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GENERAL DUAL BOOSTS IN LORENTZIAN DUAL PLANE D_1^2

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Abstract. In this paper we obtained the equations of dual boosts (rotations in the Lorentzian dual plane D_1^2) about an arbitrary point (H, K) and showed that the set of all dual translations and dual boosts is a group and the set of all dual boosts is not a group.

1. Introduction

The set $D = \{a + \varepsilon b : \varepsilon \neq 0, \varepsilon^2 = 0, a, b \in \mathbb{R}\}$ is a commutative ring with a unit. Elements of D is called as dual numbers.

Clifford [5], introduced the Dual numbers in 1873. A. P. Koltelnikov [13] applied them to describe rigid body motions in three dimension. The notion of dual angle is defined by Study [18] and Yaglom [20] described geometrical objects in three dimensional space using these numbers.

There has been many applications of dual numbers in recent years, such as; in robotics, dynamics, and kinematics ([17], [7], [16]), in computer aided geometrical design and modelling of rigid bodies, mechanism design ([2], [4], [3], [14]), in field theory ([6],[19], [1]), and in group theory ([9], [10], [11]).

Gans' ([8]) work which is on - the equations of general rotations about an arbitrary point (h, k) and the theorems about resultant of translations and rotations in the Euclidean plane- is generalized to Lorentzian plane E_1^2 in [12].

The dual plane $D^2 = \{(A_1, A_2) : A_1, A_2 \in D\}$ is a 2-dimensiaonal modul on D. In this paper we obtain the equations of general dual boosts about an arbitrary point (H, K) in Lorentzian dual plane $D_1^2 = (D^2, (+, -))$ and show that the set of all dual translations and dual boosts is a group and the set of all dual boosts is not a group.

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2. Equations of General Dual Boosts

There are four kinds of isometries in two-dimensional Dual Lorentzian plane D_1^2. They are the following :

$$\begin{pmatrix} \cosh\Phi & \sinh\Phi \\ \sinh\Phi & \cosh\Phi \end{pmatrix} \quad , \quad \begin{pmatrix} \cosh\Phi & \sinh\Phi \\ -\sinh\Phi & -\cosh\Phi \end{pmatrix}$$

$$\begin{pmatrix} -\cosh\Phi & \sinh\Phi \\ -\sinh\Phi & \cosh\Phi \end{pmatrix} \quad , \quad \begin{pmatrix} -\cosh\Phi & \sinh\Phi \\ \sinh\Phi & -\cosh\Phi \end{pmatrix}.$$

where $\Phi = \varphi + \varepsilon\varphi^*$ is the dual angle.

The ones whose determinants are $+1$ are dual boosts, the ones whose determinants are -1 are dual reflections (See [15]). Hence the dual boosts about the origin are given by the matrices

$$R_\Phi = \begin{pmatrix} \cosh\Phi & \sinh\Phi \\ \sinh\Phi & \cosh\Phi \end{pmatrix} \quad , \quad \begin{pmatrix} -\cosh\Phi & \sinh\Phi \\ \sinh\Phi & -\cosh\Phi \end{pmatrix} = B_\Phi.$$

Let $O_1 = (H, K)$, $H = h + \varepsilon h^*$, $K = k + \varepsilon k^*$ be an arbitrary point, Φ be the angle of rotation, and let
$P_1 = (X_1, Y_1)$, denote the image of $P = (X, Y)$, where $X = x + \varepsilon x^*$, $Y = y + \varepsilon y^*$, $X_1 = x_1 + \varepsilon x_1^*$, $Y_1 = y_1 + \varepsilon y_1^*$.

We express the rotation in terms of an auxiliary $\widehat{X}, \widehat{Y}-$coordinate system, with origin O_1, whose axis are parallel to X, Y axis and similarly directed.

If the coordinats of P and P_1 in the auxiliary system are $\left(\widehat{X}, \widehat{Y}\right)$ and $\left(\widehat{X}_1, \widehat{Y}_1\right)$, then we have

$$\left(\widehat{X}_1, \widehat{Y}_1\right) = \left(\widehat{X}\cosh\Phi + \widehat{Y}\sinh\Phi, \widehat{X}\sinh\Phi + \widehat{Y}\cosh\Phi\right). \tag{1}$$

Denote this dual boost by

$$\begin{array}{cccccc} f: & D_1^2 & \to & D_1^2 & & \\ & \left(\widehat{X}, \widehat{Y}\right) & \to & f\left(\widehat{X}, \widehat{Y}\right) & = & \left(\widehat{X}_1, \widehat{Y}_1\right) \end{array}$$

and (respectively denote the second dual boosts by

$$\begin{array}{cccccc} g: & D_1^2 & \to & D_1^2 & & \\ & \left(\widehat{X}, \widehat{Y}\right) & \to & g\left(\widehat{X}, \widehat{Y}\right) & = & \left(\widehat{X}_1, \widehat{Y}_1\right) \end{array}$$

where in this case

$$\left(\widehat{X}_1, \widehat{Y}_1\right) = \left(-\widehat{X}\cosh\Phi + \widehat{Y}\sinh\Phi, \widehat{X}\sinh\Phi - \widehat{Y}\cosh\Phi\right)). \tag{1'}$$

The relations between the original and the auxiliary coordinates of P, P_1 are

$$(X, Y) = \left(\widehat{X} + H, \widehat{Y} + K\right) \text{ and } (X_1, Y_1) = \left(\widehat{X}_1 + H, \widehat{Y}_1 + K\right) \tag{2}$$

substituting the values of $\widehat{X}, \widehat{Y}, \widehat{X}_1, \widehat{Y}_1$ into (1) (respectively into (1$'$)) we obtain the equations we have been seeking. Thus the dual boosts through the angle Φ about the point (H, K) has the equations

$$(X_1 - H, Y_1 - K) = ((X - H)\cosh\Phi + (Y - K)\sinh\Phi, (X - H)\sinh\Phi + (Y - K)\cosh\Phi) \tag{3}$$

(respectively

$$(X_1 - H, Y_1 - K) = (-(X - H)\cosh\Phi + (Y - K)\sinh\Phi, (X - H)\sinh\Phi - (Y - K)\cosh\Phi)\,). \tag{3$'$}$$

Hence

$$\begin{cases} (X_1, Y_1) = (X\cosh\Phi + Y\sinh\Phi + A, X\sinh\Phi + Y\cosh\Phi + B) \\ \text{where} \quad A = H(1 - \cosh\Phi) - K\sinh\Phi \\ \qquad\qquad B = K(1 - \cosh\Phi) - H\sinh\Phi \end{cases} \tag{4}$$

(respectively from (3$'$)

$$\begin{cases} (X_1, Y_1) = (-X\cosh\Phi + Y\sinh\Phi + C, X\sinh\Phi - Y\cosh\Phi + D) \\ \text{where} \quad C = H(1 + \cosh\Phi) - K\sinh\Phi \\ \qquad\qquad D = K(1 + \cosh\Phi) - H\sinh\Phi \qquad\qquad). \end{cases} \tag{4$'$}$$

Equation (4) (respectively equation (4$'$)) are seen to be the resultant TR_Φ (respectively SB_Φ) of the following dual boost. R_Φ about O and dual translation T (respectively following dual boost B_Φ about O and dual translation S)

$$\begin{aligned} (X_1, Y_1) &= R_\Phi(X, Y) = \begin{pmatrix} \cosh\Phi & \sinh\Phi \\ \sinh\Phi & \cosh\Phi \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &= (X\cosh\Phi + Y\sinh\Phi, X\sinh\Phi + Y\cosh\Phi) \end{aligned}$$

$$(X_2, Y_2) = T(X_1, Y_1) = (X_1 + A, Y_1 + B)$$

(respectively

$$\begin{aligned} (X_1, Y_1) &= B_\Phi(X, Y) = \begin{pmatrix} -\cosh\Phi & \sinh\Phi \\ \sinh\Phi & -\cosh\Phi \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \\ &= (-X\cosh\Phi + Y\sinh\Phi, Y\sinh\Phi - Y\cosh\Phi) \end{aligned}$$

$$(X_2, Y_2) = S(X_1, Y_1) = (X_1 + C, Y_1 + D)).$$

Therefore the dual boost represented by (3) is equal to TR_Φ (respectively the dual boost represented by (3$'$) is equal to SB_Φ).

Theorem 2.1. *The equations of the dual boost about an arbitrary point through a dual angle Φ is given by the equation in (4) (respectively in (4$'$)).*

Proof.

$$\begin{aligned} f: \quad D_1^2 &\rightarrow D_1^2 \\ (X, Y) &\rightarrow f(X, Y) = \begin{array}{c} (X\cosh\Phi + Y\sinh\Phi + H(1 - \cosh\Phi) - K\sinh\Phi, \\ X\sinh\Phi + Y\cosh\Phi + K(1 - \cosh\Phi) - H\sinh\Phi) \end{array} \end{aligned}$$

$$\begin{array}{llll} g: & D_1^2 & \to & D_1^2 \\ & (X,Y) & \to & g(X,Y) = \begin{array}{c}(-X\cosh\Phi + Y\sinh\Phi + H(1+\cosh\Phi) - K\sinh\Phi, \\ X\sinh\Phi - Y\cosh\Phi + K(1+\cosh\Phi) - H\sinh\Phi)\end{array}\end{array}$$

Note that for f and g we have

$$d\left(f(P), f(Q)\right) = d(P,Q), \quad d(g(P), g(Q)) = d(P,Q),$$

where

$$d(P,Q) = \left\| \overrightarrow{PQ} \right\| = \sqrt{< \overrightarrow{PQ}, \overrightarrow{PQ} >}.$$

Hence f and g are isometries. Moreover

$$f(H,K) = (H,K) \text{ and } g(H,K) = (H,K).$$

Thus f and g are the dual boosts about the point (H,K) and through a dual angle Φ. □

Using this same R (respectively B), we can also find a dual translation $T^{'}$ (respectively $S^{'}$) such that the dual boost (3) (respectively dual boost $(3^{'})$) is equal to $RT^{'}$ (respectively $BS^{'}$). To prove this we write

$$(X_1,Y_1) = T^{'}(X,Y) = \left(X + A^{'}, Y + B^{'}\right)$$

and

$$(X_2,Y_2) = R(X_1,Y_1) = (X_1\cosh\Phi + Y_1\sinh\Phi, X_1\sinh\Phi + Y_1\cosh\Phi)$$

(respectively

$$(X_1,Y_1) = S^{'}(X,Y) = \left(X + C^{'}, Y + D^{'}\right)$$

and

$$(X_2,Y_2) = B(X_1,Y_1) = (-X_1\cosh\Phi + Y_1\sinh\Phi, X_1\sinh\Phi - Y_1\cosh\Phi)\,).$$

Hence we get

$$\begin{aligned}(X_2,Y_2) &= RT^{'}(X,Y) \\ &= \left((X+A^{'})\cosh\Phi + (Y+B^{'})\sinh\Phi, (X+A^{'})\sinh\Phi + (Y+B^{'})\cosh\Phi\right)\end{aligned}$$

(respectively

$$\begin{aligned}(X_2,Y_2) &= BS^{'}(X,Y) \\ &= \left(-(X+C^{'})\cosh\Phi + (Y+D^{'})\sinh\Phi, (X+C^{'})\sinh\Phi - (Y+D^{'})\cosh\Phi\right)\,)\end{aligned}$$

or

$$\begin{aligned}(X_2,Y_2) = RT^{'}(X,Y) = (&X\cosh\Phi + Y\sinh\Phi + (A^{'}\cosh\Phi + B^{'}\sinh\Phi), \\ &X\cosh\Phi + Y\sinh\Phi + (A^{'}\cosh\Phi + B^{'}\sinh\Phi)) \qquad (5)\end{aligned}$$

(respectively

$$(X_2, Y_2) = BS^{'}(X, Y) = (-X\cosh\Phi + Y\sinh\Phi + (D^{'}\cosh\Phi - C^{'}\sinh\Phi),$$
$$X\sinh\Phi - Y\cosh\Phi + (C^{'}\sinh\Phi - D^{'}\cosh\Phi)) \ ). \quad (5^{'})$$

Now, we find $A^{'}$, $B^{'}$ (respectively $C^{'}$, $D^{'}$) so that (5) (respectively $(5^{'})$) is the same transformation as (4) (respectively $(4^{'})$) i.e. from $TR = RT^{'}$ (respectively $SB = BS^{'}$), we have

$$A^{'} = -K\sinh\Phi + H(\cosh\Phi - 1)$$
$$B^{'} = -H\sinh\Phi + K(\cosh\Phi - 1)$$

(respectively

$$C^{'} = -K\sinh\Phi - H(\cosh\Phi + 1)$$
$$D^{'} = -H\sinh\Phi - K(\cosh\Phi + 1) \ ).$$

It is readily seen that $T \neq T^{'}$ and $S \neq S^{'}$. Thus we have proved the following:

Theorem 2.2. *Any given dual boost not the identity, through a dual angle about a point other than the origin is the resultant of a dual boost R_Φ (respectively B_Φ) through Φ about the origin followed by a dual translation T, or if a dual translation $T^{'}$ followed by R_Φ (respectively B_Φ). The two dual translations are distinct, uniquely determined, and neither dual translation is I.*

Corollary 1. *If T is a dual translation and R_Φ (respectively B_Φ) is a dual boost through Φ about the origin and neither dual transformation is I, then $R_\Phi T$ (respectively $B_\Phi T$) and TR_Φ (respectively TB_Φ) are distinct dual boosts through Φ about points other than the origin.*

3. Resultants of Dual Translations and Dual Boosts

It is readily checked that the resultant of two dual translations is a dual translation and that the resultant of two dual boosts about origin $O = (0, 0)$ is a dual boost about that point i.e.

$$R_\Phi R_\Omega = R_{\Phi+\Omega}, \ B_\Phi B_\Omega = R^{-1}_{\Phi+\Omega} = R_{-(\Phi+\Omega)}$$
$$R_\Phi B_\Omega = B_{\Omega-\Phi}, B_\Omega R_\Phi = B_{\Omega-\Phi}$$

(i) Let R_Φ and R_Ω be two dual boosts about the same point (H, K) through Φ, and Ω respectively. Clearly

$$R_\Phi = TR^{'}_\Phi \text{ and } R_\Omega = R^{''}_\Omega T^{'}$$

where $R^{'}_\Phi$, $R^{''}_\Omega$ are dual boosts about $O = (0, 0)$ through Φ, and Ω respectively T and $T^{'}$ are dual translations.

Hence $R_\Phi R_\Omega = TR^{'}_\Phi R^{''}_\Omega T^{'}$. From the above discussion $R^{'''}_{\Phi+\Omega} = R^{'}_\Phi R^{''}_\Omega$ is a dual boost about $O = (0, 0)$ through $\Phi + \Omega$. Therefore $R_\Phi R_\Omega = TR^{'''}_{\Phi+\Omega}T^{'}$.

(ii) Let R_Φ and R_Ω be two dual boosts about different points (H,K) and $\left(H^{'},K^{'}\right)$. Again it is easy to see that

$$R_\Phi R_\Omega = TR^{'''}_{\Phi+\Omega}T^{'},$$

where $R^{'''}_{\alpha+\Omega}$ is a dual boost about $O=(0,0)$.

(iii) Let R_Φ and B_Ω are two dual boosts about different points (H,K) and $\left(H^{'},K^{'}\right)$. Hence we have

$$R_\Phi = TR^{'}_\Phi \text{ and } B_\Omega = B^{'}_\Omega S^{'},$$

where $R^{'}_\Phi$, $B^{'}_\Omega$ are dual boosts about $O=(0,0)$ and T and $S^{'}$ are dual translations.

Therefore $R_\Phi R_\Omega = TR^{'}_\Phi B^{'}_\Omega S^{'}$. Note that $R^{'}_\Phi B^{'}_\Omega = \beta^{''}_{\Omega-\Phi}$ is a dual boost about $O=(0,0)$. For

$$B_\Omega R_\Phi = (S^{''}\beta^{'}_\Omega)(R^{''}_\Phi T^{'}),$$

where $B^{'}_\Omega R^{''}_\Phi = \beta^{''}_{\Omega-\Phi}$ is dual boost about $O=(0,0)$. Note also that

$$R_\Phi = TR^{'}_\Phi \text{ and } B_\Omega = B^{'}_\Omega S^{''},$$

then $R_\Phi B_\Omega = TR^{'}_\Phi B^{'}_\Omega S^{''}$. $R^{'}_\Phi B^{'}_\Omega = B^{''}_{\Omega-\Phi}$ is a dual boost about $O=(0,0)$. Thus $R_\Phi B_\Omega = TB^{''}_{\Omega-\Phi}S^{''}$.

(iv) Let B_Φ and B_Ω are two dual boosts about different points.

Clearly

$$B_\Phi = SB^{'}_\Phi,\ B_\Omega = B^{''}_\Omega S^{'},$$

where $B^{'}_\Phi$, $B^{''}_\Omega$ are dual boosts about $O=(0,0)$. Hence

$$B_\Phi B_\Omega = SB^{'}_\Phi B^{''}_\Omega S^{'} = SR^{-1}_{\Phi+\Omega}S^{'} = SR_{-(\Phi+\Omega)}S^{'}.$$

$R_{-(\Phi+\Omega)}$ is a dual boost through $-(\Phi+\Omega)$ about $O=(0,0)$.

Thus in all cases problem reduces to

$$TR^{'''}_{\Phi+\Omega}T^{'},\ S^{''}B^{''}_{\Omega-\Phi}T^{'},\ TB^{''}_{\Omega-\Phi}S^{''} \text{ and } SR_{-(\Phi+\Omega)}S^{'},$$

where $R^{'''}_{\Phi+\Omega}$, $B^{''}_{\Omega-\Phi}$, $R_{-(\Phi+\Omega)}$ are dual boosts about $O=(0,0)$, S, T, $S^{'}$, $T^{'}$, $S^{''}$ are dual translations. From Corollary 1 we know that the resultant of a dual translation and a dual boosts about $O=(0,0)$ (when neither is the identify) is a dual boost about a point not O. Hence it remains to determine the nature of the resultant of (a) a dual translation and a dual boost about a point not O, and (b) two dual boosts about any points. In doing this we assume that none of these transformations is the identity.

(a) Let T (respectively S) be a dual translation, R (respectively B) a dual boost about a point not O, and consider TR (respectively SB). By theorem 2, we know that $R = T^{'}R^{'}$ (respectively $B = S^{'}B^{'}$) where $T^{'}$ (respectively $S^{'}$) is a dual translation and $R^{'}$ (respectively $B^{'}$) is a dual boost about O. Hence using the associative property of transformations, we have that

$$TR = T(T^{'}R^{'}) = (TT^{'})R^{'} = T_1R^{'}$$

(respectively

$$SB = S(S^{'}B^{'}) = (SS^{'})B^{'} = S_1B^{'} \quad)$$

where T_1 (respectively S_1) is a dual translation. Note that $T_1R^{'}$ (respectively $S_1B^{'}$) is simply the dual boost $R^{'}$ (respectively $B^{'}$) if $T_1 = I$ (respectively $S_1 = I$), and by corollary 1 it is a dual boost about a point not O if $T_1 \neq I$ (respectively $S_1 \neq I$). Thus, TR (respectively SB) is always a dual boost. In the same way RT (respectively BS) can be shown to be a dual boost.

(b) Let R, $R^{'}$ (respectively B, $B^{'}$ or respectively R, B) be dual boosts about any points. If neither point is O, then by theorem 2, $R = TR_1$ and $R^{'} = R_1^{'}T^{'}$ (respectively $B = SB_1$ and $B^{'} = B_1^{'}S^{'}$ or respectively $R = TR_1$ and $B = B_1^{'}S^{'}$) where R_1, $R_1^{'}$, B_1, $B_1^{'}$ are dual boosts about O and $T, T^{'}$, $S, S^{'}$ are dual translations. These equations are still true if one of the given dual boosts, say R (respectively B), is about O, except that then $T = I$ (respectively $S = I$). In any case then, we have

$$RR^{'} = (TR_1)(R_1^{'}T^{'}) = T(R_1R_1^{'})T^{'} = TR_2T^{'} = (TR_2)T^{'}$$

(respectively

$$BB^{'} = (SB_1)(B_1^{'}S^{'}) = S(B_1B_1^{'})S^{'} = SR_3S^{'},)$$

respectively

$$RB = (TR_1B_1^{''}S^{'}) = T(R_1B_1^{''})S^{''} = (TB_2)S^{''},)$$

where R_2, (respectively R_3) (respectively B_2) is a dual boost about O. If $R_2 = I$ (respectively $R_3 = I$ and respectively $B = I$) which occurs if R_1, $R_1^{'}$ (respectively B_1, $B_1^{'}$, respectively R_1, $B_1^{''}$) are mutually inverse then $RR^{'}$ (respectively $BB^{'}$ and RB) is clearly a dual translation. If R_2 (respectively R_3, respectively B_2) $\neq I$ then TR_2 (respectively SR_3, respectively TB_2) is a dual boost whose center is not O. Unless, $T = I$ (respectively $S = I$ or respectively $T = I$) using the discussion above it follows that $TR_2T^{'}$ (respectively $SR_3S^{'}$ respectively $TB_2S^{''}$), and hence $RR^{'}$ (respectively $BB^{'}$ and respectively RB) is a dual boost. Same argument works for BR.

Thus have proved:

Theorem 3.1. *The resultant in either order of a dual translation ($\neq I$) and a dual boost ($\neq I$) is a dual boost. The resultant of two dual boosts about any point is a dual boost or a dual translation.*

Considering the fact that the inverses of dual translations and dual boosts are dual translations and dual boosts respectively, it is clear that we have also proved:

Theorem 3.2. *The set of all dual translations and dual boosts is a group. The set off all dual boosts is not a group.*

Remark 3.3. Note that set of all R_Φ's about $O = (0,0)$ is a group. Yet set off all B_Φ's about O is not a group. Note also that a dual translation and a dual boost are generally not commutative. This was already apperant from corollary 1.

The same is true of two dual boosts,

Example 3.4. Let $R_{\pi/2}$, $B_{\pi/2}$ be dual boosts through $\pi/2+\varepsilon 0$ about $(0,0)$, $(0,1)$ respectively it is easy to see that $R_{\pi/2}B_{\pi/2} \neq B_{\pi/2}R_{\pi/2}$.

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