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MULTIPLE POSITIVE SOLUTIONS FOR A MULTI-POINT DISCRETE BOUNDARY VALUE PROBLEM

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Abstract. We study the existence and multiplicity of positive solutions for a system of nonlinear second-order difference equations subject to multi-point boundary conditions.

1. Introduction

We consider the system of nonlinear second-order difference equations

$$(S)\qquad \begin{cases} \Delta^2 u_{n-1} + f(n, v_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + g(n, u_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the multi-point boundary conditions

$$(BC)\qquad u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \ \ u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \ \ v_0 = \sum_{i=1}^{r} c_i v_{\zeta_i}, \ \ v_N = \sum_{i=1}^{l} d_i v_{\rho_i},$$

where $N \in \mathbb{N}$, $N \geq 2$, p, q, r, $l \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, and $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for $k, m \in \mathbb{N}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $\zeta_i \in \mathbb{N}$ for all $i = \overline{1, r}$, $\rho_i \in \mathbb{N}$ for all $i = \overline{1, l}$, $1 \leq \xi_1 < \ldots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$ and $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$.

Under some assumptions on f and g, we prove the existence and multiplicity of positive solutions of problem $(S)-(BC)$, by applying the fixed point index theory. By a positive solution of $(S)-(BC)$, we understand a pair of sequences $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ which satisfies (S) and (BC) and $u_n \geq 0$, $v_n \geq 0$ for all $n = \overline{0, N}$ and $\max_{n=\overline{0,N}} u_n > 0$, $\max_{n=\overline{0,N}} v_n > 0$. This problem is a generalization of the one studied in [2] where $a_i = 0$ for all $i = \overline{1, p}$ and $c_i = 0$ for all $i = \overline{1, r}$. In the last years, some multi-point boundary value problems for systems of

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nonlinear second-order difference equations which involve positive eigenvalues have been investigated. Namely, in [4] and [5], by using the Guo-Krasnosel'skii fixed point theorem, the authors give sufficient conditions for λ, μ, f and g such that the system

$$(S_1) \qquad \begin{cases} \Delta^2 u_{n-1} + \lambda s_n \widetilde{f}(n, u_n, v_n) = 0, \ \ n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + \mu t_n \widetilde{g}(n, u_n, v_n) = 0, \ \ n = \overline{1, N-1}, \end{cases}$$

with the boundary conditions (BC) has positive solutions ($u_n \geq 0$, $v_n \geq 0$ for all $n = \overline{0, N}$ and $(u, v) \neq (0, 0)$). They also study here the nonexistence of positive solutions of $(S_1) - (BC)$. In [3], the system (S) with $f(n, v_n) = \widetilde{c}_n \widetilde{\widetilde{f}}(v_n)$ and $g(n, u_n) = \widetilde{d}_n \widetilde{\widetilde{g}}(u_n)$ has been investigated with the boundary conditions $\alpha u_0 - \beta \Delta u_0 = 0$, $u_N = \sum_{i=1}^{m-2} a_i u_{\xi_i} + a_0$, $\gamma v_0 - \delta \Delta v_0 = 0$, $v_N = \sum_{i=1}^{p-2} b_i v_{\eta_i} + b_0$ where a_0, $b_0 > 0$. In this last paper, the existence of positive solutions is proved by using the Schauder fixed point theorem and the nonexistence of positive solutions is also studied.

The paper is organized as follows. In Section 2, we present some auxiliary results from [4] which investigate a boundary value problem for second-order difference equations (problem (1) − (2) below). In Section 3, we prove our main results, and an example which illustrate the obtained results is given in Section 4.

2. Auxiliary results

In this section, we present some auxiliary results from [4] related to the following second-order difference system with multi-point boundary conditions

$$\Delta^2 u_{n-1} + y_n = 0, \ \ n = \overline{1, N-1}, \tag{2.1}$$

$$u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \ \ u_N = \sum_{i=1}^{q} b_i u_{\eta_i}. \tag{2.2}$$

Lemma 2.1. *([4]) If $\Delta_1 = \left(1 - \sum_{i=1}^{q} b_i\right) \sum_{i=1}^{p} a_i \xi_i + \left(1 - \sum_{i=1}^{p} a_i\right)\left(N - \sum_{i=1}^{q} b_i \eta_i\right) \neq 0$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$ and $y_n \in \mathbb{R}$ for all $n = \overline{1, N-1}$, then the solution of (2.1)-(2.2) is given by $u_n = \sum_{j=1}^{N-1} G_1(n, j) y_j$ for all $n = \overline{0, N}$, where G_1 is defined by*

$$\begin{aligned} G_1(n, j) &= g_0(n, j) + \frac{1}{\Delta_1}\left[(N-n)\left(1 - \sum_{k=1}^{q} b_k\right) + \sum_{i=1}^{q} b_i (N - \eta_i)\right] \sum_{i=1}^{p} a_i g_0(\xi_i, j) \\ &+ \frac{1}{\Delta_1}\left[n\left(1 - \sum_{k=1}^{p} a_k\right) + \sum_{i=1}^{p} a_i \xi_i\right] \sum_{i=1}^{q} b_i g_0(\eta_i, j), \ \ n = \overline{0, N}, \ \ j = \overline{1, N-1}, \end{aligned} \tag{2.3}$$

and

$$g_0(n, j) = \frac{1}{N} \begin{cases} j(N-n), & 1 \leq j \leq n \leq N, \\ n(N-j), & 0 \leq n \leq j \leq N-1. \end{cases}$$

Lemma 2.2. *([4]) If $a_i \geq 0$ for all $i = \overline{1,p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1,q}$, $\sum_{i=1}^{q} b_i < 1$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$, then the Green's function G_1 of the problem (2.1)-(2.2), given by (2.3), satisfies $G_1(n,j) \geq 0$ for all $n = \overline{0,N}$, $j = \overline{1,N-1}$. Moreover, if $y_n \geq 0$ for all $n = \overline{1,N-1}$, then the unique solution u_n, $n = \overline{0,N}$, of the problem (2.1)-(2.2) satisfies $u_n \geq 0$ for all $n = \overline{0,N}$.*

Lemma 2.3. *([4]) Assume that $a_i \geq 0$ for all $i = \overline{1,p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1,q}$, $\sum_{i=1}^{q} b_i < 1$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$. Then the Green's function G_1 of the problem (2.1)-(2.2) satisfies the inequalities*

a) $G_1(n,j) \leq I_1(j), \ \forall\, n = \overline{0,N}, \ j = \overline{1,N-1}$, *where*

$$I_1(j) = g_0(j,j) + \frac{1}{\Delta_1}\left(N - \sum_{i=1}^{q} b_i\eta_i\right)\sum_{i=1}^{p} a_i g_0(\xi_i, j)$$
$$+\frac{1}{\Delta_1}\left(N - \sum_{i=1}^{p} a_i(N-\xi_i)\right)\sum_{i=1}^{q} b_i g_0(\eta_i, j).$$

b) For every $c \in \{1, \ldots, [N/2]\}$, we have

$$\min_{n=\overline{c,N-c}} G_1(n,j) \geq \gamma_1 I_1(j) \geq \gamma_1 G_1(n',j), \ \forall\, n' = \overline{0,N}, \ j = \overline{1,N-1},$$

where

$$\gamma_1 = \min\left\{\frac{c}{N-1}, \ \frac{c\,(1-\sum_{k=1}^{q} b_k) + \sum_{i=1}^{q} b_i(N-\eta_i)}{N - \sum_{i=1}^{q} b_i\eta_i}, \right.$$
$$\left. \frac{c\,(1-\sum_{k=1}^{p} a_k) + \sum_{i=1}^{p} a_i\xi_i}{N - \sum_{i=1}^{p} a_i(N-\xi_i)}\right\} > 0.$$

Lemma 2.4. *([4]) Assume that $a_i \geq 0$ for all $i = \overline{1,p}$, $\sum_{i=1}^{p} a_i < 1$, $b_i \geq 0$ for all $i = \overline{1,q}$, $\sum_{i=1}^{q} b_i < 1$, $1 \leq \xi_1 < \xi_2 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \eta_2 < \cdots < \eta_q \leq N-1$, $c \in \{1, \ldots, [N/2]\}$ and $y_n \geq 0$ for all $n = \overline{1,N-1}$. Then the solution u_n, $n = \overline{0,N}$, of the problem (2.1)-(2.2) satisfies the inequality $\min_{n=\overline{c,N-c}} u_n \geq \gamma_1 \max_{m=\overline{0,N}} u_m$.*

Similar results as Lemmas 2.1 - 2.4 above are also obtained for the discrete boundary value problem

$$\Delta^2 v_{n-1} + h_n = 0, \ \ n = \overline{1,N-1}, \tag{2.4}$$

$$v_0 = \sum_{i=1}^{r} c_i v_{\zeta_i}, \ \ v_N = \sum_{i=1}^{l} d_i v_{\rho_i}, \tag{2.5}$$

where $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, $c_i \geq 0$ for $i = \overline{1,r}$, $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$, $d_i \geq 0$ for $i = \overline{1,l}$ and $h_n \in \mathbb{R}$ for all $n = \overline{1,N-1}$. We denote by $\Delta_2, \ \gamma_2, \ G_2$ and I_2 the corresponding constants and functions for the problem (2.4)-(2.5) defined in a similar manner as $\Delta_1, \ \gamma_1, \ G_1$ and I_1, respectively.

3. Main results

In this section, we give sufficient conditions on f and g such that positive solutions with respect to a cone for our problem $(S)-(BC)$ exist.

We present the basic assumptions that we use in the sequel

$(A1)$ $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $a_i \geq 0$ for all $i=\overline{1,p}$, $\sum_{i=1}^{p} a_i < 1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, $b_i \geq 0$ for all $i=\overline{1,q}$, $\sum_{i=1}^{q} b_i < 1$, $1 \leq \zeta_1 < \cdots < \zeta_r \leq N-1$, $c_i \geq 0$ for all $i=\overline{1,r}$, $\sum_{i=1}^{r} c_i < 1$, $1 \leq \rho_1 < \cdots < \rho_l \leq N-1$, $d_i \geq 0$ for all $i=\overline{1,l}$, $\sum_{i=1}^{l} d_i < 1$.

$(A2)$ The functions $f,\, g : \{1,..,N-1\} \times [0,\infty) \to [0,\infty)$ are continuous and $f(n,0)=0,\ g(n,0)=0$, for all $n=\overline{1,N-1}$.

The pair of sequences $\Big((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\Big)$, $u_n \geq 0$, $v_n \geq 0$ for all $n=\overline{0,N}$ is a solution for the problem $(S)-(BC)$ if and only if $\Big((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\Big)$, $u_n \geq 0$, $v_n \geq 0$ for all $n=\overline{0,N}$ is a solution for the nonlinear system

$$\begin{cases} u_n = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) f(i,v_i), \ \ n=\overline{0,N}, \\ v_n = \displaystyle\sum_{i=1}^{N-1} G_2(n,i) g(i,u_i), \ \ n=\overline{0,N}. \end{cases} \tag{3.1}$$

Besides, the system (3.1) can be written as the nonlinear system

$$\begin{cases} u_n = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right), \ n=\overline{0,N}, \\ v_n = \displaystyle\sum_{i=1}^{N-1} G_2(n,i) g(i,u_i), \ n=\overline{0,N}. \end{cases}$$

We consider the Banach space $X=\mathbb{R}^{N+1}=\big\{u=(u_0,u_1,...,u_N),\ \ u_i\in\mathbb{R},\ \ i=\overline{0,N}\big\}$ with maximum norm $\|\cdot\|$, $\|u\| = \max_{i=\overline{0,N}} |u_i|$, for $u=(u_n)_{n=\overline{0,N}}$, and define the cone $P \subset X$ by $P=\{u\in X,\ \ u=(u_n)_{n=\overline{0,N}},\ \ u_n \geq 0,\ \ n=\overline{0,N}\}$.

We also define the operators $\mathcal{A}: P \to X$, $\mathcal{B}: P \to X$ and $\mathcal{C}: P \to X$ by

$$\mathcal{A}\left((u_n)_{n=\overline{0,N}}\right) = \left(\sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right)\right)_{n=\overline{0,N}},$$

$$\mathcal{B}\left((u_n)_{n=\overline{0,N}}\right)) = \left(\sum_{i=1}^{N-1} G_1(n,i) u_i\right)_{n=\overline{0,N}}, \ \ \mathcal{C}\left((u_n)_{n=\overline{0,N}}\right) = \left(\sum_{i=1}^{N-1} G_2(n,i) u_i\right)_{n=\overline{0,N}}.$$

Under the assumptions $(A1)$, $(A2)$, using also Lemma 2.2, it is easy to see that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are completely continuous from P to P. Thus the existence and multiplicity of positive solutions of the system $(S)-(BC)$ are equivalent to the existence and multiplicity of fixed points of the operator $\mathcal{A}$.

Theorem 3.1. *Assume that* $(A1)$ *and* $(A2)$ *hold. In addition, we suppose that there exists* $c \in \{1, ..., [N/2]\}$ *such that the following assumptions are satisfied*
$(A3)$ *There exists a positive constant* $p_1 \in (0, 1]$ *such that*

$$i)\ f_\infty^i = \liminf_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u^{p_1}} \in (0,\infty];\quad ii)\ g_\infty^i = \liminf_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u^{1/p_1}} = \infty;$$

$(A4)$ *There exists a positive constant* $q_1 \in (0, \infty)$ *such that*

$$i)\ f_0^s = \limsup_{u\to 0^+} \max_{n=\overline{1,N-1}} \frac{f(n,u)}{u^{q_1}} \in [0,\infty);\quad ii)\ g_0^s = \limsup_{u\to 0^+} \max_{n=\overline{1,N-1}} \frac{g(n,u)}{u^{1/q_1}} = 0.$$

Then the problem $(S)-(BC)$ *has at least one positive solution* $\left((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\right)$.

Proof. From assumption i) of $(A3)$, we deduce that there exist C_1, $C_2 > 0$ such that

$$f(n,u) \geq C_1 u^{p_1} - C_2,\ \ \forall\, n = \overline{1, N-1},\ \ u \in [0,\infty). \tag{3.2}$$

Then, for $u \in P$, by using (3.2), the reverse form of Cauchy inequality and Lemma 2.3, we have for $p \in (0, 1]$ that there exist $\widetilde{C}_1$, $C_3 > 0$ such that

$$(\mathcal{A}u)_n \geq \widetilde{C}_1 \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} (G_2(i,j))^{p_1} (g(j,u_j))^{p_1} \right) - C_3,\ \ \forall\, n = \overline{0,N}, \tag{3.3}$$

Now we define the cone $P_0 = \{u \in P;\ \min_{n=\overline{c,N-c}} u_n \geq \gamma\|u\|\}$, where $\gamma = \min\{\gamma_1, \gamma_2\}$ and γ_1, γ_2 are defined in Section 2. From our assumptions and Lemma 2.4, we conclude that for any $y \in P$, $y = (y_n)_{n=\overline{0,N}}$, the sequences $u = \mathcal{B}(y)$, $u = (u_n)_{n=\overline{0,N}}$ and $v = \mathcal{C}(y)$, $v = (v_n)_{n=\overline{0,N}}$ satisfy the inequalities $\min_{n=\overline{c,N-c}} u_n \geq \gamma_1\|u\| \geq \gamma\|u\|$ and $\min_{n=\overline{c,N-c}} v_n \geq \gamma_2\|v\| \geq \gamma\|v\|$. So $u = \mathcal{B}(y),\ v = \mathcal{C}(y) \in P_0$. Therefore we deduce that $\mathcal{B}(P) \subset P_0$, $\mathcal{C}(P) \subset P_0$.

We denote by $u^0 = (u_n^0)_{n=\overline{0,N}}$ the solution of the problem (2.1)-(2.2) for $y^0 = (y_n^0)_{n=\overline{1,N-1}}$, $y_n^0 = 1$ for $n = \overline{1, N-1}$. Then by Lemma 2.2, we obtain $u_n^0 = \sum_{i=1}^{N-1} G_1(n,i) \geq 0$ for all $n = \overline{0,N}$. So $u^0 = \mathcal{B}(y^0) \in P_0$.

Now let the set $M = \{u \in P;\ \text{there exists}\ \lambda \geq 0\ \text{such that}\ u = \mathcal{A}u + \lambda u^0\}$. We shall show that M is a bounded subset of X. If $u \in M$, then there exists $\lambda \geq 0$ such that $u = \mathcal{A}u + \lambda u^0$, $u = (u_n)_{n=\overline{0,N}}$, with $u_n = (\mathcal{A}u)_n + \lambda u_n^0$ for all $n = \overline{0,N}$. Then we have $u_n = (\mathcal{A}u)_n + \lambda u_n^0 = (\mathcal{B}(F(u) + \lambda y^0))_n$ for all $n = \overline{0,N}$, so $u \in P_0$, where $F : P \to P$ is defined by $(Fu)_n = f\left(n, \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i)\right)$, $n = \overline{0,N}$. Therefore $M \subset P_0$, and from the definition of P_0, we deduce

$$\|u\| \leq \frac{1}{\gamma} \min_{n=\overline{c,N-c}} u_n,\ \ \forall\, u = (u_n)_{n=\overline{0,N}} \in M. \tag{3.4}$$

From ii) of assumption $(A3)$, we conclude that for $\varepsilon_0 = 2/(\widetilde{C}_1 m_1 m_2 \gamma_1 \gamma_2^{p_1})$ there exists $C_4 > 0$ such that

$$(g(n,u))^{p_1} \geq \varepsilon_0 u - C_4, \ \ \forall\, n = \overline{c, N-c}, \ \ u \in [0,\infty), \tag{3.5}$$

where $m_1 = \sum_{i=c}^{N-c} I_1(i) > 0$ and $m_2 = \sum_{i=c}^{N-c} (I_2(i))^{p_1} > 0$.

For $u \in M$ and $n = \overline{c, N-c}$, by using Lemma 2.3 and relations (3.3), (3.5), it follows that

$$\begin{aligned}
u_n &= (\mathcal{A}u)_n + \lambda u_n^0 \geq (\mathcal{A}u)_n \\
&\geq \widetilde{C}_1 \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} (G_2(i,j))^{p_1} (g(j,u_j))^{p_1} \right) - C_3 \\
&\geq \widetilde{C}_1 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=c}^{N-c} (G_2(i,j))^{p_1} (g(j,u_j))^{p_1} \right) - C_3 \\
&\geq \widetilde{C}_1 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=c}^{N-c} \gamma_2^{p_1} (I_1(j))^{p_1} (g(j,u_j))^{p_1} \right) - C_3 \\
&\geq \widetilde{C}_1 \gamma_1 \gamma_2^{p_1} \sum_{i=c}^{N-c} I_1(i) \left(\sum_{j=c}^{N-c} (I_1(j))^{p_1} (\varepsilon_0 u_j - C_4) \right) - C_3 \\
&\geq \widetilde{C}_1 \gamma_1 \gamma_2^{p_1} \varepsilon_0 \left(\sum_{i=c}^{N-c} I_1(i) \right) \left(\sum_{j=c}^{N-c} (I_1(j))^{p_1} \right) \min_{j=\overline{c,N-c}} u_j - C_5 \\
&= \widetilde{C}_1 \gamma_1 \gamma_2^{p_1} \varepsilon_0 m_1 m_2 \min_{j=\overline{c,N-c}} u_j - C_5 = 2 \min_{j=\overline{c,N-c}} u_j - C_5,
\end{aligned}$$

where $C_5 = C_3 + \widetilde{C}_1 C_4 \gamma_1 \gamma_2^{p_1} m_1 m_2 > 0$.

Hence $\min_{n=\overline{c,N-c}} u_n \geq 2 \min_{j=\overline{c,N-c}} u_j - C_5$, and so

$$\min_{n=\overline{c,N-c}} u_n \leq C_5, \ \ \forall\, u \in M. \tag{3.6}$$

Now from relations (3.4) and (3.6), we deduce that $\|u\| \leq \min_{n=\overline{c,N-c}} u_n/\gamma \leq C_5/\gamma$ for all $u \in M$, that is M is a bounded subset of X.

Therefore there exists a sufficiently large $L > 0$ such that $u \neq \mathcal{A}u + \lambda u_0$ for all $u \in \partial B_L \cap P$ and $\lambda \geq 0$. From [1] (see also Lemma 2 from [6]), we conclude that

$$i(\mathcal{A}, B_L \cap P, P) = 0. \tag{3.7}$$

In what follows, from assumptions $(A4)$ and $(A2)$, we deduce that there exists $M_0 > 0$ and $\delta_1 \in (0,1)$ such that

$$\begin{aligned}
&f(n,u) \leq M_0 u^{q_1}, \ \forall\, n = \overline{1, N-1}, \ u \in [0,1]; \\
&g(n,u) \leq \varepsilon_1 u^{1/q_1}, \ \forall\, n = \overline{1, N-1}, \ u \in [0,\delta_1],
\end{aligned} \tag{3.8}$$

where $\varepsilon_1 = \min\{1/M_2, (1/(2M_0 M_1 M_2^{q_1}))^{1/q_1}\} > 0$, $M_1 = \sum_{j=1}^{N-1} I_1(j)$, $M_2 = \sum_{j=1}^{N-1} I_2(j)$.

Hence, for all $u \in \overline{B}_{\delta_1} \cap P$ and $i = \overline{0, N}$, we obtain

$$\sum_{j=1}^{N-1} G_2(i,j) g(j,u_j) \leq \varepsilon_1 \sum_{j=1}^{N-1} G_2(i,j) u_j^{1/q_1} \leq \varepsilon_1 \sum_{j=1}^{N-1} I_2(j) u_j^{1/q_1} \leq \varepsilon_1 M_2 \|u\|^{1/q_1} \leq 1. \tag{3.9}$$

Therefore, by (3.8) and (3.9), we conclude

$$\begin{aligned}
(\mathcal{A}u)_n &= \sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right) \\
&\leq M_0 \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right)^{q_1} \leq M_0 \varepsilon_1^{q_1} M_2^{q_1} \|u\| \sum_{i=1}^{N-1} I_1(i) \\
&= M_0 \varepsilon_1^{q_1} M_1 M_2^{q_1} \|u\| \leq \tfrac{1}{2} \|u\|, \ \ \forall\, u \in \overline{B}_{\delta_1} \cap P, \ \ n = \overline{0, N}.
\end{aligned}$$

This implies that $\|\mathcal{A}u\| \leq \|u\|/2$ for all $u \in \partial B_{\delta_1} \cap P$. From [1] (see also Lemma 1 from [6]), we deduce

$$i(\mathcal{A}, B_{\delta_1} \cap P, P) = 1. \tag{3.10}$$

Combining now (3.7) and (3.10), we obtain

$$i(\mathcal{A}, (B_L \setminus \bar{B}_{\delta_1}) \cap P, P) = i(\mathcal{A}, B_L \cap P, P) - i(\mathcal{A}, B_{\delta_1} \cap P, P) = -1.$$

We conclude that $\mathcal{A}$ has at least one fixed point $u^1 \in (B_L \setminus \bar{B}_{\delta_1}) \cap P$, $u^1 = (u_n^1)_{n=\overline{0,N}}$, that is $\delta_1 < \|u^1\| < L$. In addition, we obtain $\|v^1\| > 0$, where $v^1 = (v_n^1)_{n=\overline{0,N}}$, with $v_n^1 = \sum_{i=1}^{N-1} G_2(n,i) g(i, u_i^1)$ for all $n = \overline{0, N}$, and then $(u^1, v^1) \in P \times P$ is a positive solution of $(S) - (BC)$. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that* $(A1)$ *and* $(A2)$ *hold. In addition, we suppose that there exists* $c \in \{1, ..., [N/2]\}$ *such that the following assumptions are satisfied*
$(A5)$ *There exists a positive constant* $r_1 \in (0, \infty)$ *such that*

$$i)\ \ f_\infty^s = \limsup_{u \to \infty} \max_{n=\overline{1,N-1}} \frac{f(n,u)}{u^{r_1}} \in [0, \infty); \ \ ii)\ \ g_\infty^s = \limsup_{u \to \infty} \max_{n=\overline{1,N-1}} \frac{g(n,u)}{u^{1/r_1}} = 0;$$

$(A6)$ *The following conditions are satisfied*

$$i)\ \ f_0^i = \liminf_{u \to 0^+} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u} \in (0, \infty]; \ \ ii)\ \ g_0^i = \liminf_{u \to 0^+} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u} = \infty.$$

Then the problem $(S)-(BC)$ *has at least one positive solution* $\left((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\right)$.

Proof. By assumption $(A5)$, we deduce that there exists C_6, C_7, $C_8 > 0$ such that

$$f(n,u) \leq C_6 u^{r_1} + C_7, \ \ g(n,u) \leq \varepsilon_2 u^{1/r_1} + C_8, \ \ \forall\, n = \overline{1, N-1}, \ \ u \in [0, \infty), \tag{3.11}$$

where $\varepsilon_2 = (1/(2C_6 M_1 M_2^{r_1}))^{1/r_1}$, and M_1, M_2 are defined in the proof of Theorem 3.1.

Then for $u \in P$, by using (3.11), we conclude after some computations that

$$
\begin{aligned}
(\mathcal{A}u)_n &\le C_6 \left(\varepsilon_2 \|u\|^{1/r_1} + C_8\right)^{r_1} \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} G_2(i,j)\right)^{r_1} + M_1 C_7 \\
&\le C_6 M_1 M_2^{r_1} \left(\varepsilon_2 \|u\|^{1/r_1} + C_8\right)^{r_1} + M_1 C_7 =: \mathcal{Q}(u), \ \ \forall\, n = \overline{0,N}.
\end{aligned} \tag{3.12}
$$

Because $\lim_{\|u\|\to\infty} \mathcal{Q}(u)/\|u\| = 1/2$, then there exits a sufficiently large $R > 0$ such that

$$
\mathcal{Q}(u) \le \frac{3}{4}\|u\|, \ \ \forall\, u \in P, \ \ \|u\| \ge R. \tag{3.13}
$$

Hence, from (3.12) and (3.13), we obtain $\|\mathcal{A}u\| \le \frac{3}{4}\|u\| < \|u\|$ for all $u \in \partial B_R \cap P$, and from [1] (see also Lemma 1 from [6]), we have

$$
i(\mathcal{A}, B_R \cap P, P) = 1. \tag{3.14}
$$

On the other hand, by $(A6)$ i), we deduce that there exist $C_9 > 0$ and $\varrho_0 > 0$ such that

$$
f(n,u) \ge C_9 u, \ \ g(n,u) \ge \frac{C_0}{C_9} u, \ \ \forall\, n = \overline{1, N-1}, \ \ u \in [0, \varrho_0], \tag{3.15}
$$

where $C_0 = 1/(\gamma_1 \gamma_2 m_1 \widetilde{m}_2)$, $\widetilde{m}_2 = \sum_{j=c}^{N-c} I_2(j)$ and m_1 is defined in the proof of Theorem 3.1.

Because $g(n,0) = 0$ for all $n = \overline{1, N-1}$, and g is continuous, we conclude that there exists a sufficiently small $\delta_2 \in (0, \varrho_0)$ such that $g(n,u) \le \varrho_0 / M_2$ for all $n = \overline{1, N-1}$ and $u \in [0, \delta_2]$. Hence

$$
\sum_{i=1}^{N-1} G_2(n,i) g(i,u_i) \le \varrho_0, \ \ \forall\, u \in \bar{B}_{\delta_2} \cap P, \ \ n = \overline{0,N}. \tag{3.16}
$$

From (3.15), (3.16) and Lemma 2.3, we deduce that for any $u \in \bar{B}_{\delta_2} \cap P$, we have

$$
\begin{aligned}
(\mathcal{A}u)_n &= \sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right) \\
&\ge C_9 \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} G_2(i,j) g(j,u_j)\right) \ge C_0 \sum_{i=1}^{N-1} G_1(n,i) \left(\sum_{j=1}^{N-1} G_2(i,j) u_j\right) \\
&\ge C_0 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=1}^{N-1} G_2(i,j) u_j\right) \ge C_0 \gamma_2 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=1}^{N-1} I_2(j) u_j\right) \\
&= C_0 \gamma_2 \left(\sum_{j=1}^{N-1} I_2(j) u_j\right) \left(\sum_{i=c}^{N-c} G_1(n,i)\right) =: (\mathcal{L}u)_n, \ \ \forall\, n = \overline{0,N}.
\end{aligned}
$$

Hence, for the linear operator $\mathcal{L} : P \to P$ defined as above, we obtain

$$\mathcal{A}u \geq \mathcal{L}u, \ \ \forall\, u \in \partial B_{\delta_2} \cap P. \tag{3.17}$$

For $w^0 = (w_n^0)_n$, $w_n^0 = \sum_{i=c}^{N-c} G_1(n,i)$, $n = \overline{0,N}$, we have $w^0 \in P \setminus \{0\}$ and

$$\begin{aligned}
(\mathcal{L}w^0)_n &= C_0\gamma_2 \left[\sum_{j=1}^{N-1} I_2(j) \left(\sum_{i=c}^{N-c} G_1(j,i)\right)\right] \left(\sum_{i=c}^{N-c} G_1(n,i)\right) \\
&\geq C_0\gamma_2 \left[\sum_{j=c}^{N-c} I_2(j) \left(\sum_{i=c}^{N-c} G_1(j,i)\right)\right] \left(\sum_{i=c}^{N-c} G_1(n,i)\right) \\
&\geq C_0\gamma_1\gamma_2 \left(\sum_{j=c}^{N-c} I_2(j)\right) \left(\sum_{i=c}^{N-c} I_1(i)\right) \left(\sum_{i=c}^{N-c} G_1(n,i)\right) \\
&= \sum_{i=c}^{N-c} G_1(n,i) = w_n^0, \ \ \forall\, n = \overline{0,N}.
\end{aligned}$$

Therefore

$$\mathcal{L}w^0 \geq w^0. \tag{3.18}$$

We may suppose that $\mathcal{A}$ has no fixed point on $\partial B_{\delta_2} \cap P$ (otherwise the proof is completed). From (3.17), (3.18) and Lemma 2.3 from [6], we deduce that

$$i(\mathcal{A}, B_{\delta_2} \cap P, P) = 0. \tag{3.19}$$

Then, from (3.14) and (3.19), we obtain

$$i(\mathcal{A}, (B_R \setminus \bar{B}_{\delta_2}) \cap P, P) = i(\mathcal{A}, B_R \cap P, P) - i(\mathcal{A}, B_{\delta_2} \cap P, P) = 1.$$

We conclude that $\mathcal{A}$ has at least one fixed point in $(B_R \setminus \bar{B}_{\delta_2}) \cap P$. Thus the problem $(S)-(BC)$ has at least one positive solution $(u,v) \in P \times P$, $u = (u_n)_{n=\overline{0,N}}$, $v = (v_n)_{n=\overline{0,N}}$ ($\|u\| > 0$, $\|v\| > 0$). This completes the proof of Theorem 3.2. $\square$

Theorem 3.3. *Assume that* $(A1)$ *and* $(A2)$ *hold. In addition, we suppose that there exists* $c \in \{1, ..., [N/2]\}$ *such that* $(A3)$, $(A6)$ *and the following assumption are satisfied*
$(A7)$ *For each* $n = \overline{1, N-1}$, $f(n,u)$ *and* $g(n,u)$ *are nondecreasing with respect to* u, *and there exists a constant* $R_0 > 0$ *such that*

$$f\left(n, m_0 \sum_{i=1}^{N-1} g(i, R_0)\right) < \frac{R_0}{m_0}, \ \ \forall\, n = \overline{1, N-1},$$

where $m_0 = \max\{K_1, K_2\}$, $K_1 = \sum_{j=1}^{N-1} I_1(j)$, $K_2 = \max_{j=\overline{1,N-1}} I_2(j)$ *and* I_1, I_2 *are defined in Section 2.*

Then the problem $(S)-(BC)$ *has at least two positive solutions* (u^1, v^1), (u^2, v^2).

Proof. From Section 2, we have $0 \le G_1(n,i) \le I_1(i)$, $0 \le G_2(n,i) \le I_2(i)$ for all $n=\overline{0,N}$, $i=\overline{1,N-1}$. By using $(A7)$, for any $u \in \partial B_{R_0} \cap P$, we obtain

$$(\mathcal{A}u)_n \le \sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} I_2(j) g(j,u_j)\right)$$
$$\le \sum_{i=1}^{N-1} G_1(n,i) f\left(i, \sum_{j=1}^{N-1} I_2(j) g(j,R_0)\right) \le \sum_{i=1}^{N-1} G_1(n,i) f\left(i, m_0 \sum_{j=1}^{N-1} g(j,R_0)\right)$$
$$< \frac{R_0}{m_0} \sum_{i=1}^{N-1} G_1(n,i) \le \frac{R_0}{m_0} \sum_{i=1}^{N-1} I_1(i) \le R_0, \ \ \forall\, n = \overline{0,N}.$$

So, $\|\mathcal{A}u\| < \|u\|$ for all $u \in \partial B_{R_0} \cap P$.

By [1] (see also Lemma 1 from [6]), we deduce that

$$i(\mathcal{A}, B_{R_0} \cap P, P) = 1. \tag{3.20}$$

On the other hand, from $(A3)$, $(A6)$ and the proofs of Theorem 3.1 and Theorem 3.2, we know that there exist a sufficiently large $L > R_0$ and a sufficiently small δ_2 with $0 < \delta_2 < R_0$ such that

$$i(\mathcal{A}, B_L \cap P, P) = 0, \ \ i(\mathcal{A}, B_{\delta_2} \cap P, P) = 0. \tag{3.21}$$

From (3.20) and (3.21), we obtain

$$i(\mathcal{A}, (B_L \setminus \bar{B}_{R_0}) \cap P, P) = -1, \ \ i(\mathcal{A}, (B_{R_0} \setminus \bar{B}_{\delta_2}) \cap P, P) = 1.$$

Then $\mathcal{A}$ has at least one fixed point u^1 in $(B_L \setminus \bar{B}_{R_0}) \cap P$ and has one fixed point u^2 in $(B_{R_0} \setminus \bar{B}_{\delta_2}) \cap P$, respectively. Therefore, the problem $(S)-(BC)$ has two distinct positive solutions (u^1, v^1), $(u^2, v^2) \in P \times P$ with $\|u^i\| > 0$, $\|v^i\| > 0$ for $i = 1, 2$. The proof of Theorem 3.3 is completed. $\Box$.

4. An example

We consider the following problem

$$(S_0) \qquad \begin{cases} \Delta^2 u_{n-1} + a(v_n^\alpha + v_n^\beta) = 0, & n = \overline{1,29}, \\ \Delta^2 v_{n-1} + b(u_n^\theta + u_n^\delta) = 0, & n = \overline{1,29}, \end{cases}$$

with the multi-point boundary conditions

$$(BC_0) \qquad \begin{cases} u_0 = \frac{2}{3} u_{15}, \ \ u_{30} = \frac{1}{3} u_8 + \frac{1}{6} u_{16} + \frac{1}{4} u_{24}, \\ v_0 = \frac{1}{3} v_9 + \frac{1}{2} v_{20}, \ \ v_{30} = \frac{1}{3} v_6 + \frac{1}{4} v_{18}, \end{cases}$$

where $\alpha > 1$, $\beta < 1$, $\theta > 2$, $\delta < 1$, a, $b > 0$.

Here $N = 30$, $p = 1$, $q = 3$, $r = 2$, $l = 2$, $a_1 = 2/3$, $\xi_1 = 15$, $b_1 = 1/3$, $b_2 = 1/6$, $b_3 = 1/4$, $\eta_1 = 8$, $\eta_2 = 16$, $\eta_3 = 24$, $c_1 = 1/3$, $c_2 = 1/2$, $\zeta_1 = 9$, $\zeta_2 = 20$, $d_1 = 1/3$, $d_2 = 1/4$, $\rho_1 = 6$, $\rho_2 = 18$, $f(n,u) = a(u^\alpha + u^\beta)$ and $g(n,u) = b(u^\theta + u^\delta)$ for all $n = \overline{1,29}$, $u \ge 0$. We have $\sum_{i=1}^{1} a_i = 2/3 < 1$, $\sum_{i=1}^{3} b_i = 3/4 < 1$,

$\sum_{i=1}^{2} c_i = 5/6 < 1$, $\sum_{i=1}^{2} d_i = 7/12 < 1$, $\Delta_1 = \frac{157}{18}$, $\Delta_2 = \frac{28}{3}$. The functions I_1 and I_2 are given by

$$I_1(j) = \begin{cases} \frac{403j}{157} - \frac{j^2}{30}, & 1 \le j \le 7, \\ \frac{960}{157} + \frac{283j}{157} - \frac{j^2}{30}, & 8 \le j \le 15, \\ \frac{5280}{157} - \frac{j}{157} - \frac{j^2}{30}, & 16 \le j \le 23, \\ \frac{7440}{157} - \frac{91j}{157} - \frac{j^2}{30}, & 24 \le j \le 29, \end{cases} \qquad I_2(j) = \begin{cases} \frac{19j}{7} - \frac{j^2}{30}, & 1 \le j \le 5, \\ \frac{27}{7} + \frac{29j}{14} - \frac{j^2}{30}, & 6 \le j \le 8, \\ \frac{639}{56} + \frac{69j}{56} - \frac{j^2}{30}, & 9 \le j \le 17, \\ \frac{1125}{56} + \frac{3j}{4} - \frac{j^2}{30}, & 18 \le j \le 19, \\ \frac{2535}{56} - \frac{57j}{112} - \frac{j^2}{30}, & 20 \le j \le 29. \end{cases}$$

We obtain $K_1 = \sum_{j=1}^{29} I_1(j) \approx 461.68046709$, $K_2 = \max_{j=\overline{1,29}} I_2(j) \approx 22.78928571$, $m_0 = K_1$. The functions $f(n,u)$ and $g(n,u)$ are nondecreasing with respect to u for any $n = \overline{1,29}$, and for $c = 1$ and $p = 1/2$ the assumptions $(A3)$ and $(A6)$ are satisfied; indeed we have

$$f_\infty^i = \lim_{u\to\infty} \frac{a(u^\alpha + u^\beta)}{u^{1/2}} = \infty, \quad g_\infty^i = \lim_{u\to\infty} \frac{b(u^\theta + u^\delta)}{u^2} = \infty,$$
$$f_0^i = \lim_{u\to 0^+} \frac{a(u^\alpha + u^\beta)}{u} = \infty, \quad g_0^i = \lim_{u\to 0^+} \frac{b(u^\theta + u^\delta)}{u} = \infty.$$

We take $R_0 = 1$ and then $\sum_{i=1}^{29} g(i, R_0) = 58b$ and $f\left(n, m_0 \sum_{i=1}^{29} g(i,1)\right) = f(n, 58bm_0)) = a[(58bm_0)^\alpha + (58bm_0)^\beta]$ for all $n = \overline{1,29}$. If $a[(58bm_0)^\alpha + (58bm_0)^\beta] < \frac{1}{m_0}$, then the assumption $(A7)$ is satisfied. For example, if $\alpha = 3/2$, $\beta = 1/2$, $b = 1/(58m_0) \approx 3.734 \cdot 10^{-5}$ and $a < 1/(2m_0)$ $(a < 1.083 \cdot 10^{-3})$, then the above inequality is satisfied. By Theorem 3.3, we deduce that the problem $(S_0) - (BC_0)$ has at least two positive solutions.

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