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APPROXIMATION BY THE BIVARIATE COMPLEX BASKAKOV-STANCU OPERATORS

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ABSTRACT. In this paper we study the approximation properties of the Stancu type bivariate generalization of the complex Baskakov operators. We obtain a Voronovskaja type result with quantitative estimates for bivariate complex Baskakov-Stancu operators attached to analytic functions having suitable exponential growth on compact polydisks. Also we give the exact order of approximation.

1. INTRODUCTION

In the present paper we deal with the following type complex Baskakov operators relating to divided difference of an analytic function f. For a complex valued function f defined on $[R,\infty)\cup\overline{D}_R$ with $D_R=\{z\in\mathbb{C}:|z|<R\}$, the complex Baskakov operators defined by

$$W_n(f)(z)=\sum_{j=0}^{\infty}\frac{n(n+1)...(n+j-1)}{n^j}\left[0,1/n,...,j/n;f\right]z^j \tag{1.1}$$

were studied in [8. pp.124-134]. Here the function $f:[R,\infty)\cup\overline{D}_R\to\mathbb{C}$ is analytic in D_R and all its derivatives bounded on $[0,\infty)$ by the same constant and also has an expontential growth condition for all $z\in D_R$ and for $j=0$, one takes $n(n+1)...(n+j-1)=1$. In [8], the Voronovskaja type results with a quantitative estimate. The exact order of simultaneous approximation for these operators were given. Considering the real Baskakov operators defined in [4], the classical complex Baskakov operator is defined by

$$B_n(f)(z)=(1+z)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{z}{1+z}\right)^k f\left(\frac{k}{n}\right), z\in\mathbb{C}.$$

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If $z=x, x\in\mathbb{R}$ and $x\geq 0$ then $W_n(f)(z)=B_n(f)(z)$. But if $x<0$ then $W_n(f)(z)$ may be different from $B_n(f)(z)$ [8. page 124]. Therefore $W_n(f)(z)$ and $B_n(f)(z)$ do not necessarily coincide for all $z\in\mathbb{C}$. In [8], the approximation properties of these operators were studied separately, under different hypothesis on f and $z\in\mathbb{C}$. Furthermore, bivariate form of the operators $W_n(f)$ given by (1.1) was introduced and the results in univariate case were extended to the bivariate case for the analytic functions on polydisks [8.pp.172-180].

The Stancu type generalization of the complex Baskakov operators were studied by Gal et.all. [6] which are defined as follows

$$W_n^{\alpha,\beta}(f)(z)=\sum_{j=0}^{\infty}\frac{n(n+1)...(n+j-1)}{(n+\beta)^j}\left[\frac{\alpha}{n+\beta},\frac{\alpha+1}{n+\beta},...,\frac{\alpha+j}{n+\beta};f\right]z^j, z\in\mathbb{C} \tag{1.2}$$

where $0\leq\alpha\leq\beta$ and α,β are real numbers with independent of n, $[x_0,x_1,...,x_n;f]$ denotes the divided difference of the function f on the distinct points $x_0,x_1,...,x_n$. In [6], Voronoskaja type result with quantitative estimates and convergence results for the operators (1.2) attached the analytic functions on compact disks were obtained. The similar results for complex Bernstein-Stancu polynomials in [7],[9], and for complex Durrmeyer-Stancu and genuine Durrmeyer-Stancu operators in [13] and [10] were obtained. In case of real variables, some approximation properties of the Baskakov and Baskakov-Stancu operators were investigated in [1], [5],[11],[12] and [14].

The aim of the present paper is to investigate the approximation properties of bivariate complex Baskakov-Stancu operators of tensor product kind. We extend the approximation results from the univariate case, obtained in [6] for the complex Baskakov-Stancu operators, to the bivariate case.

First we present a few concepts in the bivariate case which are natural extensions of the usual concepts in the univariate case. Let $D_{R_j}:=\{z_j\in\mathbb{C}:|z_j|<R_j,\ j=1,2\}$ and $D_{R_1}\times D_{R_2}$ denotes an open polydisk (of center 0 and radius R) where $R=(R_1,R_2)$ and $|z_1|\leq r_1, |z_2|\leq r_2, r_1<R_1$ with $r_2<R_2$. Let also $\overline{D}_{R_1}\times\overline{D}_{R_2}=\left\{(z_1,z_2)\in\mathbb{C}^2:|z_j|\leq R_j,\ j=1,2\right\}$ denotes the closed polydisk.

We defined the bivariate complex Baskakov-Stancu operators as follows

$$W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(z_1,z_2)=\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{n(n+1)...(n+\nu-1)}{(n+\beta)^{\nu}}\frac{m(m+1)...(m+\mu-1)}{(m+\delta)^{\mu}}\times$$

$$\left[\frac{\alpha}{n+\beta},\frac{\alpha+1}{n+\beta},...,\frac{\alpha+\nu}{n+\beta};\left[\frac{\gamma}{m+\delta},\frac{\gamma+1}{m+\delta},...,\frac{\gamma+\mu}{m+\delta};f(.,.)\right]_{z_2}\right]_{z_1}z_1^{\nu}z_2^{\mu} \tag{1.3}$$

where $f:\left([R_1,+\infty)\cup\overline{D}_{R_1}\right)\times\left([R_2,+\infty)\cup\overline{D}_{R_2}\right)\to\mathbb{C}$ is analytic in $D_{R_1}\times D_{R_2}$ and f has all partial derivatives bounded on $[0,+\infty)\times[0,+\infty)$, by the same constant, and satisfies an exponential growth condition, namely $|f(z_1,z_2)|\leq Me^{A_1|z_1|+A_2|z_2|}$, for all $z_1 \in \overline{D}_{R_1}$, $z_2 \in \overline{D}_{R_2}$ and for $\nu = 0, \mu = 0$, $n(n+1)...(n+\nu-1) = 1, m(m+1)...(m+\mu-1) = 1$.

In this paper, we would like to obtain the exact order of approximation for the operators given by (1.3) on compact polydiscs. First we give the order of approximation and the Voronovskaja type theorems with quantitative estimate for the operators $W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)$ defined by (1.3). These results allow us to obtain the exact order in approximation by the operators $W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)$.

2. Auxiliary Results

In order to establish the next results, we need the following auxiliary lemmas.

Lemma 2.1. *([6] Lemma1) For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, $z \in \mathbb{C}$, let us define*

$$V_n^{\alpha,\beta}(e_k, z) = \sum_{\nu=0}^{\infty} \frac{n(n+1)...(n+\nu-1)}{(n+\beta)^{\nu}} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k\right] z^{\nu},$$

where $e_k(z) = z^k$. Then $V_n^{\alpha,\beta}(e_0, z) = 1$ and we have the following recurrence relation:

$$V_n^{\alpha,\beta}(e_{k+1}, z) = \frac{z(1+z)}{n+\beta}\left(V_n^{\alpha,\beta}(e_k, z)\right)' + \frac{nz+\alpha}{n+\beta} V_n^{\alpha,\beta}(e_k, z).$$

As a result,

$$V_n^{\alpha,\beta}(e_1, z) = \frac{nz+\alpha}{n+\beta}, \quad V_n^{\alpha,\beta}(e_2, z) = \frac{n(n+1)z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)z^2}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Throughout the paper we use the two dimensional test functions $e_{i,j} : [0, +\infty) \times [0, +\infty) \to \mathbb{R}, e_{i,j}(x_1, x_2) = e_i(x_1)e_j(x_2)$ with $e_i(x_1) = x_1^i$ and $e_j(x_2) = x_2^j$ for $i, j \in \{0, 1, 2\}$.

Lemma 2.2. *([6] Lemma 3) For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, $z \in \mathbb{C}$ and $|z| \leq r$, $r \geq 1$ then we have*

$$\left|V_n^{\alpha,\beta}(e_k, z)\right| \leq (k+1)! r^k.$$

Lemma 2.3. *For all $n, m \in \mathbb{N} \cup \{0\}$ we have*

$$\begin{aligned}
W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{0,0})(x_1, x_2) &= 1 \\
W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{1,0})(x_1, x_2) &= \frac{nx_1+\alpha}{n+\beta} \\
W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{0,1})(x_1, x_2) &= \frac{mx_2+\gamma}{m+\delta} \\
W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{2,0})(x_1, x_2) &= \frac{n(n+1)x_1^2}{(n+\beta)^2} + \frac{nx_1(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \\
W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{0,2})(x_1, x_2) &= \frac{m(m+1)x_2^2}{(m+\delta)^2} + \frac{mx_2(1+2\gamma)}{(m+\delta)^2} + \frac{\gamma^2}{(m+\delta)^2}
\end{aligned}$$

where $e_{i,j}(i,j=0,1,2)$ are the test functions.

Proof. Considering Lemma 1 and using Barbosu method in [2], [3], it can be easily proved. So we omit the details of proof. □

Lemma 2.4. *Let $f:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ has all the partial derivatives bounded by the same constant in $[0,+\infty)\times[0,+\infty)$, then $\left\{W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)\right\}$ uniformly converges to f on $[0,r_1]\times[0,r_2]$, for $r_1,r_2>0$.*

Proof. Considering Lemma 3 we obtain

$$\begin{aligned}
\lim_{n,m\to\infty}\left\|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{0,0})-e_{0,0}\right\|_{r_1,r_2} &= 0,\\
\lim_{n,m\to\infty}\left\|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{1,0})-e_{1,0}\right\|_{r_1,r_2} &= 0,\\
\lim_{n,m\to\infty}\left\|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{0,1})-e_{0,1}\right\|_{r_1,r_2} &= 0,
\end{aligned}$$

$$\lim_{n,m\to\infty}\left\|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{2,0}+e_{0,2})-(e_{2,0}+e_{0,2})\right\|_{r_1,r_2}=0.$$

Hence by Volkov's theorem in [15], we reach the desired result. □

Lemma 2.5. *For all $\nu,\mu\in\mathbb{N}\cup\{0\}$, $0\le\alpha\le\beta$, $0\le\gamma\le\delta$ and $|z_1|\le r_1, |z_2|\le r_2$ and $r_1,r_2\ge1$ we have*

$$\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{\nu,\mu})(z_1,z_2)\right|\le(\nu+1)!\,(\mu+1)!r_1^{\nu}r_2^{\mu}.$$

Proof. Using the equality $e_{k,j}(z_1,z_2)=e_k(z_1)\,e_j(z_2)$ and by the definition of $W_{n,m}^{\alpha,\beta,\gamma,\delta}$, from Lemma 2 we get the result. □

3. Approximation by bivariate complex Baskakov-Stancu operators

In this section we will give some convergence results with quantitative estimates for the operators $W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)$.

Theorem 3.1. *Let $n_0,\ m_0\in\mathbb{N}$ and $3\le n_0<2R_1<\infty$, $3\le m_0<2R_2<\infty$ and $\nu,\mu\in\mathbb{N}\cup\{0\}$, $0\le\alpha\le\beta$, $0\le\gamma\le\delta$. Suppose that $f:\left([R_1,+\infty)\cup\overline{D}_{R_1}\right)\times\left([R_2,+\infty)\cup\overline{D}_{R_2}\right)\to\mathbb{C}$ has all the partial derivatives bounded by the same constant in $[0,+\infty)\times[0,+\infty)$, analytic in $D_{R_1}\times D_{R_2}$, that means $f(z_1,z_2)=\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}c_{\nu,\mu}z_1^{\nu}z_2^{\mu}$, for all $|z_1|\le R_1$, $|z_2|\le R_2$ and suppose that there exist $M>0$ and $A_i\in\left(\frac{1}{R_i},1\right)$, $i=1,2$, with the property that $|c_{\nu,\mu}|\le M\frac{A_1^{\nu}A_2^{\mu}}{\nu!\mu!}$, for all $\nu,\mu=0,1,2,...$, (which implies $|f(z_1,z_2)|\le Me^{A_1|z_1|+A_2|z_2|}$, for all $z_1\in D_{R_1}$, $z_2\in D_{R_2}$). If $1\le r_1<\min\left\{\frac{n_0}{2},\frac{1}{A_1}\right\}$, $1\le r_2<\min\left\{\frac{m_0}{2},\frac{1}{A_2}\right\}$ then for all $|z_1|\le r_1$, $|z_2|\le r_2$ and $n>n_0$, $m>m_0$ the sequence of the operators $\left\{W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)\right\}$ is uniformly converges to f on $\overline{D}_{r_1}\times\overline{D}_{r_2}$ for all $n\ge n_0, m\ge m_0$.*

Proof. Using the results of Lemma 4 and page 159 in [6] we have

$$\begin{aligned}\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(z_1,z_2)\right| &\leq \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} c_{\nu,\mu} W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{\nu,\mu})(z_1,z_2) \\ &\leq M\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}(\nu+1)(\mu+1)(r_1A_1)^{\nu}(r_2A_2)^{\mu}<\infty\end{aligned}$$

where the last series is convergent for all $n,m\in\mathbb{N}$, $|z_1|\leq r_1$, $|z_2|\leq r_2$, $n\geq n_0$, $m\geq m_0$ with $1\leq r_1<\min\left\{\frac{n_0}{2},\frac{1}{A_1}\right\}$, $1\leq r_2<\min\left\{\frac{m_0}{2},\frac{1}{A_2}\right\}$.

On the other hand, from Lemma 3 we have

$$\lim_{n,m\to\infty} W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(x_1,x_2)=f(x_1,x_2)$$

for all $(x_1,x_2)\in[0,r_1]\times[0,r_2]$ and by the classical Vitali's theorem we arrive at $\left\{W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)\right\}$ is uniformly converges to f on $\overline{D}_{r_1}\times\overline{D}_{r_2}$ for all $n\geq n_0, m\geq m_0$. □

Now, we can give the following estimate in approximation for $W_{n,m}^{\alpha,\beta}(f)$ to f..

Theorem 3.2. *Let $0\leq\alpha\leq\beta,\ 0\leq\gamma\leq\delta$. Suppose that the hypotheses are same on the function f and on the constants n_0, m_0, R_1, R_2, M, A_1, A_2 in the statement of Theorem 1. Then*

(i): *Suppose that $1\leq r_1<\min\left\{\frac{n_0}{2},\frac{1}{A_1}\right\}, 1\leq r_2<\min\left\{\frac{m_0}{2},\frac{1}{A_2}\right\}$. Then for all $|z_1|\leq r_1$, $|z_2|\leq r_2$ and $n>n_0$, $m>m_0$ we have*

$$\begin{aligned}&\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(z_1,z_2)-f(z_1,z_2)\right| \\ \leq\ & \frac{\alpha+\beta r_1}{n+\beta}D_{r_1,r_2}(f)+\frac{A_{r_1}(f)}{n+\beta}+\frac{\alpha B_{r_1}(f)}{n+\beta}+\frac{\beta C_{r_1}(f)}{n+\beta} \\ &+\frac{\gamma+\delta r_2}{m+\delta}F_{r_1,r_2}(f)+\frac{A_{r_2}(f)}{m+\delta}+\frac{\gamma B_{r_2}(f)}{m+\delta}+\frac{\delta C_{r_2}(f)}{m+\delta}\end{aligned}$$

where

$$\begin{aligned}
D_{r_1,r_2}(f) &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\, r_1^{\nu-1}r_2^{\mu} < +\infty,\\
F_{r_1,r_2}(f) &= \sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty}|c_{\nu,\mu}|\, r_2^{\mu-1}r_1^{\nu}\,(\nu+1)! < +\infty,\\
A_{r_1}(f) &= (1+r_1)\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\,\nu\,(\nu+1)!r_1^{\nu-1}r_2^{\mu} < +\infty,\\
A_{r_2}(f) &= (1+r_2)\sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty}|c_{\nu,\mu}|\,\mu\,(\mu+1)!r_2^{\mu-1} < +\infty,\\
B_{r_1}(f) &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\,\nu r_2^{\mu} < +\infty,\\
B_{r_2}(f) &= \sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty}|c_{\nu,\mu}|\,\mu r_2^{\mu-1} < +\infty,\\
C_{r_1}(f) &= \sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\,\nu r_2^{\mu}r_1^{\nu} < +\infty,\\
C_{r_2}(f) &= \sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty}|c_{\nu,\mu}|\,\mu r_2^{\mu} < +\infty.
\end{aligned}$$

(ii): *Let* $\nu_1,\nu_2\in\mathbb{N}$ *be with* $\nu_1+\nu_2\geq 1$ *and* $1\leq r_1<r_1^*<\min\left\{\frac{n_0}{2},\frac{1}{A_1}\right\}$, $1\leq r_2<r_2^*<\min\left\{\frac{m_0}{2},\frac{1}{A_2}\right\}$ *be arbitrary fixed. Then for all* $|z_1|\leq r_1$, $|z_2|\leq r_2$, $n>n_0$ *and* $m>m_0$ *we have*

$$\left|\frac{\partial^{\nu_1+\nu_2}W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)-\frac{\partial^{\nu_1+\nu_2}f}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)\right|$$

$$\leq C_{r_1^*,r_2^*,n,m}^{\alpha,\beta,\gamma,\delta}(f)\frac{\nu_1!}{(r_1^*-r_1)^{\nu_1+1}}\frac{\nu_2!}{(r_2^*-r_2)^{\nu_2+1}}$$

where the constant

$$C^{\alpha,\beta,\gamma,\delta}_{r_1^*,r_2^*,n,m}(f)$$

$$= \frac{\alpha+\beta r_1^*}{n+\beta}\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\left(r_1^*\right)^{\nu-1}\left(r_2^*\right)^{\mu}+\frac{A_{r_1^*}(f)}{n+\beta}+\frac{\alpha B_{r_1^*}(f)}{n+\beta}+\frac{\beta C_{r_1^*}(f)}{n+\beta}$$

$$+\frac{\gamma+\delta r_2^*}{m+\delta}\sum_{\nu=0}^{\infty}\sum_{\mu=1}^{\infty}|c_{\nu,\mu}|\left(r_2^*\right)^{\mu-1}\left[\left(r_1^*\right)^{\nu}(\nu+1)!\right]+\frac{A_{r_2^*}(f)}{m+\delta}+\frac{\gamma B_{r_2^*}(f)}{m+\delta}+\frac{\delta C_{r_2^*}(f)}{m+\delta}$$

and $A_{r_1^*}(f)$, $A_{r_2^*}(f)$, $B_{r_1^*}(f)$, $B_{r_2^*}(f)$, $C_{r_1^*}(f)$, $C_{r_2^*}(f)$ *is given as the above.*

Proof. (i) Denote $e_{\nu,\mu}(z_1,z_2)=e_\nu(z_1).e_\mu(z_2)$ where $e_\nu(t)=t^\nu$. From Lemma 2 we get

$$\begin{aligned}&\left|W^{\alpha,\beta,\gamma,\delta}_{n,m}(f)(z_1,z_2)-f(z_1,z_2)\right|\\ \leq&\left|\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}c_{\nu,\mu}W^{\alpha,\beta,\gamma,\delta}_{n,m}(e_{\nu,\mu})(z_1,z_2)-\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}c_{\nu,\mu}e_{\nu,\mu}(z_1,z_2)\right|\\ \leq&\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\left|W^{\alpha,\beta,\gamma,\delta}_{n,m}(e_{\nu,\mu})(z_1,z_2)-e_{\nu,\mu}(z_1,z_2)\right|\end{aligned}$$

taking into account the estimate

$$\left|V_n^{\alpha,\beta}(e_k)(z)-e_k(z)\right|\leq r^{k-1}\frac{\alpha+\beta r}{n+\beta}+\frac{r\,(1+r)}{n+\beta}k\,(k+1)!r^{k-2}+\frac{k\alpha}{n+\beta}r^{k-1}+\frac{k\beta}{n+\beta}r^{k}$$

for $k\in\mathbb{N}$, $|z|\leq r$, $r\geq 1$,in the proof of Theorem1 in [6] for all $|z_1|\leq r_1$ and $|z_2|\leq r_2$ and using Lemma 1 we obtain

$$\begin{aligned}\left|W^{\alpha,\beta,\gamma,\delta}_{n,m}(e_{\nu,\mu})(z_1,z_2)-e_{\nu,\mu}(z_1,z_2)\right| &= \left|W_n^{\alpha,\beta}(e_\nu)(z_1).W_m^{\gamma,\delta}(e_\mu)(z_2)-z_1^\nu z_2^\mu\right|\\ &\leq \left|W_n^{\alpha,\beta}(e_\nu)(z_1).W_m^{\gamma,\delta}(e_\mu)(z_2)-W_n^{\alpha,\beta}(e_\nu)(z_1).z_2^\mu\right|\\ &\quad+\left|W_n^{\alpha,\beta}(e_\nu)(z_1).z_2^\mu-z_1^\nu z_2^\mu\right|\\ &\leq \left|W_n^{\alpha,\beta}(e_\nu)(z_1)\right|.\left|W_m^{\gamma,\delta}(e_\mu)(z_2)-z_2^\mu\right|\\ &\quad+\left|z_2^\mu\right|.\left|W_n^{\alpha,\beta}(e_\nu)(z_1)-z_1^\nu\right|\\ &\leq r_1^\nu\,(\nu+1)!\left[r_2^{\mu-1}\frac{\gamma+\delta r_2}{m+\delta}\right.\\ &\quad\left.+\frac{r_2\,(1+r_2)}{m+\delta}\mu\,(\mu+1)!r_2^{\mu-2}+\frac{\mu\gamma}{m+\delta}r_2^{\mu-1}+\frac{\mu\delta}{m+\delta}r_2^{\mu}\right]\\ &\quad+r_2^\mu\left[r_1^{\nu-1}\frac{\alpha+\beta r_2}{n+\beta}\right.\\ &\quad\left.+\frac{r_1\,(1+r_1)}{n+\beta}\nu\,(\nu+1)!r_1^{\nu-2}+\frac{\nu\alpha}{n+\beta}r_1^{\nu-1}+\frac{\nu\beta}{n+\beta}r_1^{\nu}\right]\end{aligned}$$

which from the conditions on the coeffients $c_{\nu,\mu}$ implies

$$\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(z_1,z_2)-f(z_1,z_2)\right|$$

$$\begin{aligned}
&\leq \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(e_{\nu,\mu})(z_1,z_2)-e_{\nu,\mu}(z_1,z_2)\right| \\
&\leq \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\left\{r_1^{\nu}(\nu+1)!\left[r_2^{\mu-1}\frac{\gamma+\delta r_2}{m+\delta}+\frac{r_2(1+r_2)}{m+\delta}\mu(\mu+1)!r_2^{\mu-2}+\frac{\mu\gamma}{m+\delta}r_2^{\mu-1}\right.\right. \\
&\quad\left.+\frac{\mu\delta}{m+\delta}r_2^{\mu}\right]+r_2^{\mu}\left[r_1^{\nu-1}\frac{\alpha+\beta r_1}{n+\beta}+\frac{r_1(1+r_1)}{n+\beta}\nu(\nu+1)!r_1^{\nu-2}+\frac{\nu\alpha}{n+\beta}r_1^{\nu-1}\right. \\
&\quad\left.\left.+\frac{\nu\beta}{n+\beta}r_1^{\nu}\right]\right\} \\
&= \frac{\alpha+\beta r_1}{n+\beta}D_{r_1,r_2}(f)+\frac{A_{r_1}(f)}{n+\beta}+\frac{\alpha B_{r_1}(f)}{n+\beta}+\frac{\beta C_{r_1}(f)}{n+\beta} \\
&\quad+\frac{\gamma+\delta r_2}{m+\delta}F_{r_1,r_2}(f)+\frac{A_{r_2}(f)}{m+\delta}+\frac{\gamma B_{r_2}(f)}{m+\delta}+\frac{\delta C_{r_2}(f)}{m+\delta}
\end{aligned}$$

which proves (i).

Here, the analyticity of f implies that the series $D_{r_1,r_2}(f)$, $F_{r_1,r_2}(f)$, $B_{r_1}(f)$, $B_{r_2}(f)$, $C_{r_1}(f)$, $C_{r_2}(f)$ are convergent and the convergency of $A_{r_1}(f)$, $A_{r_2}(f)$ follows from $|c_{\nu,\mu}|\leq M\dfrac{A_1^{\nu}A_2^{\mu}}{\nu!\mu!}$.

(ii) Now we give the rate of convergence in simultaneous approximation. Let $1\leq r_1<r_1^*<R_1, 1\leq r_2<r_2^*<R_2$. By the Cauchy's formula we get

$$\left|\frac{\partial^{\nu_1+\nu_2}W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)-\frac{\partial^{\nu_1+\nu_2}f}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)\right|$$

$$\leq\frac{\nu_1!\nu_2!}{(2\pi i)^2}\int\limits_{|u_2-z_2|=r_2^*}\int\limits_{|u_1-z_1|=r_1^*}\frac{\left|W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(u_1,u_2)-f(u_1,u_2)\right|}{|u_1-z_1|^{\nu_1+1}|u_2-z_2|^{\nu_2+1}}du_1du_2$$

passing to absolute value with $|z_1|\leq r_1,|z_2|\leq r_2$ and taking into account that $|u_1-z_1|=r_1^*-r_1,|u_2-z_2|=r_2^*-r_2$, by applying the estimate in (i) we obtain

$$\left|\frac{\partial^{\nu_1+\nu_2}W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)-\frac{\partial^{\nu_1+\nu_2}f}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}}(z_1,z_2)\right|$$

$$\leq C_{r_1^*,r_2^*,n,m}^{\alpha,\beta,\gamma,\delta}(f)\frac{\nu_1!}{(r_1^*-r_1)^{\nu_1+1}}\frac{\nu_2!}{(r_2^*-r_2)^{\nu_2+1}}$$

which proves the theorem. □

The second result is about the Voronovskaja-type theorem for operator (1.3). This Voronovskaja-type formula will be the product of the parametric extensions generated by the Voronovskaja-type formula in univariate case in Theorem 2 [6]. Thus, for $f(z_1, z_2)$ defining the parametric extensions of Voronoskaja formula by

$$\begin{aligned} {}_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2) \quad : &= W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - f(z_1,z_2) \\ &\quad - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2), \\ {}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2) \quad : &= W_m^{\gamma,\delta}(f)(z_1,.)(z_2) - f(z_1,z_2) \\ &\quad - \frac{\gamma-\delta z_2}{m+\delta}\frac{\partial f}{\partial z_2}(z_1,z_2) - \frac{z_2(1+z_2)}{2m}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2). \end{aligned}$$

their product gives

$$\begin{aligned} & {}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2) \circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2) \\ = \quad & W_m^{\gamma,\delta}\left[W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - f(z_1,z_2) - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right] \\ & - \left[W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - f(z_1,z_2) - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right] \\ & - \frac{z_2(1+z_2)}{2m}\left[W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial f}{\partial z_1}\right)(z_1,z_2)\right. \\ & \left. - \frac{z_1(1+z_1)}{2n}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial^2 f}{\partial z_1^2}\right)(z_1,z_2)\right] \\ : \quad & = E_1 - E_2 - E_3. \end{aligned}$$

By simple calculation we can write

$$\begin{aligned} & {}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2) \circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2) \\ = \quad & W_{n,m}^{\alpha,\beta,\gamma,\delta}(f)(z_1,z_2) - W_m^{\gamma,\delta}(f)(z_1,.)(z_2) \\ & - \frac{\alpha-\beta z_1}{n+\beta}W_m^{\gamma,\delta}\left(\frac{\partial f}{\partial z_1}(z_1,.)\right)(z_2) - \frac{z_1(1+z_1)}{2n}W_m^{\gamma,\delta}\left(\frac{\partial^2 f}{\partial z_1^2}(z_1,.)\right)(z_2) \\ & - W_n^{\alpha,\beta}(f)(.,z_2)(z_1) + f(z_1,z_2) + \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) + \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\ & - \frac{z_2(1+z_2)}{2m}W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1) + \frac{z_2(1+z_2)}{2m}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) \\ & + \frac{z_2(1+z_2)}{2m}\frac{\alpha-\beta z_1}{n+\beta}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial f}{\partial z_1}\right)(z_1,z_2) + \frac{z_1(1+z_1)}{2n}\frac{z_2(1+z_2)}{2m}\frac{\partial^4 f}{\partial z_1^2 \partial z_2^2}(z_1,z_2) \end{aligned}$$

from which can be derived the commutativity property

$${}_{z_2}L_m(f)(z_1,z_2) o_{z_1} L_n(f)(z_1,z_2) =_{z_1} L_n(f)(z_1,z_2) o_{z_2} L_m(f)(z_1,z_2).$$

The Voronovskaja's theorem can be stated as follows.

Theorem 3.3. *Let $0 \leq \alpha \leq \beta, 0 \leq \gamma \leq \delta$. Suppose that the hypothesis on the function f and on the constants n_0, m_0, R_1, R_2, M, A_1, A_2 in the statement of Theorem (1) hold and let $1 \leq r_1 < \min\left\{\frac{n_0}{2}, \frac{1}{A_1}\right\}$, $1 \leq r_2 < \min\left\{\frac{m_0}{2}, \frac{1}{A_2}\right\}$ be fixed. For all $n > n_0$, $m > m_0$ and $|z_1| \leq r_1$, $|z_2| \leq r_2$ we have the following Voronovskaja-type result*

$$\left|{}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2) \circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2)\right|$$

$$\leq M_{1,r_1,r_2}(f)\left[\frac{1}{n^2}+\frac{1}{m^2}\right] + \sum_{k=2}^{6} M_{k,r_1,r_2}(f)\left[\frac{1}{(n+\beta)^2}+\frac{1}{(m+\delta)^2}\right]$$

where

$$M_{1,r_1,r_2}(f) \quad : \quad = 16M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}(r_1A_1)^{\nu}(r_2A_2)^{\mu}(\nu-1)(\nu-2)^2(\mu+1) < +\infty,$$

$$M_{2,r_1,r_2}(f) \quad : \quad = \alpha^2 M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}(r_1A_1)^{\nu-2}(r_2A_2)^{\mu}\frac{(\nu-1)}{2}(\mu+1) < +\infty,$$

$$M_{3,r_1,r_2}(f) \quad : \quad = 2\alpha M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}(r_1A_1)^{\nu-2}(r_2A_2)^{\mu}\nu^2(\mu+1)r_1 < +\infty,$$

$$M_{4,r_1,r_2}(f) \quad : \quad = \left(\frac{\beta^2}{2}+2\beta\right)M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}(r_1A_1)^{\nu-2}(r_2A_2)^{\mu}\nu^2(\nu+1)(\mu+1)r_1^2 < +\infty,$$

$$M_{5,r_1,r_2}(f) \quad : \quad = \alpha\beta M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}\frac{(r_1A_1)^{\nu-2}}{(\nu-2)!}(r_2A_2)^{\mu}(\mu+1)r_1 < +\infty,$$

$$M_{6,r_1,r_2}(f) \quad : \quad = \beta^2 M\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}\frac{(r_1A_1)^{\nu-2}}{(\nu-2)!}(r_2A_2)^{\mu}(\mu+1)r_1^2 < +\infty.$$

Proof. By the hypothesis we can write $f(z_1,z_2) = \sum_{\nu=0}^{\infty} f_\nu(z_2)z_1^\nu$, where $f_\nu(z_2) = \sum_{\mu=0}^{\infty} c_{\nu,\mu}z_2^\mu$. It follows $\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) = \sum_{\nu=2}^{\infty} f_\nu(z_2)\nu(\nu-1)z_1^{\nu-2}$ and $\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) =$

$\sum_{\nu=0}^{\infty} \frac{\partial^2 f_\nu}{\partial z_2^2}(z_2) z_1^\nu$, where $\frac{\partial^2 f_\nu}{\partial z_2^2}(z_2) = \sum_{\mu=2}^{\infty} c_{\nu,\mu}\mu(\mu-1) z_2^{\mu-2}$,

$$\begin{aligned}
\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial f}{\partial z_1}\right)(z_1,z_2) &= \frac{\partial^2}{\partial z_2^2}\left(\sum_{\nu=1}^{\infty} \nu z_1^{\nu-1} f_\nu(z_2)\right) \\
&= \sum_{\nu=1}^{\infty} \nu z_1^{\nu-1} \frac{\partial^2 f_\nu}{\partial z_2^2}(z_2) \\
&= \sum_{\nu=1}^{\infty}\sum_{\mu=2}^{\infty} c_{\nu,\mu}\nu z_1^{\nu-1}\mu(\mu-1) z_2^{\mu-2}
\end{aligned}$$

$\frac{\partial f}{\partial z_1}(z_1,z_2) = \sum_{\nu=1}^{\infty} \nu z_1^{\nu-1} f_\nu(z_2)$. This implies that $W_n^{\alpha,\beta}(f)(.,z_2)(z_1) = \sum_{\nu=0}^{\infty} f_\nu(z_2) W_n^{\alpha,\beta}(e_\nu)(z_1)$ and

$$\begin{aligned}
& W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - f(z_1,z_2) - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\
= & W_n(f)(.,z_2)(z_1) - f(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\
& + W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - W_n(f)(.,z_2)(z_1) - \frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2)
\end{aligned}$$

$$\begin{aligned}
= & \sum_{\nu=2}^{\infty} f_\nu(z_2) W_n(e_\nu)(z_1) - \sum_{\nu=0}^{\infty} f_\nu(z_2) z_1^\nu - \frac{z_1(1+z_1)}{2n}\sum_{\nu=2}^{\infty} f_\nu(z_2)\nu(\nu-1) z_1^{\nu-2} \\
& + \sum_{\nu=2}^{\infty} f_\nu(z_2) W_n^{\alpha,\beta}(e_\nu)(z_1) - \sum_{\nu=2}^{\infty} f_\nu(z_2) W_n(e_\nu)(z_1) - \frac{\alpha-\beta z_1}{n+\beta}\sum_{\nu=1}^{\infty} \nu z_1^{\nu-1} f_\nu(z_2) \\
= & \sum_{\nu=2}^{\infty} f_\nu(z_2)\left[W_n(e_\nu)(z_1) - e_\nu(z_1) - \frac{z_1(1+z_1)}{2n}\nu(\nu-1) z_1^{\nu-2}\right] \\
& + \sum_{\nu=2}^{\infty} f_\nu(z_2)\left[W_n^{\alpha,\beta}(e_\nu)(z_1) - W_n(e_\nu)(z_1) - \frac{\alpha-\beta z_1}{n+\beta}\nu z_1^{\nu-1}\right]
\end{aligned}$$

Applying $W_m^{\gamma,\delta}$ to the last expression with respect to z_2, we obtain

$$\begin{aligned}
E_1 &= \sum_{\nu=2}^{\infty} W_m^{\gamma,\delta}(f_\nu)(z_2)\left[W_n(e_\nu)(z_1) - e_\nu(z_1) - \frac{z_1(1+z_1)}{2n}\nu(\nu-1)z_1^{\nu-2}\right] \\
&\quad + \sum_{\nu=2}^{\infty} W_m^{\gamma,\delta}(f_\nu)(z_2)\left[W_n^{\alpha,\beta}(e_\nu)(z_1) - W_n(e_\nu)(z_1) - \frac{\alpha-\beta z_1}{n+\beta}\nu z_1^{\nu-1}\right] \\
&= \sum_{\nu=2}^{\infty}\left(\sum_{\mu=0}^{\infty} c_{\nu,\mu} W_m^{\gamma,\delta}(e_\mu)(z_2)\right)\left[W_n(e_\nu)(z_1) - e_\nu(z_1) - \frac{z_1(1+z_1)}{2n}\nu(\nu-1)z_1^{\nu-2}\right] \\
&\quad + \sum_{\nu=2}^{\infty}\left(\sum_{\mu=0}^{\infty} c_{\nu,\mu} W_m^{\gamma,\delta}(e_\mu)(z_2)\right)\left[W_n^{\alpha,\beta}(e_\nu)(z_1) - W_n(e_\nu)(z_1) - \frac{\alpha-\beta z_1}{n+\beta}\nu z_1^{\nu-1}\right]
\end{aligned}$$

Passing to absolute value with $|z_1| \leq r_1$and $|z_2| \leq r_2$ and taking into account the estimates in proofs of Theorem 2.4.2 in [8. pp.175-176], it follows

$$\begin{aligned}
|E_1| &\leq \sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\, r_2^{\mu}(\mu+1)!\left[\frac{16 r_1^{\nu}\nu!(\nu-1)(\nu-2)^2}{n^2}\right] \\
&\quad + \sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty}|c_{\nu,\mu}|\, r_2^{\mu}(\mu+1)!\left[\frac{(\nu-1)\nu!\alpha^2}{2(n+\beta)^2} r_1^{\nu-2} + \frac{2\alpha\nu^2\nu!}{(n+\beta)^2} r_1^{\nu-1}\right. \\
&\quad \left. + \frac{\nu^2(\nu+1)!}{(n+\beta)^2}\left(\frac{\beta^2}{2}+2\beta\right) r_1^{\nu} + \frac{\nu(\nu-1)\alpha\beta}{(n+\beta)^2} r_1^{\nu-1} + \frac{\nu(\nu-1)\beta^2}{(n+\beta)^2} r_1^{\nu}\right] \\
&\leq \frac{1}{n^2}\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} 16M (A_1 r_1)^{\nu}(A_2 r_2)^{\mu}(\nu-1)(\nu-2)^2(\mu+1) \\
&\quad + \frac{1}{(n+\beta)^2}\sum_{\nu=2}^{\infty}\sum_{\mu=0}^{\infty} M\frac{(A_1 r_1)^{\nu-2}(A_2 r_2)^{\mu}}{\nu!}(\mu+1)\left[\frac{(\nu-1)\nu!\alpha^2}{2} + 2\alpha\nu^2\nu! r_1\right. \\
&\quad \left. + \nu^2(\nu+1)!\left(\frac{\beta^2}{2}+2\beta\right) r_1^2 + \nu(\nu-1)\alpha\beta r_1 + \nu(\nu-1)\beta^2 r_1^2\right]
\end{aligned}$$

Similarly,

$$\begin{aligned}
|E_2| &\leq \frac{1}{n^2}\sum_{\mu=0}^{\infty}\sum_{\nu=2}^{\infty} 16M\frac{(r_1 A_1)^{\nu}(r_2 A_2)^{\mu}}{\mu!}(\nu-1)(\nu-2)^2 \\
&\quad + \frac{1}{(n+\beta)^2}\sum_{\mu=0}^{\infty}\sum_{\nu=2}^{\infty} M\frac{(A_1 r_1)^{\nu-2}(r_2 A_2)^{\mu}}{\nu!\mu!}\left[\frac{(\nu-1)\nu!\alpha^2}{2} + 2\alpha\nu^2(\nu+1)! r_1\right. \\
&\quad \left. + \left(\frac{\beta^2}{2}+2\beta\right)\nu! r_1^2 + \nu(\nu-1)\alpha\beta r_1 + \nu(\nu-1)\beta^2 r_1^2\right]
\end{aligned}$$

Then

$$W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1) = \sum_{\nu=0}^{\infty}\left(\frac{\partial^2 f_\nu}{\partial z_2^2}(.,z_2)\right)W_n^{\alpha,\beta}(e_\nu)(z_1) = \sum_{\nu=0}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}\mu(\mu-1)z_1^{\mu-2}W_n^{\alpha,\beta}(e_\nu)(z_1)$$

and

$$\left[W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1)-\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) -\frac{\alpha-\beta z_1}{n+\beta}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial f}{\partial z_1}\right)(z_1,z_2)-\frac{z_1(1+z_1)}{2n}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial^2 f}{\partial z_1^2}\right)(z_1,z_2)\right]$$

$$= \sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}\mu(\mu-1)z_2^{\mu-2}W_n^{\alpha,\beta}(e_\nu)(z_1)-\sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}\mu(\mu-1)z_2^{\mu-2}(e_\nu)(z_1) -\frac{\alpha-\beta z_1}{n+\beta}\sum_{\nu=1}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}\nu z_1^{\nu-1}\mu(\mu-1)z_2^{\mu-2}-\frac{z_1(1+z_1)}{2n}\sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}z_1^{\nu-2}\mu(\mu-1)$$

$$= \sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}c_{\nu,\mu}\mu(\mu-1)z_2^{\mu-2}\left[W_n^{\alpha,\beta}(e_\nu)(z_1)-(e_\nu)(z_1)-\frac{\alpha-\beta z_1}{n+\beta}\nu z_1^{\nu-1} -\frac{z_1^{\nu-1}(1+z_1)\mu(\mu-1)}{2n}\right]$$

which again by Theorem 2.4.2 in [8], implies

$$|E_3| \le \frac{r_2(1+r_2)}{2m}\frac{16M}{n^2}\sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}\frac{(r_1A_1)^\nu(r_2A_2)^\mu}{\mu!}\mu(\mu-1)(\nu-1)(\nu-2)^2 +\frac{r_2(1+r_2)}{2m}\frac{1}{(n+\beta)^2}\sum_{\nu=2}^{\infty}\sum_{\mu=2}^{\infty}M\frac{(r_1A_1)^{\nu-2}(r_2A_2)^\mu}{\nu!\mu!}\mu(\mu-1)\left[\frac{(\nu-1)\nu!\alpha^2}{2} +2\alpha\nu^2 r_1(\nu+1)!+\left(\frac{\beta^2}{2}+2\beta\right)\nu! r_1^2+\nu(\nu-1)\alpha\beta r_1+\nu(\nu-1)\beta^2 r_1^2\right]$$

Interchanging above the places of n and m, by reason of symmetry, we get a similar order of approximation for $\left|{}_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2)\circ_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2)\right|$.

In conclusion if we use the commutativity property of ${}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2)\circ_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2)$,

$$\left|{}_{z_2}L_m^{\gamma,\delta}(f)(z_1,z_2)\circ_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2)\right| \le |E_1|+|E_2|+|E_3| \le M_{1,r_1,r_2}(f)\left[\frac{1}{n^2}+\frac{1}{m^2}\right]+\sum_{k=2}^{6}M_{k,r_1,r_2}(f)\left[\frac{1}{(n+\beta)^2}+\frac{1}{(m+\delta)^2}\right]$$

where the series $M_{i,r_1,r_2}(f), i = 1,2,3,4,5,6$ given by the statement are convergent due to $|c_{\nu,\mu}| \leq M \dfrac{A_1^{\nu} A_2^{\mu}}{\nu!\mu!}$. □

The following theorem will be useful to find exact order of approximation by $W_{n,n}^{\alpha,\beta}(f)$.

Theorem 3.4. *Let $0 \leq \alpha \leq \beta, 0 \leq \gamma \leq \delta$. Suppose that $n_0 = m_0$ and the hypothesis on the function f and on the constants n_0, m_0, R_1, R_2, M, A_1, A_2 in the statement of Theorem 1 hold and let $1 \leq r_1 < \min\left\{\dfrac{n_0}{2}, \dfrac{1}{A_1}\right\}$, $1 \leq r_2 < \min\left\{\dfrac{m_0}{2}, \dfrac{1}{A_2}\right\}$ be fixed. Denoting $\|f\|_{r_1,r_2} = \sup\{|f(z_1,z_2)|; |z_1| \leq r_1; |z_2| \leq r_2\}$ and f is not a solution of the complex partial differantial equation*

$$(\alpha - \beta z_1)\frac{\partial f}{\partial z_1}(z_1,z_2) + \frac{z_1(1+z_1)}{2}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) + (\alpha - \beta z_2)\frac{\partial f}{\partial z_2}(z_1,z_2)$$

$$+\frac{z_2(1+z_2)}{2}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) = 0, |z_1| \leq R_1, |z_2| \leq R_2 \tag{3.1}$$

then for all $n > n_0$ we have

$$\left\|W_{n,n}^{\alpha,\beta}(f) - f\right\|_{r_1,r_2} \geq \frac{K_{r_1,r_2,f}^{\alpha,\beta}}{n}$$

where $K_{r_1,r_2,f}^{\alpha,\beta}$ depends only on f, α, β, r_1, r_2.

Proof. For all $|z_1| \leq r_1$, $|z_2| \leq r_2$ and $n \in \mathbb{N}$, we can write

$$W_{n,n}^{\alpha,\beta}(f)(z_1,z_2) - f(z_1,z_2)$$

$$\begin{aligned}
= \quad & \frac{2}{n}\left\{\frac{z_2(1+z_2)}{4}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) + \frac{z_1(1+z_1)}{4}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right.\\
& +\frac{n(\alpha-\beta z_2)}{2(n+\beta)}\frac{\partial f}{\partial z_2}(z_1,z_2) + \frac{n(\alpha-\beta z_1)}{2(n+\beta)}\frac{\partial f}{\partial z_1}(z_1,z_2)\\
& \left.+\frac{2}{n}\left[\frac{n^2}{4}{}_{z_2}L_n^{\alpha,\beta}(f)(z_1,z_2) \circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2)\right] + R_n(f)(z_1,z_2)\right\}
\end{aligned}$$

where

$$\begin{aligned}
R_n(f)(z_1,z_2) &= W_n^{\alpha,\beta}(f)(.,z_2)(z_1) - f(z_1,z_2) \\
&\quad -\frac{\alpha-\beta z_1}{n+\beta}\frac{\partial f}{\partial z_1}(z_1,z_2) - \frac{z_1(1+z_1)}{2n}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) \\
&\quad +W_n^{\alpha,\beta}(f)(z_1,.)(z_2) - f(z_1,z_2) \\
&\quad -\frac{\alpha-\beta z_2}{n+\beta}\frac{\partial f}{\partial z_2}(z_1,z_2) - \frac{z_2(1+z_2)}{2n}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) \\
&\quad +\frac{z_2(1+z_2)}{2n}\left[W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_2^2}(.,z_2)\right)(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)\right] \\
&\quad +\frac{z_1(1+z_1)}{2n}\left[W_n^{\alpha,\beta}\left(\frac{\partial^2 f}{\partial z_1^2}(z_1,.)\right)(z_2) - \frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right] \\
&\quad -\frac{z_1(1+z_1)}{2n}\frac{z_2(1+z_2)}{2n}\frac{\partial^4 f}{\partial z_1^2\partial z_2^2}(z_1,z_2)
\end{aligned}$$

$$+\frac{\alpha-\beta z_1}{n+\beta}\left[W_n^{\alpha,\beta}\left(\frac{\partial f}{\partial z_1}(z_1,.)\right)(z_2) - \frac{\partial f}{\partial z_1}(z_1,z_2)\right]$$
$$-\frac{z_2(1+z_2)}{2n}\frac{\alpha-\beta z_1}{n+\beta}\frac{\partial^2}{\partial z_2^2}\left(\frac{\partial f}{\partial z_1}\right)(z_1,z_2)$$

By using Theorem 2.4.3 in [8] and Theorem 3 in [6] immediate that $\|R_n(f)\|_{r_1,r_2} \to 0$ as $n\to\infty$. Also, by Theorem (3) we obtain

$$\frac{n^2}{4}\left\|{}_{z_2}L_n^{\alpha,\beta}(f)(z_1,z_2)\circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2)\right\|_{r_1,r_2}$$
$$\le \frac{M_{1,r_1,r_2}(f)}{2} + \frac{n^2}{2(n+\beta)^2}\sum_{k=2}^{6} M_{k,r_1,r_2}(f)$$

which implies

$$\frac{2}{n}\left\|\frac{n^2}{4}\left[{}_{z_2}L_n^{\alpha,\beta}(f)(z_1,z_2)\circ_{z_1} L_n^{\alpha,\beta}(f)(z_1,z_2)\right] + R_n(f)\right\|_{r_1,r_2} \to 0, \text{as } n\to\infty.$$

Denoting

$$\begin{aligned}
H(z_1,z_2) &= \frac{z_2(1+z_2)}{4}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2) + \frac{n(\alpha-\beta z_2)}{2(n+\beta)}\frac{\partial f}{\partial z_2}(z_1,z_2) \\
&\quad +\frac{z_1(1+z_1)}{4}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2) + \frac{n(\alpha-\beta z_1)}{2(n+\beta)}\frac{\partial f}{\partial z_1}(z_1,z_2)
\end{aligned}$$

and taking into the inequalities

$$\|F+G\|_{r_1,r_2} \ge \left|\|F\|_{r_1,r_2} - \|G\|_{r_1,r_2}\right| \ge \|F\|_{r_1,r_2} - \|G\|_{r_1,r_2}$$

it follows

$$\begin{aligned}\left\|W_{n,n}^{\alpha,\beta}(f)-f\right\|_{r_1,r_2} &= \Bigg\| \frac{2}{n}\left\{\frac{z_2(1+z_2)}{4}\frac{\partial^2 f}{\partial z_2^2}(z_1,z_2)+\frac{z_1(1+z_1)}{4}\frac{\partial^2 f}{\partial z_1^2}(z_1,z_2)\right.\\ &\quad+\frac{n(\alpha-\beta z_2)}{2(n+\beta)}\frac{\partial f}{\partial z_2}(z_1,z_2)+\frac{n(\alpha-\beta z_1)}{2(n+\beta)}\frac{\partial f}{\partial z_1}(z_1,z_2)\\ &\quad\left.+\frac{2}{n}\left[\frac{n^2}{4}{}_{z_2}L_n^{\alpha,\beta}(f)(z_1,z_2)\circ_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2)\right]+R_n(f)(z_1,z_2)\right\}\Bigg\|_{r_1,r_2}\\ &\geq \frac{2}{n}\left\{\|H\|_{r_1,r_2}-\left\|\frac{2}{n}\frac{n^2}{4}{}_{z_2}L_n^{\alpha,\beta}(f)(z_1,z_2)\circ_{z_1}L_n^{\alpha,\beta}(f)(z_1,z_2)+R_n(f)\right\|_{r_1,r_2}\right\}\\ &\geq \frac{2}{n}\frac{1}{2}\|H\|_{r_1,r_2}=\frac{1}{n}\|H\|_{r_1,r_2}\end{aligned}$$

for all $n\geq n_0$, with n_0 depending only on f, r_1 and r_2. We used here that by hypothesis we have $\|H\|_{r_1,r_2}\geq 0$.For $n\in\{1,2,...,n_0-1\}$ we obviously have $\left\|W_{n,n}^{\alpha,\beta}(f)-f\right\|_{r_1,r_2}\geq \frac{{}^nN_{r_1,r_2,f}^{\alpha,\beta}}{n}$ with ${}^nN_{r_1,r_2,f}^{\alpha,\beta}=n.\left\|W_{n,n}^{\alpha,\beta}(f)-f\right\|_{r_1,r_2}>0$, which finally implies $\left\|W_{n,n}^{\alpha,\beta}(f)-f\right\|_{r_1,r_2}\geq \frac{K_{r_1,r_2,f}^{\alpha,\beta}}{n}$ for all $n\in\mathbb{N}$, where $K_{r_1,r_2,f}^{\alpha,\beta}=\min\left\{\|H\|_{r_1,r_2}^1, N_{r_1,r_2,f}^{\alpha,\beta}, {}^2N_{r_1,r_2,f}^{\alpha,\beta},...,{}^{n_0-1}N_{r_1,r_2,f}^{\alpha,\beta}\right\}$. □

Combining Theorem 2 with Theorem 4 we obtain the following exact order.

Corollary 1. *If f is not a solution of equation (3.1), then the exact order in approximation by the bivariative complex Baskakov-Stancu operator $W_{n,n}^{\alpha,\beta}(f)$ in $\frac{1}{n}$.*

Note that, for $\alpha=\beta=0,\gamma=\delta=0$, the Theorems 2,3 and 4 become the results in the book [8. pp. 172-179].

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