbb{R}$ such that β satisfies $\mu + \nu\beta + \delta\beta^2 = 0$, $\beta \in \mathbb{R}$. Then there is a unique solution of IVP (1)-(2) defined on $x \in \mathbb{R}$, $t \in [0,\infty)$. Moreover, $u(x,t) \in [0,\beta]$.*

Proof. Eq. (1) is a special case of Eq. (3). The function $f(u) = -\mu u - \nu u^2 - \delta u^3$ is Lipschitz continuous on the interval $[0, \beta]$ and the Lipschitz constant is $c = \left|\mu + 2\beta\nu + 3\beta^2\delta\right|$. So, due to Theorem 1, the Cauchy problem (1)-(2) has a unique bounded solution $u(x,t)$ defined on $x \in \mathbb{R}$ and $t \in [0, \infty)$. Also $u(x,t) \in [0, \beta]$.

3. Stability and Periodicity

Definition 1. Let $u(x,t)$ be the solution of IVP (3)-(4). Then $u(x,t)$ is said to be a *stable solution* if given an $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\bar{u}_0(x)$ satisfies

$$||\bar{u}_0(x) - u_0(x)|| < \delta,$$

the solution $\bar{u}(x,t)$ with $\bar{u}(x,0) = \bar{u}_0(x)$ of equation (1) satisfies

$$||\bar{u}(x,t) - u(x,t)|| < \varepsilon$$

for all $t \geq 0$. If the solution $u(x,t)$ is not stable, then it is said to be *unstable*. The solution $u(x,t)$ is said to be *locally asymptotically stable* if it is *stable* and, in addition,

$$||\bar{u}(x,t) - u(x,t)|| \to 0, \text{ as } t \to \infty.$$

To study the asymptotic stability and instability of the equilibrium solution $u(x,t) \equiv 0$ of KPP equation with subject to traveling wave solutions of KPP equation, we first of all have to find these kinds of solutions. To do this, we apply the wave transform

$$u(x,t) = U(\xi), \ \xi = x - \omega t \tag{6}$$

to Equation (1), where ω represent the wave speed. Then we obtain second order nonlinear ordinary differential equation

$$U'' + \omega U' - \mu U - \nu U^2 - \delta U^3 = 0. \tag{7}$$

If $\omega > 0$ ($\omega < 0$), then $U(x - \omega t)$ represents a wave traveling to the right (left). If we introduce the new dependent variables $X(\xi)$ and $Y(\xi)$ as

$$X(\xi) = U(\xi), \ Y(\xi) = U'(\xi), \tag{8}$$

then Eq. (7) reduce to the first-order system of ordinary differential equations in X and Y as follow

$$\begin{cases} X' = Y, \\ Y' = -\omega Y + \mu X + \nu X^2 + \delta X^3. \end{cases} \tag{9}$$

So, the stability of (7) is equivalent to the stability of the system (9).

Remark 1. We note that system (9) has at most three critical (equilibrium) points. If $\nu^2 < 4\delta\mu$, then (0,0) is only critical point. If $\nu^2 = 4\delta\mu$, then there are two critical points: (0,0) and $(-\frac{\nu}{2\delta}, 0)$. If $\nu^2 > 4\delta\mu$, then there are three equilibrium points: (0,0), $(\frac{-\nu-\sqrt{\nu^2-4\delta\mu}}{2\delta}, 0)$ and $(\frac{-\nu+\sqrt{\nu^2-4\delta\mu}}{2\delta}, 0)$. Hence the possible equilibrium solutions of Eq. (1) are $u = 0, \ u = -\frac{\nu}{2\delta}, \ u = \frac{-\nu\pm\sqrt{\nu^2-4\delta\mu}}{2\delta}$.

Now we can prove the following results.

Theorem 3. *The equilibrium point* $(0,0)$ *of system (9) is locally asymptotically stable iff* $\omega > 0$ *and* $\mu < 0$.
Proof. Since

$$\lim_{(X,Y)\to(0,0)} \frac{0}{\sqrt{X^2+Y^2}} = \lim_{(X,Y)\to(0,0)} \frac{\nu X^2 + \delta X^3}{\sqrt{X^2+Y^2}} = 0,$$

(0,0) is a simple critical point of system (9). On the other hand, (0,0) is also the unique equilibrium point of the linear system

$$\begin{cases} X' = Y \\ Y' = \mu X - \omega Y. \end{cases} \tag{10}$$

The characteristic equation of linear system (10) is

$$\lambda^2 + \omega\lambda - \mu = 0. \tag{11}$$

Since $\omega > 0$ and $\mu < 0$, both characteristic roots of (11) have negative real parts. So, it is clear that the equilibrium point (0,0) of system (10) is asymptotically stable as $\xi \to +\infty$. Due to the qualitative theory of ordinary differential equation, there is an asymptotical equivalance between linear system (10) and perturbed system (9). Therefore the zero solution of (9) is also asymptotically stable as $\xi \to +\infty$.
Theorem 4. *Under the conditions of Theorem 3, the zero solution of KPP equation* $u(x,t) \equiv 0$ *is asymptotically stable.*
Proof Repeating the proof of Theorem 3 and considering (6) and (8), the proof is completed.
Theorem 5. *The equilibrium point* $(0,0)$ *of system (10) is unstable iff either* $\omega < 0$ *or* $\mu > 0$.
Proof From (11), at least one eigenvalue of (11) is positive or has positive real part iff either $\omega < 0$ or $\mu > 0$. Thus the proof is completed.
Remark 2. Due to the above study, certain stability and instability regions for the zero solution of KPP equation and as well as the types of it can be given in the $\omega\mu-$ plane. For this, in Fig. 1 the $\omega\mu-$ plane is divided into six subregions as follows:

In Fig. 1, shaded regions show that the zero solution $u(x,t) \equiv 0$ of KPP equation is asymptotical stable. In other regions, $u(x,t) \equiv 0$ is unstable. On the other hand, the types of the equilibrium point $u(x,t) \equiv 0$ can be identified as in ordinary differential equations: It is called a saddle point in regions I and II, a node point in regions III and VI, a spiral point in regions IV and V.
Now, we can state a negative criter for the periodicity of Eq. (1).
Theorem 6. *KPP equation has no periodic traveling wave solution.*
Proof. We have already showed that all traveling wave solutions of KPP equation come from system (9). Now, let us demonstrate the second hands of system (9) as

$$F(X,Y) = Y, \; G(X,Y) = -\omega Y + \mu X + \nu X^2 + \delta X^3$$

respectively. Then,

$$\frac{\partial F}{\partial X} + \frac{\partial G}{\partial Y} = -\omega.$$

Since $\omega \neq 0$, $\frac{\partial F}{\partial X} + \frac{\partial G}{\partial Y}$ is always positive or negative for all X, Y. Therefore, due to well known Bendixon theorem [19], system (9) has no closed trajectory in $XY-$phase plane. This means that Eq. (7) does not have any periodic solutions. So, KPP equation has no periodic traveling wave solutions.

Remark 3. Due to Theorem 6, there is no periodic solution of KPP equation. But, in paper [8], the traveling wave solutions that obtained in [15]

$$u(\xi) = \mp \frac{\sqrt{-2\delta\Delta}}{2\delta} \tan \frac{1}{2}\sqrt{-\Delta}\xi - \frac{\nu}{2\delta}$$

and

$$u(\xi) = \pm \frac{\sqrt{-2\delta\Delta}}{2\delta} \cot \frac{1}{2}\sqrt{-\Delta}\xi - \frac{\nu}{2\delta}$$

have been refered as periodic solutions of KPP equation. As a matter of the fact that, they can not be solutions of KPP equation for everywhere. Because, they are not defined at the points $\xi = \frac{\pi}{\sqrt{-\Delta}} + \frac{2k\pi}{\sqrt{-\Delta}}$, and $\xi = \frac{2k\pi}{\sqrt{-\Delta}}$, $k \in \mathbb{Z}$, respectively.

4. Traveling Wave Solutions of KPP Equation

In Section 3, we showed that all traveling wave solutions of KPP equation are equivalent to the solutions of system (9). Because the component $X(\xi)$ of any solution $(X(\xi), Y(\xi))$ of (9) is equal to $U(\xi)$ which indicates the traveling wave solutions of KPP equation.

According to the qualitative theory of differential equations if we can find two first independent integrals of system (9), then the general solutions of (9) can be expressed explicitly and so can all kinds of traveling wave solutions of KPP equation. However, it is generally difficult to find even one of the first integrals. Because there is not any systematic way to tell us how to find these integrals. So, our aim is to obtain at least one first integral of system (9). To do this, we will apply the Division Theorem which is based on the Hilbert-Nullsellensatz Theorem [10]. Now, we recall the Division Theorem for two variables in the complex domain $\mathbb{C}$.

Division Theorem. *Suppose that P(w,z) and Q(w,z) are polynomials in* $\mathbb{C}[w,z]$ *and P(w,z) is irreducible in* $\mathbb{C}[w,z]$*; if Q(w,z) vanishes at all zero points of P(w,z), then there exist a polynomial H(w,z) in* $\mathbb{C}[w,z]$ *such that,*

$$Q(w,z) = P(w,z)H(w,z).$$

According to the first integral method, we assume that $(X(\xi), Y(\xi))$ is a non-trivial solution of (9) and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i \tag{12}$$

is an irreducible polynomial in the complex domain $\mathbb{C}$ such that

$$Q(X(\xi),Y(\xi))=\sum_{i=0}^{m}a_i(X(\xi))Y(\xi)^i=0 \tag{13}$$

where $a_i(X)$ $(i=0,1,...,m)$ are polynomials of X and $a_m(X)\neq 0$. Equation (12) is called the first integral of (9). According to the Division Theorem, there exists a polynomial $g(X)+h(X)Y$ in the complex domain $\mathbb{C}$ such that

$$\frac{dQ}{d\xi}=\frac{\partial Q}{\partial X}\frac{dX}{d\xi}+\frac{\partial Q}{\partial Y}\frac{dY}{d\xi}=(g(X)+h(X)Y)\sum_{i=0}^{m}a_i(X)Y^i. \tag{14}$$

We consider two different cases for (12) $m=1$ and $m=2$.

Case 1. $m=1$

Equating the coefficients of Y^i on both sides of equation (14), we have

$$a_1'(X) = h(X)a_1(X), \tag{15a}$$
$$a_0'(X) = (\omega+g(X))a_1(X)+h(X)a_0(X), \tag{15b}$$

$$a_1(X)[\mu X+\nu X^2+\delta X^3]=g(X)a_0(X). \tag{15c}$$

Since $a_i(X)$ are polynomials, from (15a) we deduce that $a_1(X)$ is constant and $h(X)=0$. For simplification we take $a_1(X)=1$. Hence (15) can be rewritten as

$$a_0'(X)=\omega+g(X), \qquad (16a)$$
$$\mu X+\nu X^2+\delta X^3=g(X)a_0(X) \qquad (16b)$$

Balancing the degrees of $a_0(X)$ and $g(x)$, we conclude that $\deg g(X)=1$ only. Assume that

$$g(X)=AX+B \tag{17}$$

where $A,\ B\in\mathbb{C}$. Then, from (16a)

$$a_0(X)=\frac{A}{2}X^2+(B+\omega)X+C \tag{18}$$

where C is an arbitrary integration constant. Substituting (17) and (18) into (16b) and setting all coefficients of X^i $(i=0,1,2,3)$ to be zero, we obtain

$$A_1=\sqrt{2\delta},\qquad B_1=\frac{2\nu}{3\sqrt{2\delta}}-\frac{2\omega}{3},\qquad C=0,\qquad \mu_1=\frac{2\nu^2}{9\delta}-\frac{2\nu\omega}{9\sqrt{2\delta}}-\frac{2\omega^2}{9} \tag{19a}$$

$$A_1=-\sqrt{2\delta},\qquad B_1=-\frac{2\nu}{3\sqrt{2\delta}}-\frac{2\omega}{3},\qquad C=0,\qquad \mu_2=\frac{2\nu^2}{9\delta}+\frac{2\nu\omega}{9\sqrt{2\delta}}-\frac{2\omega^2}{9}. \tag{19b}$$

Using the conditions (19a-b) in equation (13), we have

$$Y+\frac{\sqrt{2\delta}}{2}X^2+(\frac{2\nu}{3\sqrt{2\delta}}+\frac{\omega}{3})X=0 \tag{20a}$$

and

$$Y - \frac{\sqrt{2\delta}}{2} X^2 + (-\frac{2\nu}{3\sqrt{2\delta}} + \frac{\omega}{3})X = 0. \tag{20b}$$

Solving Eqs. (20a) and (20b) with subject to Y and substituting them into Eq. (9), we obtain the following exact solutions of KPP equation, respectively,

$$u_1(x,t) = (\frac{\nu}{3\delta} + \frac{\omega}{3\sqrt{2\delta}})[\coth(\frac{\nu}{3\sqrt{2\delta}} + \frac{\omega}{6})(x - \omega t + \xi_0) - 1] \tag{21}$$

$$u_2(x,t) = (\frac{\nu}{3\delta} - \frac{\omega}{3\sqrt{2\delta}})[\coth(-\frac{\nu}{3\sqrt{2\delta}} + \frac{\omega}{6})(x - \omega t + \xi_0) - 1] \tag{22}$$

where ξ_0 is an arbitrary constant.

Case 2. $m = 2$.

By equating the coefficients of Y^i on both sides of (14) we have

$$\begin{aligned}
&a_2'(X) = h(X)a_2(X), && (23a)\\
&a_1'(X) = (2\omega + g(X))a_2(X) + h(X)a_1(X), && (23b)\\
&a_0'(X) = -2a_2(\mu X + \nu X^2 + \delta X^3) + (\omega + g(X))a_1(X) + h(X)a_0(X), && (23c)\\
&a_1(X)[\mu X + \nu X^2 + \delta X^3] = g(X)a_0(X). && (23d)
\end{aligned}$$

Since $a_i(X)$ are polynomials, from (23a), we deduce that $a_2(X)$ is constant and $h(X) = 0$. Again, let us take $a_2(X) = 1$. Thus the system can be rewritten as follow

$$\begin{aligned}
&a_1'(X) = 2\omega + g(X), && (24a)\\
&a_0'(X) = -2(\mu X + \nu X^2 + \delta X^3) + (\omega + g(X)a_1(X), && (24b)\\
&a_1(X)[\mu X + \nu X^2 + \delta X^3] = g(X)a_0(X). && (24c)
\end{aligned}$$

Balancing the terms of $a_0(X)$, $a_1(X)$ and $g(X)$, we conclude that either $\deg g(X) = 0$ or $\deg g(X) = 1$.

Let us consider the case of $\deg g(X) = 0$, that is,

$$g(x) = A \tag{25}$$

where $A \neq 0$. Then, from (24a-b), we get

$$a_1(X) = (2\omega + A)X + B, \tag{26}$$

$$a_0(X) = -\frac{\delta}{2}X^4 - \frac{2\nu}{3}X^3 + [\omega^2 + \frac{\omega A}{2} - \mu + \omega A + \frac{A^2}{2}]X^2 + (B\omega + AB)X + C \tag{27}$$

where B and C are integration constants. Let us substitute $a_0(X)$, $a_1(X)$ and $g(X)$ into (24c) and equate the all coefficients of X^i $(i = 0, 1, 2, 3, 4)$ to the zero. Therefore, it follows

$$A = -\frac{6\omega}{5}, \quad B = 0, \quad \mu = -\frac{6\omega^2}{25}, \quad \delta = 0, \quad C = 0. \tag{28}$$

Combining (28), (12) and (9), we find two differential equations as

$$X' + \frac{2\omega}{5}X + \sqrt{\frac{2\nu}{3}}X^{3/2} = 0, \tag{29a}$$

$$X' + \frac{2\omega}{5}X - \sqrt{\frac{2\nu}{3}}X^{3/2} = 0. \tag{29b}$$

These equations have the following solutions, respectively,

$$X(\xi) = \frac{\frac{4\omega^2}{25\nu}}{(-\sqrt{\frac{2\nu}{3}} + e^{\frac{2\omega}{5}(\xi+\xi_0)})^2}, \tag{30a}$$

$$X(\xi) = \frac{\frac{4\omega^2}{25\nu}}{(\sqrt{\frac{2\nu}{3}} + e^{\frac{2\omega}{5}(\xi+\xi_0)})^2}. \tag{30b}$$

By

$$\frac{e^\eta}{1+e^\eta} = \frac{1}{2}[\tanh\frac{\eta}{2} + 1] \text{ and } \frac{e^\eta}{1-e^\eta} = -\frac{1}{2}[\coth\frac{\eta}{2} + 1],$$

the above solutions (30a) and (30b) that are the solitary wave solutions of KPP equation with $\delta = 0$ can be rewritten as, respectively,

$$u_3(x,t) = \frac{3\omega^2}{50\nu}(\coth\frac{\omega}{10}(x - \omega t + \xi_0) - 1)^2 \tag{31a}$$

$$u_4(x,t) = \frac{3\omega^2}{50\nu}(\tanh\frac{\omega}{10}(x - \omega t + \xi_0) - 1)^2 \tag{31b}$$

where ξ_0 is an arbitrary constant.
We note that in the case of $\delta = 0,\ \mu = -1,\ \nu = 1$, the KPP equation reduces to Fisher equation. Hence from (31a-b), some exact solutions of Fisher equation are obtained as follows

$$u(x,t) = \frac{1}{4}[\coth(\frac{x}{2\sqrt{6}} \pm \frac{5}{12}t + \xi_0) \pm 1]^2$$

$$u(x,t) = \frac{1}{4}[\tanh(\frac{x}{2\sqrt{6}} \pm \frac{5}{12}t + \xi_0) \pm 1]^2.$$

Now we assume that $\deg g(X) = 1$; that is, $g(X) = AX + B$. Then, from (24a-b) we find

$$a_1 = \frac{A}{2}X^2 + (B + 2\omega)X + C, \tag{32a}$$

$$a_0 = (\frac{A^2}{8} - \frac{\delta}{2})X^4 + (\frac{5A\omega}{6} - \frac{2\nu}{3} + \frac{AB}{2})X^3 \tag{32b}$$
$$+(\frac{3B\omega}{2} + \omega^2 - \mu + \frac{AC}{2} + \frac{B^2}{2})X^2 + (C\omega + BC)X + D$$

where C, D are arbitrary integration constants. Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ into (24c) and setting all the coefficients of powers X to be zero, we obtain the following nonlinear algebraic system

$$\begin{aligned}
&\tfrac{A\delta}{2} = \tfrac{A^3}{8} - \tfrac{A\delta}{2}\\
&\tfrac{A\nu}{2} + (B+2\omega) = A(\tfrac{5A\omega}{6} - \tfrac{2\nu}{3} + \tfrac{AB}{2}) + B(\tfrac{A^2}{8} - \tfrac{\delta}{2})\\
&\tfrac{A\mu}{2} + (B+2\omega)\nu + C\delta = A(\tfrac{3B\omega}{2} + \omega^2 - \mu + \tfrac{AC}{2} + \tfrac{B^2}{2}) + B(\tfrac{5A\omega}{6} - \tfrac{2\nu}{3}\tfrac{AB}{2})\\
&(B+2\omega)\mu + C\nu = AC(\omega+B) + B(\tfrac{3B\omega}{2} + \omega^2 - \mu + \tfrac{AC}{2} + \tfrac{B^2}{2})\\
&C\mu + AD + BC(\omega+B) = 0\\
&BD = 0
\end{aligned}$$

which has the solution

$$A = \pm 2\sqrt{2\delta}, \quad B = \frac{\nu A}{3\delta} - \frac{4\omega}{3}, \quad C = 0, \quad D = 0, \quad \mu = \frac{2\nu^2}{9\delta} - \frac{2\omega^2}{9} - \frac{2\nu\omega}{9A}. \quad (33)$$

Putting (33) into (13), we obtain the same equations as (20a) and (20b). So we have the same exact solutions as (21) and (22).

5. Conclusion

In this work, we showed that the zero solution $u(x,t) = 0$ of KPP equation is asymptotically stable if $\omega > 0$ and $\mu < 0$ and it is unstable if either $\omega < 0$ or $\mu > 0$. After that we proved that KPP equation has no periodic solution. Finally, we obtained some new exact traveling wave solutions of KPP equation that are different from those in [5-8]. For a verification of Theorem 4, let us choose the parameters $\omega,\ \nu,\ \delta$ and μ as $\omega = 1,\ \nu = 1,\ \delta = 2,\ \mu = -\frac{2}{9}$. Then from (21), we have the solution $u_1(x,t) = -\frac{1}{3} + \frac{1}{3}\coth(\frac{x-t}{3})$ which is plotted in Fig. 2. This solution goes to the zero as $x - t \to \infty$. This case is agree with the asymptotic stability of the zero solution. Indeed, the values $\omega = 1,\ \mu = -\frac{2}{9}$ come from the asymptotic stability region VI.

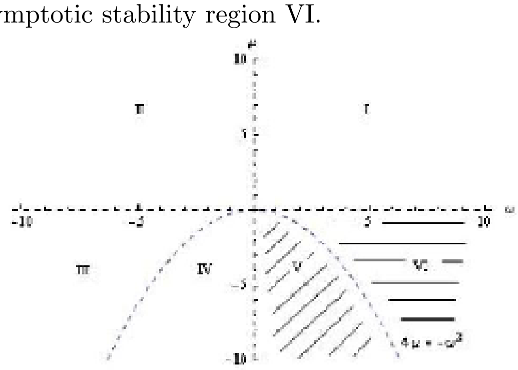

Fig.1 Regions of stability

References

[1] Abbasbandy S., Shirzadi A., The first integral method for modified Benjamin-Bona-Mahony equation, Commun. Nonlinear Sci. Numer. Simul., 15 (2010),1759–1764.
[2] Adomian G., The generalized Kolmogorov-Petrovskii-Piskunov equation, Foundation of Pyhsics Letters, 8 (1995), 99-101.
[3] Ali A. H. A., Raslan K. R., The first integral method for solving a system of nonlinear partial differential equations, Int. J. Nonlinear Sci., 5 (2008),111–119.
[4] Allen L. J. S., An Introduction to Mathematical Biology, 2007, Pearson.
[5] Branco J.R., Ferreira J.A., Oliveira P. Numerical methods for the generalized Fisher–Kolmogorov–Petrovskii–Piskunov equation, Applied Numerical Mathematics, 57 (2007), 89-102.
[6] Britton N. F. Reaction-Diffusion Equations and Their Applications to Biology, 1986, Academic Press, New York.
[7] Deng X., Exact peaked wave solution of CH-γ equation by the first-integral method, Appl. Math. Comput., 206 (2008), 806–809.
[8] Feng J., Li W., Wan Q., Using $\frac{G'}{G}$-expansion method to seek the traveling wave solution of Kolmogorov–Petrovskii–Piskunov equation, Applied Mathematics and Computation, 217 (2011), 5860-5865.
[9] Feng Z.S., The first-integral method to the Burgers–KdV equation, J. Phys. A, 35 (2002), 343–350.
[10] Hong W. P., Jung Y.D., Auto-bäclund transformation and analytic solutions for general variable coefficient KdV equation, Phys. Lett. A, 257 (1999), 149–152.
[11] Khater A.H., Malfliet W., Callebaut D.K, Kamel E.S., The tanh method, a simple transformation and exact analytical solutions for nonlinear reaction diffusion equations, Chaos Solitons Fractals, 14(3) (2002), 513 - 522.
[12] Kolmogorov A. N., Petrovskii I. G., Piskunov N. S., Etude de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Moscow Univ. Math. Bull., 1 (1937), 1–25.
[13] Liu C., The relation between the kink-type solution and the kink-bell-type solution of nonlinear evolution equations, Physics Letters A, 312 (2003), 41-48.
[14] Lu B., Zhang H.Q., XIE F.D., Travelling wave solutions of nonlinear partial equations by using the first integral method, Appl. Math. Comput., 216 (2010),1329-1336.
[15] Ma W. X., Fuchssteiner B., Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation, Int. J. Non-Linear Mech., 31 (1996), 329-338.
[16] Murray J. D., Mathematical Biology I: An Introduction, 2002, Springer, Berlin.
[17] Öğün A., Kart C., Exact solutions of Fisher and generalized Fisher equations with variable coefficients, Acta Math. Appl. Sin. Engl Ser., 23 (2007), 563-568.
[18] Raslan K. R., The first integral method for solving some important nonlinear partial differential equations, Nonlinear Dynam., 53 (2008), 281–286.
[19] Sımmons G. F., Differential Equations, 1989, McGraw-Hill, New York, pp. 341.
[20] Taghizadeh N., Mirzazadeh M., Farahrooz F., Exact solutions of the nonlinear Schrödinger equation by the first integral method, J. Math. Anal. Appl., 374 (2011), 549–553.

Current address: Arzu Öğün Ünal; Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY

E-mail address: aogun@science.ankara.edu.tr

URL: http://communications.science.ankara.edu.tr/index.php?series=A1