Commun.Fac.Sci.Univ.Ank.Series A1
Volume 62, Number 2, Pages 17–27 (2013)
ISSN 1303–5991

WATADA'S FUZZY PORTFOLIO SELECTION MODEL AND ITS APPLICATION

GULTAC EROGLU INAN, AYSEN APAYDIN

Abstract. Portfolio selection has been originally proposed by H.M. Markowitz 1952. The Markowitz's approcach to the portfolio selection has some difficulties. For example, an aspiration level given by decision makers aren't taken into consideration in the Markowitz approach. In this paper, Watada's Fuzzy Portfolio Selection Model Based on Decision Maker's Aspiration Level is presented basically. A numerical example of the Watada's Portfolio Selection Problem is given as the application. Data is taken as the closing prices of eight securities that cycling in İstanbul Stock Exchange (IMKB) between the dates of 01.06.2009-01.06.2010. Finally, it can be said that,the aspiration levels have been provided with a high membership degree so we can say that the successfull portfolio has been formed. During the study, Excel, Minitab and Lingo software programmes were used.

1. Introduction

A major step in the direction of the quantitative management of portfolio was made by Harry Markowitz in his paper "Porftolio Selection" published in 1952 in the Journal of Finance. The ideas introduced in this article have come to build the foundations of what is now popularly referred to as mean-variance analysis, mean-variance optimization, and Modern Portfolio Theory.

Markowitz reasoned that investors should decide on the basis of a trade-off between risk and ecpected retun. Expected return of a security is defined as the expected price change plus any additional income over the time horizon considered, such as dividend payments, divided by the beginning price of the security. He suggested that risk should be measured by the variance of returns-the average squared deviation around the expected return. He argued that for any given level of expected return, a rational investor would choose the portfolio with minimum variance from amongst the set of all possible portfolios. (Fabozzi et al 2007).

Received by the editors Oct. 5, 2012; Accepted: Sep. 8, 2013.
2000 *Mathematics Subject Classification.* Primary 91B28; Secondary 90C70.
Key words and phrases. Portfolio Analysis, Portfolio Risk, Expected Return, Fuzzy Set, Fuzzy Portfolio Selection.

The term "fuzzy" was proposed by Zadeh (1962). Zadeh (1965) formally published the famous paper "Fuzzy Sets". The fuzzy set theory is developed to improve the oversimplified model, thereby developing a more robust and flexible model in order to solve real-world complex systems involving human aspects. Furthermore, it helps the decision maker not only to consider the existing alternatives under given constraints (optimize a given system), but also to develop new alternatives. The fuzzy set theory has been applied in many fields, such as operations research, management science, control theory, expert system, human behavior (Lai-Hwang 1992).

An investor is faced with a choice from an enormous number of assets, such as stocks and bonds. It seems very difficult to decide which securities should be selected because of the inherent existence of uncertainty. Similar to other decision problems in an uncertainty environment, portfolio selection problems can also be modeled as fuzzy programming problems (Tanaka and Guo1998).

Tanaka et al (2000) proposed the two kinds of portfolio selection models based on fuzzy probabilities and possibility distributions rather than probability distributions in Markowitz's model.They gave a numerical example of a portfolio selection probem. In the study fuzzy probabilities and possibility distributions were obtained depending on possibility grades associated with security data. Based on a fuzzy probability and a possibility distribution, portfolios were selected to minimize the variance of the return of a portfolio in a fuzzy probability model and the spread of the return of a portfolio in a possibility model.

Watada (2001) formed a portfolio selection model by using fuzzy decision theory. This model is directly related to mean–variance model. First a decision maker defines, for each of an expected return and risk, a necessity level which requires all feasible solutions should hold, that is, a minimum requirement and a sufficiency level at which a decision maker is satisfied with a solution. Then membership functions are constructed for each of an expected return and risk.

Huang (2007)* made a new definition of risk for random fuzzy portfolio selection. The aim of the study is to solve the portfolio selection problem when security returns contain both randomness and fuzziness.In the study a new optimal portfolio selection model is proposed and a new hybrid intelligent algorithm is designed for solving this new problem.

Huang (2007)** proposed the two new types of fuzzy mean variance models. based on credibility measure. In the study, security returns were regarded as fuzzy variables.He provided a hybrid intelligent algorithm to give a general solution. As result, it was shown that the proposed model was effective by the numerical examples.

Hasuike et al (2009) formulated random fuzzy portfolio selection problems as nonlinear programming problems based on both stochastic and fuzzy programming approaches. In the study several portfolio selection problems including probabilistic future returns with ambiguous expected returns are assumed as random fuzzy variables. They transformed the problems into equivalent deterministic quadratic programming problems. They used the probabilistic change constraints, possibility measure and fuzzy goals.

2. Portfolio Analysis

In 1952, Markowitz published his pioneering work which paved the foundation of the modern portfolio analysis. It combines probability and optimization techniques to model the behaviour of investment under uncertainty. The investors are assumed to strike a balance between maximizing the return and minimizing the risk of their investment. The return is quantified by the mean, and the risk is charecterized by the variance, of a portfolio of assets. The two objectives of an investor are thus to maximize the expected value of return and to minimize the variance of a porrtfolio.

The return of a portfolio is equal to the weighted mean of securities in the portfolio. The weight of each return is a ratio of security in a portfolio. The return of a portfolio is defined as,

$$\begin{aligned} r_p &= \sum_{i=1}^{n} R_i X_i = R_1X_1 + R_2X_2 + ... + R_nX_n \\ &= (R_1, ..., R_n) \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \qquad (2.1) \\ &= R^t X \end{aligned}$$

Where R_i is the return of i^{th} security $(i = 1, 2, ..., n)$ and X_i is the weight of the i^{th} security respectively.

Expected return of a portfolio is defined as

$$\begin{aligned} E(r_p) &= E\left(\sum_{i=1}^{n} R_i X_i\right) = \sum_{i=1}^{n} r_i X_i \\ &= r_1X_1 + r_2X_2 + ... + r_nX_n \qquad (2.1) \\ &= (r_1, ..., r_n) \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \qquad (2.2) \\ &= r^t X \end{aligned}$$

The variance of the portfolio is defined as,

$$\begin{aligned} Var\left(r_p\right) &= Var\left(X^t R\right) \\ &= X^t \Sigma X \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j Cov\left(ij\right) \end{aligned} \tag{2.3}$$

Then portfolio risk is as follows

$$\sigma_p = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j Cov\left(ij\right)} \tag{2.4}$$

(Wang and Zhu 2002).

3. Markowitz's Portfolio Selection Model

Markowitz considered possibility and optimization techniques together for modelling the behaviour of a model under uncertainty. Return matrix, mean vector and variance-covariance matrix are fundamental components in Markowitz model.

Assume that there are n securities denoted by $S_i\,(i = 1, ..., n)$, the return of the security S_i is denoted as R_i and the proportion of total investment funds devoted to this security is denoted as X_i.Thus $\sum_{i=1}^{n} X_i = 1$.

Since $R_i\,(i = 1, ..., n)$vary from time to time, those are assumed to be random variables which can be represented by the pair of the average vector and covariance matrix. At the discrete time $(k = 1, ..., m)$ (for m period) n kinds of returns are denoted as a vector

$$\left(R_{k1}, ..., R_{kn}\right)^t, \quad k = 1, ..., m$$

The total data over m periods are denoted as the following the return matrix

$$R^t = \begin{pmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{m1} & \cdots & R_{mn} \end{pmatrix} \tag{3.1}$$

where R_{ki} is the weight of $i.\,(i = 1, ..., n)$ securities in $k.\,(k = 1, ..., m)$ period.

Then average vector of returns over in m period is denoted as $\mu = \left(\mu_1, ..., \mu_n\right)^t$ and is written as

$$\mu = \begin{pmatrix} \sum_{k=1}^{m} \frac{R_{k1}}{m} \\ \vdots \\ \sum_{k=1}^{m} \frac{R_{kn}}{m} \end{pmatrix} \tag{3.2}$$

Also the corresponding variance covariance matrix $S = \left[S_{ij}^2\right]$ can be written as

$$S_{ij}^2 = \sum_{k=1}^{m} \frac{\left(R_{ki} - \mu_i\right)\left(R_{kj} - \mu_j\right)}{m-1} \tag{3.3}$$

Therefore random variables can be represented by the average vector μ and the covariance matrix S, denoted as (μ, S). Since the variance is regarded as the risk of investment, the best investment is one with the minimum variance for a given level of return.This leads to the following quadratic programming problem,

$$\begin{array}{rcl} & & \min X^t S X \\ \mu^t X & \geq & \alpha \\ \sum_{i=1}^{n} X_i & = & 1 \\ X_i & \geq & 0,\ i = 1, 2, ..., n \end{array} \tag{3.4}$$

Where α represents the minimum expected return the investor would accept (Tanaka et. al 2000)

4. Watada Fuzzy Portfolio Selection Model

Fuzzy set theory permits the gradual assessment of the membership of the elements in relation to a set; this is described with the aid of a membership function $\mu_A(x) \rightarrow [0, 1]$. Fuzzy sets are an extension of classical set theory since, for a certain universe, a membership function may act as an indicator function, mapping all elements to either 1 or 0, as in the classical notation.

Many problems in management have mainly been studied from optimizing points of view. As the management is much influenced by the disturbance of a social and economical circumstances, optimization approach is not always the best. It is because under such influences, many problems are ill-structured. Therefore, a satisfaction approach may be much better than an optimization one. In this discussion, it is acceptable that the aspiration level on the treated problem is resolved on the base past experiences and knowledge possessed by a decision maker, in the case where the aspiration level of a decision maker should be considered to solve a problem from the perspective of satisfaction strategy. Therefore, it is more natural that the vague aspiration level of a decision maker is denoted as a fuzzy number.

Watada (2001) presented a portfolio selection model using the fuzzy decision principle. The model is directly related to the mean variance model, where the satisfaction degree for an expected return rate and corresponding risk are described by membership functions. The larger the expected return is, the better its portfolio is.

First a decision maker defines, for each of an expected return rate and risk, a necessity level which requires all feasible solutions should hold, that is, a minimum requirement and a sufficiency level at which a decision maker is satisfied with a solution. Then membership functions are constructed for each of an expected return rate and risk. The larger the expected return rate is, the better its portfolio is. The trapezoidal membership function can be defined as follows

$$\mu_E\left(E\left(r_p\right)\right)=\begin{cases} 1 & , \quad E_U \leq E\left(r_p\right) \\ 1+\frac{E(r_p)-E_U}{E_U-E_L} & , \quad E_L \leq E\left(r_p\right) \leq E_U \\ 0 & , \quad E\left(r_p\right) \leq E_L \end{cases} \tag{4.1}$$

where;

E_L : necessity degree
E_U : sufficiency degree

The trapezoidal membership function of the goal for an expected return rate is given at Figure1.

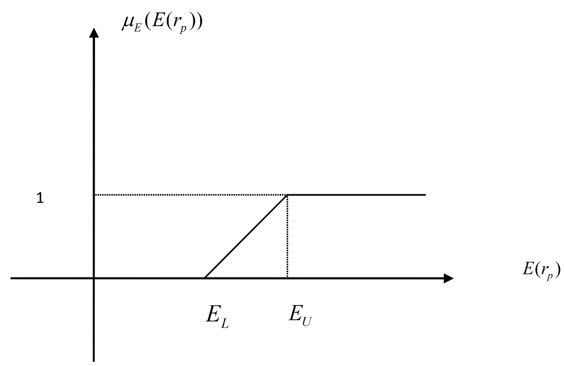

FIGURE 1. The trapezoidal membership function of the goal for an expected return rate (Watada 2001).

The less the grade of risk, the better its portfolio is. The trapezoidal membership function should be defined as

$$\mu_V\left(Var\left(r_p\right)\right)=\begin{cases} 1 & , \quad Var\left(r_p\right) \leq V_U \\ 1-\frac{Var(r_p)-V_U}{V_L-V_U} & , \quad V_U \leq Var\left(r_p\right) \leq V_L \\ 0 & , \quad V_L \leq Var\left(r_p\right) \end{cases} \tag{4.2}$$

where,

V_L : sufficiency degree
V_U : necessity degree

The trapezoidal membership function of the goal for an expected return rate is given at Figure2.

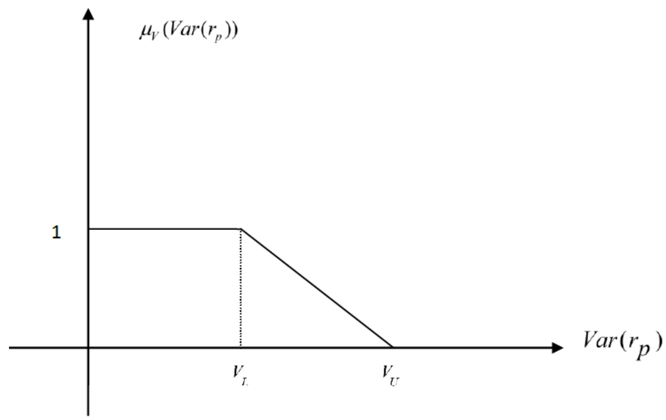

FIGURE 2. The trapezoidal membership function of the goal for risk

Watada (2001).

In terms of Bellman-Zadeh's maximization principle (Bellman-Zadeh 1970), the two objcetive portfolio selection can be written as a fuzzy multi objective mathematical programming problem in the following:

$$\begin{aligned} & \max \lambda \\ \lambda & \leq \mu_E(E(r_p)) \\ \lambda & \leq \mu_V(Var(r_p)) \\ \lambda & \geq 0 \end{aligned} \tag{4.3}$$

$\lambda = \min(\mu_E(E(r_p)),\ \mu_V(Var(r_p)))$ denotes a membership grade. Substituting 4.1 and 4.2 into 4.3, the following optimization problem is obtained

$$\begin{aligned} & \max \lambda \\ Var(r_p) + (V_L - V_U)\lambda & \leq V_L \\ E(r_p) + (E_L - E_U)\lambda & \geq E_L \\ \sum_{i=1}^{n} X_i & = 1 \\ \lambda & \geq 0 \\ X_i & \geq 0 \end{aligned} \tag{4.4}$$

(Watada 2001).

5. Application

During the study,Watada's Fuzzy Portfolio Selection Model Based on Decision Maker aspiration level was used as a method. As the data set, the 253 daily closing prices of eight securities that cycling in Istanbul Stock Exchange 100 index between the dates 01.06.2009-01.06.2010 were taken (Resource : www.imkb.gov.tr). The securities taken from the sectors as health, electronic goods, clothing, beverage(cola) , banking, beverage(water), glass, gsm.Returns of the securities were calculated according to the explanation $\ln(\frac{P_t}{P_{t-1}})$ of the closing prices. Where P_t is defined as,

P_t : "closing price of $t.$ day"

Firstly, correlation matrix was obtained concerning with the securities. When the correlation values were examined, it seemed that there were not securities which were related to each other (with 1 correlation value).

Correlation matrix was calculated as,

$$\rho = \begin{bmatrix} 1 & 0.336 & 0.334 & 0.185 & 0.283 & 0.295 & 0.170 & 0.213 \\ 0.336 & 1 & 0.327 & 0.225 & 0.312 & 0.305 & 0.214 & 0.417 \\ 0.334 & 0.327 & 1 & 0.164 & 0.358 & 0.421 & 0.209 & 0.326 \\ 0.185 & 0.225 & 0.164 & 1 & 0.212 & 0.239 & 0.163 & 0.326 \\ 0.283 & 0.312 & 0.358 & 0.212 & 1 & 0.338 & 0.273 & 0.255 \\ 0.295 & 0.305 & 0.421 & 0.239 & 0.338 & 1 & 0.316 & 0.380 \\ 0.170 & 0.214 & 0.209 & 0.163 & 0.273 & 0.316 & 1 & 0.153 \\ 0.213 & 0.417 & 0.326 & 0.326 & 0.255 & 0.380 & 0.153 & 1 \end{bmatrix}$$

At the next stage, average return vector μ^t, variance covariance matrix S were calculated on the returns. During the calculation, Excel, Minitab and Lingo software programmes were used.

$$\mu^t = [\,0.001754155 \quad 0.004336165 \quad 0.003405774 \quad 0.002120309 \quad 0.001602629 \quad 0.000323686 \quad 0.002306364 \quad 0.000276042\,]$$

$$S = \begin{bmatrix} 0.00046082 & & & & & & & \\ 0.00021381 & 0.00088064 & & & & & & \\ 0.00022677 & 0.00030732 & 0.00100032 & & & & & \\ 0.00009063 & 0.00015232 & 0.00011823 & 0.00052246 & & & & \\ 0.00015011 & 0.00022869 & 0.00027968 & 0.00011990 & 0.00061134 & & & \\ 0.00013997 & 0.00019967 & 0.00029450 & 0.00012092 & 0.00018448 & 0.00048827 & & \\ 0.00011870 & 0.00020714 & 0.00021532 & 0.00012097 & 0.00021930 & 0.00022732 & 0.00105926 & \\ 0.00008533 & 0.00023140 & 0.00019299 & 0.00008673 & 0.00011802 & 0.00015711 & 0.00009292 & 0.00034955 \end{bmatrix}$$

In the study, $E(r_p)$ and $Var(r_p)$ and predicted values were used in order to estimate the Watada's model E_U, E_L, V_U, V_L necessity and sufficiency level. These predicted values were calculated by supposing that the weight of each security was equal in portfolio.

$$\begin{aligned} E\left(r_p\right) &= \frac{1}{8}\left(0.001754155+...+0.000276042\right) \\ &= 0.002016 \end{aligned}$$

$$\begin{aligned} Var\left(r_p\right) &= \begin{bmatrix} \frac{1}{8} & \cdots & \frac{1}{8} \end{bmatrix} S \begin{bmatrix} \frac{1}{8} \\ \vdots \\ \frac{1}{8} \end{bmatrix} \\ &= 0.000237 \end{aligned}$$

Then, being based on these predicted values, for the expected return and variance, the arbitrary necessity and sufficiency levels of the decision maker were defined and the related membership functions were constructed.In the study these V_L, V_U and E_L, E_U aspiration degress were chosen randomly to be close to the predicted values of $E\left(r_p\right)$and $Var\left(r_p\right)$.

Let's define E_L, E_U degress as $E_L = 0.001$ and $E_U = 0.003$ randomly.Under these values, membership function for portfolio expected return is constructed as;

$$\mu_E\left(E\left(r_p\right)\right) = \begin{cases} 1 & , \quad 0.001 \leq E\left(r_p\right) \\ 1+\frac{E(r_p)-0.003}{0.002} & , \quad 0.001 \leq E\left(r_p\right) \leq 0.003 \\ 0 & , \quad E\left(r_p\right) \leq 0.004 \end{cases}$$

Let's define V_L, V_U degress as $V_L = 0.0005$ and $V_U = 0.0002$ randomly.Under these values, membership function for portfolio variance is constructed as;

$$\mu_V\left(Var\left(r_p\right)\right) = \begin{cases} 1 & , \quad Var\left(r_p\right) \leq 0.0002 \\ 1+\frac{0.0002-Var(r_p)}{0.0003} & , \quad 0.0002 \leq Var\left(r_p\right) \leq 0.0005 \\ 0 & , \quad 0.0005 \leq Var\left(r_p\right) \end{cases}$$

The increase or the decrease in the V_L, V_U and E_L, E_U values influences the optimal solution. The changes may cause increase or dicrease at the expected portfolio return and the portfolio variance. To give an example;when V_L and V_U degress are taken as $V_L = 0.0006$ and $V_U = 0.0003$, the portfolio variance is calculated as $Var\left(r_p\right) = 0.000339$.

The problem (5.1) was formed by replacing input values in Watada model (4.4).

$$\begin{aligned} & \max\lambda \\ Var\left(r_p\right)+0.0003\lambda &\leq 0.0005 \\ E\left(r_p\right)-0.002\lambda &\geq 0.001 \\ \sum_{i=1}^{8} X_i &= 1 \\ \lambda &\geq 0 \\ X_i &\geq 0, \quad i=1,...,8 \end{aligned} \tag{5.1}$$

In the last stage, the problem was solved by Lingo software programme.

6. Results

By solving the problem (5.1), weights and membership degree λ were obtained as

TABLE 1. Membership degree λ and weights of each security obtained by solving the problem (5.1)

λ	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0.80	0.2075	0.2136	0.1124	0.2846	0.0623	0.0000	0.0950	0.0243

Based on these weights,

$$\begin{aligned} \lambda &= \min\left(\mu_E\left(E\left(r_p\right)\right), \mu_V\left(Var\left(r_p\right)\right)\right) = 0.80 \\ E\left(r_p\right) &= 0.0026 \\ Var\left(r_p\right) &= 0.000256 \end{aligned}$$

were calculated.

TABLE 2. Membership degree λ and weights of each security in the situation that $V_L = 0.0006$ and $V_U = 0.0003$ (6.1)

λ	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0.98	0.1506	0.3079	0.1562	0.2803	0.0846	0	0.0963	0

$$\begin{aligned} \lambda &= \min\left(\mu_E\left(E\left(r_p\right)\right), \mu_V\left(Var\left(r_p\right)\right)\right) = 0.98 \\ E\left(r_p\right) &= 0.003 \\ Var\left(r_p\right) &= 0.000339 \end{aligned}$$

In the Classical Markowitz Model, for $\alpha = 0.001$ the solution was obtained as

TABLE 3. The solution for the Classical Markowitz Model (α=0.001)

X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
0.2238	0	0	0.2010	0.0931	0.0934	0.0567	0.3318

$$\begin{aligned} E\left(r_p\right) &= 0.001221 \\ Var\left(r_p\right) &= 0.000183 \end{aligned}$$

7. Conclusion

When the Table 1 is examined, it is seems that the investor should invest approximately %20.75 of his money in 1st security, %21. 36 of his money in 2nd security %11.24 of his money in 3 security, %28.46 of his money in 4th security, %6.23 of his money in 5th security, %9.50 of his money in 7th security and %2.43 of his money in 8th security.

According to these obtaining weights , the expected portfolio return was calculated as $E\left(r_p\right) = 0.0026$, the portfolio variance was obtained as $Var\left(r_p\right) = 0.000256$

The aim function value $\lambda = \min\left(\mu_E\left(E\left(r_p\right)\right), \mu_V\left(Var\left(r_p\right)\right)\right) = 0.80$ shows that, the model is successful about maximizing the λ membership degree. It seems that the aspiration level has been formed with a high membership degree so a successful portfolio has been formed.

It can be said that; applying the Fuzzy Method on the IMKB data set can be accepted as innovation and originality.

If we need to compare the solution of Watada Fuzzy Model with the solution of Markowitz Classical Model, it is observed that the expected return of the WFM is larger than the MCM return.At the same time, its variance is larger than the MCM variance. The risk aversion investor should prefer the Markowitz Classical Model.

Anahtar Kelimeler: Portföy analizi, portföy riski, beklenen getiri, bulanık küme, bulanık portföy seçimi.

References

[1] Bellman, R.E., Zadeh, L.A. 1970. Decision making in a fuzzy environment, Management Science 17, B141-B164.

[2] Fabozzi, F., Kolm, P., Pachamanova, D., Focardi, S. 2007. Robust Portfolio Optimization and Management, Wiley.

[3] *Huang,.X. 2007. A new perspective for optimal portfolio selection with random fuzzy returns. Information Sciences, 5404-5414.

[4] **Huang,.X. 2007. Portfolio selection with fuzzy returns. Journal of Intelligent and fuzzy systems-IOS Press.

[5] Hasuike, T., Katagiri H., Ishii H. 2009. Portfolio selection problems with random fuzzy variable returns. Fuzzy Sets and Systems, 2579-2596.

[6] Lai, Y.J, Hwang, C.L. 1992. Fuzzy Mathematical Programming (Methods and Applications), Springer-Verlag Berlin, Germany.

[7] Markowitz, H.M., 1952. Portfolio Selection. The Journal of Finance. New york, 77-91.

[8] Tanaka, H., Guo, P., 1998. Possibilistic Data Analysis for Operation Research.

[9] Tanaka, H., Guo, P. and Turksen, I.B. 2000. Portfolio selection based on fuzzy probabilities and possibility distributions. Fuzzy Sets and Systems, 111, 387-397.

[10] Wang, S. and Zhu, S. 2002. On Fuzzy Portfolio Selection Problems.Fuzzy Optimization and Decision Making, 1, 361-377.

[11] Watada, J., 2001. Fuzzy portfolio model for decision making in investment, Dynamical Aspects in Fuzzy Decision Making, 141-162, Physica-Verlag, Heidelberg.

[12] Zadeh, L.A. 1962. From circuit theory to system theory, Proceedings of Institute of Radio Engineering 856-865.

[13] Zadeh, L.A. 1965. Fuzzy Sets, Information and Control 8, 338-353.

Current address: Ankara University Faculty of Science, Department of Statistics, Ankara, Turkey

E-mail address: `geroglu@science.ankara.edu.tr, apaydin@science.ankara.edu.tr`

URL: `http://communications.science.ankara.edu.tr/index.php?series=A1`