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CONE CONVERGENCE FOR MULTIPLE SEQUENCES*

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Abstract. The aim of this paper is to introduce a new type convergence which is useful when a d-multiple sequence is not convergent in some usual senses.

1. Introduction

Main purpose of this paper is to introduce a new type convergence which especially can be thought to be useful when a multidimensional sequence is not convergent. Though the new idea could be, explained in and applied to many subjects of functional analysis including multiple sequences related to convergence types such as statistical convergence, ideal convergence and to matrix transformations between sequence spaces and so on. For the sake of clarity we introduce this notion in some plain part of the notion of statistical convergence.

Let $\mathbb{N}^d$ be the set of d-tuples $\mathbf{k} := (k_1, k_2, \ldots, k_d)$ with nonnegative integers for coordinates k_j, where d is a fixed positive integer. Note that $\mathbb{N}^d$ is partially ordered by agreeing $\mathbf{k} \leq \mathbf{n}$ if and only if $k_j \leq n_j$ for each integer j (see [8]). A function $x : \mathbb{N}^d \to \mathbb{R}(\mathbb{C})$ is called a real (complex) d-multiple sequence. If $d = 2$ then a function $x : \mathbb{N}^2 \to \mathbb{R}(\mathbb{C})$ is called a real (complex) double sequence. The definition of the convergence of double sequences was given by Pringsheim in [3]. Remember that a double sequence (x_{nk}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{nk} - L| < \varepsilon$ whenever $n, k \geq N(\varepsilon)$ [2, 4, 5, 6].

The idea of statistical convergence was first presented by Fast in [1] . The notion of statistically convergent double sequences has been studied by many authors (see for instance, [5, 6, 7, 8]). Regarding these works, to be adopted to the definition of

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the density of a subset E of $\mathbb{N}^2$, the density $\rho(E)$ of any subset of $E \subseteq \mathbb{N}^d$ can be given by

$$\rho(E) = \lim_{\min K_i \to \infty} \frac{1}{K_1 \cdots K_d} \sum_{k_1 \leq K_1} \cdots \sum_{k_d \leq K_d} \chi_E(k_1, \ldots, k_d), \quad (i = 1, 2, \ldots, d)$$

provided the limit exists.

Now to recall the definition of a cone let $\mathbb{R}^d_{\geq}$ denote the set of d-tuples $\mathbf{x} := (x_1, x_2, \ldots, x_d)$ with nonnegative reals for coordinates x_j. Suppose that $\mathbf{x_1} := (x_{\mathbf{11}}, x_{\mathbf{12}}, \ldots, x_{\mathbf{1}d})$, $\mathbf{x_2} := (x_{\mathbf{21}}, x_{\mathbf{22}}, \ldots, x_{\mathbf{2}d})$, ..., $\mathbf{x_d} := (x_{\mathbf{d1}}, x_{\mathbf{d2}}, \ldots, x_{\mathbf{dd}}) \in \mathbb{R}^d_{\geq}$ are given such that $\mathbf{x_i}$ and $\mathbf{x_j}$ are not co-linear for $\mathbf{i} \neq \mathbf{j}$. Then the set

$$\sigma = \mathbb{R}^d_{\geq}\mathbf{x_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{x_d} = \left\{\boldsymbol{\alpha}_1\mathbf{x_1} + \cdots + \boldsymbol{\alpha}_d\mathbf{x_d} : \ \boldsymbol{\alpha}_i \in \mathbb{R}^1_{\geq} \text{ for } i = 1, \ldots, d\right\}$$

is called the cone generated by $\mathbf{x}_1, \mathbf{x_2}, \ldots, \mathbf{x_d}$. A cone is said to be pointed if it includes the null vector $\mathbf{0}$.

For a given cone $\sigma = \mathbb{R}^d_{\geq}\mathbf{x_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{x_d}$ and $\mathbf{u} := (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d_{\geq}$, the shift of σ with respect to $\mathbf{u}$ is defined to be the set $\mathbf{u} + \sigma$.

2. Main results

Definition 2.1. Let $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ be a d -multiple sequence of (real or complex) numbers and $\sigma = \mathbb{R}^d_{\geq}\mathbf{s_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{s_d}$ be a fixed cone. Then the d-multiple sequence $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is called σ- Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $|x_{\mathbf{n}} - x_{\mathbf{k}}| < \varepsilon$ whenever $\mathbf{n} \geq \mathbf{k} \geq \mathbf{N}$ and $\mathbf{n}, \mathbf{k} \in \sigma$.

Definition 2.2. A d-multiple sequence $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is said to be σ-bounded if there exists $M > 0$ such that $|x_{\mathbf{k}}| < M$ for all $\mathbf{k} \in \sigma$.

Definition 2.3. Let $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ be a d -multiple sequence of (real or complex) numbers and $\sigma = \mathbb{R}^d_{\geq}\mathbf{s_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{s_d}$ be a fixed cone. Then the d-multiple sequence $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is called σ- convergent to a number L if for each $\varepsilon > 0$ there exists $\mathbf{N} \in \mathbb{N}^d$ such that $|x_{\mathbf{k}} - L| < \varepsilon$ whenever $\mathbf{k}\,(\in \sigma) \geq \mathbf{N}$.

If $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is σ-convergent to a real number L we denote this by $\sigma - \lim\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = L$ or $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma} L$.

Note that every double sequence, which is convergent in Pringsheim's sense, is convergent with respect to the fixed cone $\sigma = \mathbb{R}^2_{\geq}(1, 0) + \mathbb{R}^2_{\geq}(0, 1)$. More generally, every d-multiple sequence, which is convergent in Pringsheim's sense, is convergent with respect to the fixed cone $\sigma = \mathbb{R}^d_{\geq}(1, 0, \ldots, 0) + \mathbb{R}^d_{\geq}(0, 1, , 0, \ldots, 0) + \cdots + \mathbb{R}^d_{\geq}(0, 0, , 0, \ldots, 1)$.

Example 2.4. Let $\sigma = \mathbb{R}^2_{\geq}(1, 0) + \mathbb{R}^2_{\geq}(1, 1)$ and

$$x_{\mathbf{k}:=(k_1,k_2)} = \begin{cases} k_1 & , \quad k_1 \leq k_2, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^2\right) \xrightarrow{\sigma} 0$. On the other hand it is obvious that this double sequence is not convergent in Pringsheim's sense.

Due to simplicity, the proofs of the following proposition and some of the theorems are omitted.

Proposition 1. *If* $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ *is* σ*-convergent then its limit is unique.*

Theorem 2.5. *If a d-multiple sequence is* σ*-convergent then it is* σ *-bounded. But, the converse of this is not true in general.*

Theorem 2.6. *Let* $\sigma - \lim\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = L_1$ *and* $\sigma - \lim\left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = L_2$. *Then,* $\sigma - \lim\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = L_1 + L_2$ *and* $\sigma - \lim\left(c\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)\right) = cL$ *for all scalars c.*

Lemma 2.7. *If* σ_1 *and* σ_2 *are any two pointed cones and* $\sigma_3 = \sigma_1 \cap \sigma_2 \neq \phi$ *then* σ_3 *is also a pointed cone.*

Using Lemma 2.7, we have the following:

Theorem 2.8. *If* $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_1} a$, $\left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_2} b$ *and* $\sigma_3 = \sigma_1 \cap \sigma_2 \neq \emptyset$ *has non-empty interior then* $\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_3} a + b$.

Remark 2.9. If $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_1} a$, $\left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_2} b$ and $\sigma_3 = \sigma_1 \cap \sigma_2$ has an empty interior then we may not have $\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \xrightarrow{\sigma_3} a + b$ in general for any pointed cone σ_3.

We can see this by the following example.

Example 2.10. Let $\sigma_1 = \mathbb{R}^2_{\geq}(1,2) + \mathbb{R}^2_{\geq}(0,1)$, $\sigma_2 = \mathbb{R}^2_{\geq}(1,1) + \mathbb{R}^2_{\geq}(1,0)$ and

$$x_{\mathbf{k}:=(k_1,k_2)} = \begin{cases} 1 & , \quad k_1 \geq 2k_2, \\ 0 & , \quad \text{otherwise}, \end{cases} \qquad y_{\mathbf{k}:=(k_1,k_2)} = \begin{cases} 2 & , \quad k_1 \leq k_2, \\ 0 & , \quad \text{otherwise}. \end{cases}$$

Then

$$x_{\mathbf{k}:=(k_1,k_2)} + y_{\mathbf{k}:=(k_1,k_2)} = \begin{cases} 1 & , \quad k_1 \geq 2k_2 \\ 0 & , \quad k_2 < k_1 < 2k_2 \\ 2 & , \quad k_1 \leq k_2 \end{cases}$$

and we have $\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^2\right) \xrightarrow{\sigma_1} 1$, $\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^2\right) \xrightarrow{\sigma_2} 2$ and $\left(x_{\mathbf{k}} + y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^2\right) \xrightarrow{\sigma_3} 0$, where $\sigma_3 = \mathbb{R}^2_{\geq}(1,2) + \mathbb{R}^2_{\geq}(1,1)$.

Definition 2.11. A subset E of $\mathbb{N}^d$ is said to have density $\rho_\sigma(E)$ with respect to the fixed cone $\sigma = \mathbb{R}^d_{\geq}\mathbf{x_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{x_d}$ if the following limit exists.

$$\rho_\sigma(E) = \lim_{\min K_i \to \infty} \frac{1}{K_1 \cdots K_d} \sum_{k_1 \leq K_1} \cdots \sum_{k_d \leq K_d} \chi_E(k_1, \ldots, k_d);$$

where $K_i, k_i \in \sigma$ with $i = 1, 2, \ldots, d$.

Definition 2.12. A d-multiple sequence $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is said to be σ-statistically convergent to L if for every $\varepsilon > 0$, $\rho_\sigma\left(\{(k_1, \ldots, k_d) : |x_{k_1 \ldots k_d} - L| \geq \varepsilon\}\right) = 0$.

Definition 2.13. Let $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ and $y = \left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ be two d -multiple sequences and $\sigma = \mathbb{R}^d_{\geq}\mathbf{x_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{x_d}$ be a fixed cone. Then we say that $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = \left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ for almost all $\mathbf{k} \in \sigma$ if

$$\delta_d\left(\left\{\mathbf{k} \in \mathbb{N}^d \cap \sigma : \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \neq \left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)\right\}\right) = 0.$$

Definition 2.14. Let $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ be a d -multiple sequence. A subset D of $\mathbb{R}^d$ said to contain $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ for almost all $\mathbf{k}$ if

$$\delta_d\left(\left\{\mathbf{k} \in \mathbb{N}^d \cap \sigma : \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) \notin D\right\}\right) = 0.$$

Theorem 2.15. *A d-multiple sequence $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is σ-statistically convergent if and only if it is σ-statistically Cauchy.*

Proof. Since the necessity is obvious, we only prove the sufficiency. Let $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ be a σ- statistically Cauchy sequence. Choose $\varepsilon = 1$, then there exist $k_1^1, k_2^1, \ldots, k_d^1$ such that the closed circle U_1 of diameter 2 units with center at $k_1^1, k_2^1, \ldots, k_d^1$ contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. Now for $\varepsilon = 1/2$ there exist $k_1^2, k_2^2, \ldots,$ k_d^2 such that the closed circle U^2 of diameter 1 unit with center at $x_{k_1^2 k_2^2 \cdots k_d^2}$ contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. Take $U_2 = U_1 \cap U^2$ then U_2 which is closed subset of $\mathbb{R}^d$ with diameter less than or equal to 1 unit such that U_2 contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. Take $\varepsilon = 2^{-2}$, then there exist $k_1^3, k_2^3, \ldots, k_d^3$ such that the closed circle U^3 of diameter 1/2 unit with center at $x_{k_1^3 k_2^3 \cdots k_d^3}$ contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. If we choose $U_3 = U_2 \cap U^3$ then U_3 is closed subset of $\mathbb{R}^d$ with diameter less than or equal to 1/2 unit such that U_3 contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. Following this way, we have a sequence (U_n) of closed subsets of $\mathbb{R}^d$ such that

(i) $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$.
(ii) $diam U_n \leq 2^{2-n}$ for all $n \in \mathbb{N}$.

Then $\bigcap_{n=1}^{\infty} U_n$ contains one point. Let us call this point as L. Then $L \in U_n$ for all $n \in \mathbb{N}$. If we choose m such that $2^{-m} < \varepsilon$ then U_n contains $x_{\mathbf{k}}$ for almost all $\mathbf{k} \in \sigma$. This means $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is statistically convergent to L. □

Now we are ready to give the following cone d-multiple analogues of the result in [7].

Theorem 2.16. *Let $x = \left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is a d -multiple sequence and $\sigma = \mathbb{R}^d_{\geq}\mathbf{x_1} + \cdots + \mathbb{R}^d_{\geq}\mathbf{x_d}$ be a fixed cone. Then the following statements are equivalent:*

(i) $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is σ-statistically convergent to ℓ.
(ii) $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ is σ-statistically Cauchy.
(iii) There exists a subsequence $\left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ of $\left(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right)$ such that $\sigma - \lim\left(y_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d\right) = \ell$.

Conclusion

As is mentioned at the beginning of the article this new type convergence can be applied to many subjects of functional analysis including multiple sequences related to convergence types such as statistical convergence, ideal convergence and so on and to matrix transformations between sequence spaces including multiple sequences. So, application area of this new type convergence is enormous for further works.

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