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# EFFECT OF GENERALIZED RELATIVE ORDER ON THE GROWTH OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their generalized relative orders and generalized relative lower orders.

## 1. Introduction

Let $f$ be an entire function defined on set of all finite complex numbers $\mathbb{C}$. The maximum modulus $Mf(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $M_f(r) = \max_{|z|=r} |f(z)|$. If $f$ is non-constant entire then $M_f(r)$ is strictly increasing and continuous and therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. On the other hand the maximum term $\mu_f(r)$ of $f$ can be defined in the following way:

$$\mu_f(r) = \max_{n\geq 0} (|a_n| r^n)$$

whose inverse is also a increasing function of $r$.

The ratios $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ and $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ are called the growth of $f$ with respect to $g$ in terms of their maximum moduli and the maximum term respectively. And the study of comparative growth properties of entire functions which is one of a prominent branch of the value distribution theory of entire functions is the prime concern of the paper. Our notations are standard within the theory of Nevanlinna's value distribution of entire functions and therefore we do not explain those in detail as available in [15]. In the sequel the following two notations are

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used:

$$\begin{aligned}\log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ;\\ \log^{[0]} x &= x\end{aligned}$$

and

$$\begin{aligned}\exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ;\\ \exp^{[0]} x &= x.\end{aligned}$$

Taking this into account the *generalized order* (respectively, *generalized lower order*) of an entire function $f$ as introduced by Sato [11] is given by:

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}$$

$$\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log\log M_{\exp z}(r)} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r}\right)$$

where $l \geq 1$.

These definitions extend the definitions of *order* $\rho_f$ and *lower order* $\lambda_f$ of an entire function $f$ since for $l = 2$, these correspond to the particular case $\rho_f^{[2]} = \rho_f(2,1) = \rho_f$ and $\lambda_f^{[2]} = \lambda_f(2,1) = \lambda_f$.

Using the inequality

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \quad \{cf. \text{ [13] }\} \text{ for } 0 \leq r < R,$$

the growth indicator $\rho_f$ ( respectively $\lambda_f$) and consequently $\rho_f^{[l]}$ ( respectively $\lambda_f^{[l]}$) are reformulated as:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \quad \left(\text{respectively } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r}\right)$$

and

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r} \quad \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_f(r)}{\log r}\right)$$

where $l \geq 1$.

For any two entire functions $f$ and $g$, Bernal {[1], [2]} introduced the definition of relative order of $f$ with respect to $g$, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r},\end{aligned}$$

which avoid comparing growth just with $\exp z$ to determine *order* of entire functions as we see in the earlier and naturally this definition coincides with the classical one [14] for $g = \exp z$.

Similarly, one can define the relative lower order of $f$ with respect to $g$ denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r} .$$

In the case of relative order, it therefore seems reasonable to state suitably an alternative definition of relative order of entire function in terms of its maximum terms. Datta and Maji [6] introduced such a definition in the following way:

**Definition 1.** [6] *The relative order $\rho_g(f)$ and the relative lower order $\lambda_g(f)$ of an entire function $f$ with respect to another entire function $g$ are defined as follows:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} .$$

Lahiri and Banerjee [10] gave a more generalized concept of relative order in the following way:

**Definition 2.** [10] *If $l \geq 1$ is a positive integer, then the $l$- th generalized relative order of $f$ with respect to $g$, denoted by $\rho_f^l(g)$ is defined by*

$$\begin{aligned} \rho_g^{[l]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} . \end{aligned}$$

*Clearly $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.*

*Likewise, one can define the generalized relative lower order of $f$ with respect to $g$ denoted by $\lambda_g^{[l]}(f)$ as*

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r} .$$

In terms of maximum terms of entire functions, Definition 2 can be reformulated as:

**Definition 3.** *For any positive integer $l \geq 1$, the growth indicators$\rho_g^{[l]}(f)$ and $\lambda_g^{[l]}(f)$ for an entire function $f$ are defined as:*

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1} \mu_f(r)}{\log r} .$$

In fact, Lemma 6 states the equivalence of Definition 2 and Definition 3.

For entire functions, the notions of the growth indicators such as *order* is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* and consequently the *generalized relative orders* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the growth of composite entire functions needs to be modified on the basis of their *relative order* some of which has been explored in [4], [5], [6], [7], [8] and [9]. In this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their *generalized relative orders (* respectively *generalized relative lower orders).*

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.** [12] *Let $f$ and $g$ be any two entire functions Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right) .$$

**Lemma 2.** [12] *If $f$ and $g$ are any two entire functions with $g(0) = 0$. Then for all sufficiently large values of $r$,*

$$\mu_{f \circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right) .$$

**Lemma 3.** [3] *If $f$ and $g$ are two entire functions then for all sufficiently large values of $r$,*

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \leq M_{f \circ g}(r) \leq M_f(M_g(r)) .$$

**Lemma 4.** [2] *Suppose that $f$ be an entire function and $\alpha > 1$, $0 < \beta < \alpha$. Then for all sufficiently large $r$,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

**Lemma 5.** [6] *If $f$ be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large $r$,*

$$\mu_f(\alpha r) \geq \beta\mu_f(r) .$$

**Lemma 6.** *Definition 2 and Definition 3 are equivalent.*

*Proof.* Taking $R = \alpha r$ in the inequalities $\mu_h(r) \leq M_h(r) \leq \frac{R}{R-r}\mu_h(R)$ $\{cf.$ [13] $\}$, for $0 \leq r < R$ we obtain that

$$M_h^{-1}(r) \leq \mu_h^{-1}(r)$$

and

$$\mu_h^{-1}(r) \leq \alpha M_h^{-1}\left(\frac{\alpha r}{(\alpha-1)}\right).$$

Since $M_h^{-1}(r)$ and $\mu_h^{-1}(r)$ are increasing functions of $r$, then for any $\alpha > 1$ it follows from the above and the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{\alpha}{\alpha-1}\mu_f(\alpha r)$ $\{cf.$ [13] $\}$ that

$$M_h^{-1}M_f(r) \leq \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] \tag{1}$$

and

$$\mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)}M_f(r)\right]. \tag{2}$$

Therefore in view of Lemma 5 we have from (1) that

$$M_h^{-1}M_f(r) \leq \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\,\alpha}{(\alpha-1)}\cdot r\right] .$$

Thus from above we get that

$$\begin{aligned} \frac{\log^{[l]} M_g^{-1}M_f(r)}{\log r} &\leq \frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)}\cdot r\right]}{\log r} \\ i.e., \ \frac{\log^{[l]} M_g^{-1}M_f(r)}{\log r} &\leq \frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)}\cdot r\right]}{\log\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)}\cdot r\right]+O(1)} \end{aligned}$$

$$\begin{aligned} i.e. \ \ \rho_g^{[l]}(f) &= \limsup_{r\to\infty}\frac{\log^{[l]} M_g^{-1}M_f(r)}{\log r} \\ &\leq \limsup_{r\to\infty}\frac{\log^{[l]} \mu_h^{-1}\mu_f\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)}\cdot r\right]}{\log\left[\frac{(2\alpha-1)\alpha}{(\alpha-1)}\cdot r\right]+O(1)} \end{aligned}$$

$$i.e., \ \rho_g^{[l]}(f) \leq \limsup_{r\to\infty}\frac{\log^{[l]} \mu_h^{-1}\mu_f(r)}{\log r} \tag{3}$$

and accordingly

$$\lambda_g^{[l]}(f) \leq \liminf_{r\to\infty}\frac{\log^{[l]} \mu_h^{-1}\mu_f(r)}{\log r} \ . \tag{4}$$

Similarly, in view of Lemma 4 it follows from (2) that

$$\mu_h^{-1}\mu_f(r) \leq \alpha M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]$$

and from above we obtain that

$$\begin{aligned}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} &\leq \frac{\log^{[l]}\alpha M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]}{\log r} \\ i.e., \frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} &\leq \frac{\log^{[l]}M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]+O(1)}{\log\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]+O(1)}\end{aligned}$$

$$\begin{aligned} i.e.\ \rho_g^{[l]}(f) &= \limsup_{r\to\infty}\frac{\log^{[l]}M_h^{-1}M_f\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]+O(1)}{\log\left[\left(\frac{2\alpha-1}{\alpha-1}\right)\cdot r\right]+O(1)} \\ &\geq \limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r}\end{aligned}$$

$$i.e.,\ \rho_g^{[l]}(f)\geq\limsup_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r} \tag{5}$$

and consequently

$$\lambda_g^{[l]}(f)\geq\liminf_{r\to\infty}\frac{\log^{[l]}\mu_h^{-1}\mu_f(r)}{\log r}\ . \tag{6}$$

Combining (3), (5) and (4), (6) we obtain that

$$\rho_g^{[l]}(f)=\limsup_{r\to\infty}\frac{\log^{[l]}\mu_g^{-1}\mu_f(r)}{\log r}\ \text{and}\ \lambda_g^{[l]}(f)=\liminf_{r\to\infty}\frac{\log^{[l]}\mu_g^{-1}\mu_f(r)}{\log r}\ .$$

This proves the lemma. □

## 3. Main Results

In this section we present the main results of the paper.

**Theorem 1.** *Let $f$, $g$ and $h$ be any three entire functions such that $\lambda_g^{[q]}<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ where $p$ and $q$ are any two positive integers with $p>1$ and $q>2$. Then*

$$(i)\ \liminf_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-q+1]}\mu_h^{-1}\mu_f(r)}=0$$

*and*

$$(ii)\ \liminf_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p-q+1]}M_h^{-1}M_f(r)}=0\ .$$

*Proof.* Since $\mu_h^{-1}(r)$ is an increasing function of $r$, taking $R = \beta r\,(\beta > 1)$ in Lemma 1 and in view of Lemma 5 it follows for a sequence of values of $r$ tending to infinity that

$$\begin{aligned} \mu_{f\circ g}(r) &\leq \frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r)\right) \\ i.e.,\ \mu_{f\circ g}(r) &\leq \mu_f\left(\frac{(2\alpha-1)\,\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right) \end{aligned}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) \leq \log^{[p]}\mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\,\alpha\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right) \tag{7}$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}(\beta r)^{\lambda_g^{[q]}+\varepsilon} + O(1). \tag{8}$$

Again from Definition 3, we obtain for all sufficiently large values of $r$ that

$$\log^{[p-q+1]}\mu_h^{-1}\mu_f(r) \geqslant \exp^{[q-2]} r^{\lambda_h^{[p]}(f)-\varepsilon}\ . \tag{9}$$

Now in view of (8) and (9), we get for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p-q+1]}\mu_h^{-1}\mu_f(r)} \leq \frac{\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}(\beta r)^{\lambda_g^{[q]}+\varepsilon}+O(1)}{\exp^{[q-2]} r^{\lambda_h^{[p]}(f)-\varepsilon}}. \tag{10}$$

Since $\lambda_g^{[q]} < \lambda_h^{[p]}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that $\lambda_g^{[q]}+\varepsilon < \lambda_h^{[p]}(f)-\varepsilon$ and therefore, first part of the theorem follows from (10).

As $M_h^{-1}(r)$ is an increasing function of $r$, by similar reasoning as above the second part of the theorem follows from the second part of Lemma 3 and therefore its proof is omitted. □

**Remark 1.** *If we take $\rho_g^{[q]} < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ instead of $\lambda_g^{[q]} < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and the other conditions remain the same, the conclusion of Theorem 1 remains valid with "limit inferior" replaced by "limit".*

**Theorem 2.** *Let $f$, $g$ and $h$ be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\lambda_g^{[q]} < \infty$ where $p, q$ are any integers with $p \geq 1$ and $q \geq 2$. Then for every positive constant $A$ and each $\alpha \in (-\infty, \infty)$,*

$$(i)\ \liminf_{r\to\infty}\frac{\left\{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q-1]} r^A)} = 0 \text{ if } A > (1+\alpha)\lambda_g^{[q]}$$

*and*

$$(ii)\ \liminf_{r\to\infty}\frac{\left\{\log^{[p]} M_h^{-1}M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} M_h^{-1}M_f(\exp^{[q-1]} r^A)} = 0 \text{ if } A > (1+\alpha)\lambda_g^{[q]}\ .$$

*Proof.* If $1+\alpha \leq 0$, then the theorem is obvious. We consider $1+\alpha > 0$.
Now from the definition of generalized relative lower order, we get for all sufficiently large values of $r$ that

$$\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q-1]} r^A) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} r^A \ . \tag{11}$$

Therefore we get from (8) and (11), for a sequence of values of $r$ tending to infinity that

$$\frac{\left\{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q-1]} r^A)} \leq \frac{\left(\rho_h^{[p]}(f) + \varepsilon\right)^{1+\alpha} \cdot \exp^{[q-2]}(\beta r)^{\left(\lambda_g^{[q]}+\varepsilon\right)(1+\alpha)}}{\left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} r^A}$$

$$\times \left[1 + \frac{O(1)}{\left(\rho_h^{[p]}(f) + \varepsilon\right)^{1+\alpha} \cdot \exp^{[q-2]}(\beta r)^{\left(\lambda_g^{[q]}+\varepsilon\right)(1+\alpha)}}\right]^{(1+\alpha)} , \tag{12}$$

where we choose $0 < \varepsilon < \min\left\{\lambda_h^{[p]}(f), \frac{A}{1+\alpha} - \lambda_g^{[q]}\right\}$. So from (12) we obtain that

$$\liminf_{r\to\infty} \frac{\left\{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q-1]} r^A)} = 0 \ .$$

This proves the first part of the theorem.
Similarly, the second part of the theorem can be carried out using the same technique as above and with the help of Lemma 3. Therefore its proof is omitted. □

In view of Theorem 2, the following theorem can be carried out:

**Theorem 3.** *Let $f$, $g$, $h$ and $k$ be any four entire functions with $\rho_h^{[p]}(f) < \infty$, $\lambda_g^{[q]} < \infty$ and $\lambda_k^{[m]}(g) > 0$ where $p, q, m$ are any three integers with $p \geq 1$, $q \geq 2$ and $m \geq 1$. Then for every positive constant $A$ and each $\alpha \in (-\infty, \infty)$,*

$$(i) \ \liminf_{r\to\infty} \frac{\left\{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)\right\}^{1+\alpha}}{\log^{[m]} \mu_k^{-1} \mu_g(\exp^{[q-1]} r^A)} = 0 \ \text{if } A > (1+\alpha)\lambda_g^{[q]}$$

*and*

$$(ii) \ \liminf_{r\to\infty} \frac{\left\{\log^{[p]} M_h^{-1} M_{f \circ g}(r)\right\}^{1+\alpha}}{\log^{[m]} M_k^{-1} M_g(\exp^{[q-1]} r^A)} = 0 \ \text{if } A > (1+\alpha)\lambda_g^{[q]} \ .$$

The proof is omitted.

**Theorem 4.** *Let $f, g, h, k, l, b$ and $a$ be any seven entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$, $\rho_a^{[s]}(g) < \infty$ and $\rho_g^{[n]} < \lambda_k^{[q]}$ where $p, q, m, n, s$ are all positive integers with $p \geq 1$, $m \geq 1$, $s \geq 1$ $n \geq 2$, $q \geq 2$ and $q \geq n$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) + \log^{[s]} \mu_a^{-1}\mu_g(r)} = \infty$$

*and*

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[m]} M_b^{-1}M_{l\circ k}(r)}{\log^{[p]} M_h^{-1}M_{f\circ g}(r) + \log^{[s]} M_a^{-1}M_g(r)} = \infty\ .$$

*Proof.* Since $\mu_b^{-1}(r)$ is an increasing function of $r$, it follows from Lemma 2 and Lemma 5 for all sufficiently large values of $r$ that

$$\begin{aligned}
\log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r) &\geq \log^{[m]} \mu_b^{-1}\mu_l\left(\frac{1}{24}\mu_k\left(\frac{r}{4}\right) - \frac{|k(0)|}{3}\right) \\
i.e.,\ \log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r) &\geq \left(\lambda_b^{[m]}(l) - \varepsilon\right)\log\left(\frac{1}{24}\mu_k\left(\frac{r}{4}\right) - \frac{|k(0)|}{3}\right) \\
i.e.,\ \log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r) &\geq \left(\lambda_b^{[m]}(l) - \varepsilon\right)\log\mu_k\left(\frac{r}{4}\right) + O(1)
\end{aligned}$$

$$i.e.,\ \log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r) \geq \left(\lambda_b^{[m]}(l) - \varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_k^{[q]}-\varepsilon} + O(1)\,. \tag{13}$$

Also for any $\beta > 1$, it follows from (7) for all sufficiently large values of $r$ that

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f) + \varepsilon\right)\exp^{[n-2]}(\beta r)^{\rho_g^{[n]}+\varepsilon} + O(1). \tag{14}$$

Further from the definition of generalized relative order, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[s]} \mu_a^{-1}\mu_g(r) \leq \left(\rho_a^{[s]}(g) + \varepsilon\right)\log r\ . \tag{15}$$

Since $\rho_g^{[n]} < \lambda_k^{[q]}$, we can choose $\varepsilon\,(> 0)$ in such a manner that

$$\rho_g^{[n]} + \varepsilon < \lambda_k^{[q]} - \varepsilon\ . \tag{16}$$

Therefore combining (13), (14) and (15) and in view of (16), we get for all sufficiently large values of $r$ that

$$\begin{aligned}
&\frac{\log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) + \log^{[s]} \mu_a^{-1}\mu_g(r)} \\
\geqslant\ &\frac{\left(\lambda_b^{[m]}(l) - \varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_k^{[q]}-\varepsilon} + O(1)}{\left(\rho_a^{[s]}(g) + \varepsilon\right)\log r + \left(\rho_h^{[p]}(f) + \varepsilon\right)\exp^{[n-2]}(\beta r)^{\rho_g^{[n]}+\varepsilon} + O(1)} \\
&i.e.,\ \lim_{r\to\infty} \frac{\log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) + \log^{[s]} \mu_a^{-1}\mu_g(r)} = \infty\ .
\end{aligned}$$

Thus the first part of the theorem follows from above.
Similarly, the second part of the theorem can be deduced with the help of Lemma 3 and therefore the proof is omitted. □

**Theorem 5.** *Let $f, g, h, k, l$ and $b$ be any six entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$ and $\rho_g^{[n]} < \lambda_k^{[q]}$ where $p, q, m, n$ are all positive integers with $p \geq 1$, $m \geq 1$, $n \geq 2$, $q \geq 2$ and $q \geq n$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[m]} \mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) + \log^{[p]} \mu_h^{-1}\mu_f(r)} = \infty$$

*and*

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[m]} M_b^{-1}M_{l\circ k}(r)}{\log^{[p]} M_h^{-1}M_{f\circ g}(r) + \log^{[p]} M_h^{-1}M_f(r)} = \infty\ .$$

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 4.

**Theorem 6.** *Let $f, g, h, k, l, b$ and $a$ be any seven entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$, $\rho_a^{[s]}(g) < \infty$ and $\rho_g^{[n]} < \lambda_k^{[q]}$ where $p, q, m, n, s$ are all positive integers with $p \geq 1$, $m \geq 1$, $s \geq 1$, $n \geq 2$, $q \geq 2$ and $q \geq n$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[m-p]} \mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r) \cdot \log^{[s-p]} \mu_a^{-1}\mu_g(r)} = \infty \text{ if } p = \min\{m, p, s\}$$

$$(ii)\ \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p-m]} \mu_h^{-1}\mu_{f\circ g}(r) \cdot \log^{[s-m]} \mu_a^{-1}\mu_g(r)} = \infty \text{ if } m = \min\{m, p, s\}$$

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[m-s]} \mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p-s]} \mu_h^{-1}\mu_{f\circ g}(r) \cdot \mu_a^{-1}\mu_g(r)} = \infty \text{ if } s = \min\{m, p, s\}$$

*and*

$$(iv)\ \lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r) \cdot \log^{[s-p]} M_a^{-1}M_g(r)} = \infty \text{ if } p = \min\{m, p, s\}$$

$$(v)\ \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1}M_{f\circ g}(r) \cdot \log^{[s-m]} M_a^{-1}M_g(r)} = \infty \text{ if } m = \min\{m, p, s\}$$

$$(vi)\ \lim_{r\to\infty} \frac{\log^{[m-s]} M_b^{-1}M_{l\circ k}(r)}{\log^{[p-s]} M_h^{-1}M_{f\circ g}(r) \cdot M_a^{-1}M_g(r)} = \infty \text{ if } s = \min\{m, p, s\}\ .$$

*Proof.* From (15) it follows for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[s-1]} \mu_a^{-1}\mu_g(r) \leq r^{\left(\rho_a^{[s]}(g)+\varepsilon\right)}\ . \tag{17}$$

**Case I.** Let $p = \min\{m, p, s\}$.
Therefore combining (13), (14) and (17) and in view of (16), we get for all sufficiently large values of $r$ that

$$\frac{\log^{[m-p]} \mu_b^{-1} \mu_{l \circ k}(r)}{\mu_h^{-1} \mu_{f \circ g}(r) \cdot \log^{[s-p]} \mu_a^{-1} \mu_g(r)}$$

$$\geqslant \frac{\exp^{[p]}\left[\left(\lambda_b^{[m]}(l) - \varepsilon\right) \exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_k^{[q]} - \varepsilon} + O(1)\right]}{\exp^{[p-1]} r^{\left(\rho_a^{[s]}(g) + \varepsilon\right)} \cdot \exp^{[p]}\left[\left(\rho_h^{[p]}(f) + \varepsilon\right) \exp^{[n-2]}(\beta r)^{\rho_g^{[n]} + \varepsilon} + O(1)\right]}$$

$$i.e., \lim_{r \to \infty} \frac{\log^{[m-p]} \mu_b^{-1} \mu_{l \circ k}(r)}{\mu_h^{-1} \mu_{f \circ g}(r) \cdot \log^{[s-p]} \mu_a^{-1} \mu_g(r)} = \infty .$$

Thus the first part of the theorem follows from above.
**Case II.** Let $m = \min\{m, p, s\}$.
Then combining (13), (14) and (17) and in view of (16), we obtain for all sufficiently large values of $r$ that

$$\frac{\mu_b^{-1} \mu_{l \circ k}(r)}{\log^{[p-m]} \mu_h^{-1} \mu_{f \circ g}(r) \cdot \log^{[s-m]} \mu_a^{-1} \mu_g(r)}$$

$$\geqslant \frac{\exp^{[m]}\left[\left(\lambda_b^{[m]}(l) - \varepsilon\right) \exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_k^{[q]} - \varepsilon} + O(1)\right]}{\exp^{[m-1]} r^{\left(\rho_a^{[s]}(g) + \varepsilon\right)} \cdot \exp^{[m]}\left[\left(\rho_h^{[p]}(f) + \varepsilon\right) \exp^{[n-2]}(\beta r)^{\rho_g^{[n]} + \varepsilon} + O(1)\right]}$$

$$i.e., \lim_{r \to \infty} \frac{\mu_b^{-1} \mu_{l \circ k}(r)}{\log^{[p-m]} \mu_h^{-1} \mu_{f \circ g}(r) \cdot \log^{[s-m]} \mu_a^{-1} \mu_g(r)} = \infty,$$

which is the second part of the theorem.
**Case III.** Let $s = \min\{m, p, s\}$.
Now combining (13), (14) and (17) and in view of (16), it follows for all sufficiently large values of $r$ that

$$\frac{\log^{[m-s]} \mu_b^{-1} \mu_{l \circ k}(r)}{\log^{[p-s]} \mu_h^{-1} \mu_{f \circ g}(r) \cdot \mu_a^{-1} \mu_g(r)}$$

$$\geqslant \frac{\exp^{[s]}\left[\left(\lambda_b^{[m]}(l) - \varepsilon\right) \exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_k^{[q]} - \varepsilon} + O(1)\right]}{\exp^{[s-1]} r^{\left(\rho_a^{[s]}(g) + \varepsilon\right)} \cdot \exp^{[s]}\left[\left(\rho_h^{[p]}(f) + \varepsilon\right) \exp^{[n-2]}(\beta r)^{\rho_g^{[n]} + \varepsilon} + O(1)\right]}$$

$$i.e., \lim_{r \to \infty} \frac{\log^{[m-s]} \mu_b^{-1} \mu_{l \circ k}(r)}{\log^{[p-s]} \mu_h^{-1} \mu_{f \circ g}(r) \cdot \mu_a^{-1} \mu_g(r)} = \infty .$$

Thus the third part of the theorem is established.
Analogously using the same technique, the remaining parts of the theorem follows from Lemma 3 and therefore their proofs are omitted. □

In view of Theorem 6, the following theorem can be carried out and therefore its proof is omitted:

**Theorem 7.** *Let $f, g, h, k, l$ and $b$ be any six entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$ and $\rho_g^{[n]} < \lambda_k^{[q]}$ where $p, q, m$ are all positive integers with $p \geq 1$, $m \geq 1$, $n \geq 2$, $q \geq 2$ and $q \geq n$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[m-p]} \mu_b^{-1} \mu_{l\circ k}(r)}{\mu_h^{-1} \mu_{f\circ g}(r) \cdot \mu_h^{-1} \mu_f(r)} = \infty \text{ if } p = \min\{m, p\}$$

$$(ii)\ \lim_{r\to\infty} \frac{\mu_b^{-1} \mu_{l\circ k}(r)}{\log^{[p-m]} \mu_h^{-1} \mu_{f\circ g}(r) \cdot \log^{[p-m]} \mu_h^{-1} \mu_f(r)} = \infty \text{ if } m = \min\{m, p\}$$

*and*

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1} M_{l\circ k}(r)}{M_h^{-1} M_{f\circ g}(r) \cdot M_h^{-1} M_f(r)} = \infty \text{ if } p = \min\{m, p\}$$

$$(iv)\ \lim_{r\to\infty} \frac{M_b^{-1} M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1} M_{f\circ g}(r) \cdot \log^{[p-m]} M_h^{-1} M_f(r)} = \infty \text{ if } m = \min\{m, p\}.$$

**Remark 2.** *If we consider $\rho_g^{[n]} < \rho_k^{[q]}$ or $\lambda_g^{[n]} < \lambda_k^{[q]}$ instead of $\rho_g^{[n]} < \lambda_k^{[q]}$ in Theorem 4, Theorem 5, Theorem 6 and Theorem 7 and the other conditions remain the same, the conclusion of Theorem 4, Theorem 5, Theorem 6 and Theorem 7 remains valid with "limit superior " replaced by " limit".*

**Theorem 8.** *Let $f$, $g$, $h$ and $k$ be any four entire functions such that (i) $\rho_h^{[p]}(f \circ g) < \infty$ and (ii) $\lambda_k^{[q]}(g) > 0$ where $p, q$ are any two positive integers. Then*

$$(i)\ \lim_{r\to\infty} \frac{\left[\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)\right]^2}{\log^{[q-1]} \mu_k^{-1} \mu_g(\exp(r)) \cdot \log^{[q]} \mu_k^{-1} \mu_g(r)} = 0$$

*and*

$$(ii)\ \lim_{r\to\infty} \frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^2}{\log^{[q-1]} M_k^{-1} M_g(\exp(r)) \cdot \log^{[q]} M_k^{-1} M_g(r)} = 0\ .$$

*Proof.* For any arbitrary positive $\varepsilon$, we have for all sufficiently large values of $r$ that

$$\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f \circ g) + \varepsilon\right) \log r\ . \tag{18}$$

Again for all sufficiently large values of $r$ we get that

$$\log^{[q]} \mu_k^{-1} \mu_g(r) \geq \left(\lambda_k^{[q]}(g) - \varepsilon\right) \log r\ . \tag{19}$$

Similarly, for all sufficiently large values of $r$ we have

$$\begin{aligned}\log^{[q]}\mu_k^{-1}\mu_g(\exp(r)) &\geq \left(\lambda_k^{[q]}(g)-\varepsilon\right)r \\ i.e.,\ \log^{[q-1]}\mu_k^{-1}\mu_g(\exp(r)) &\geq \exp\left[\left(\lambda_k^{[q]}(g)-\varepsilon\right)r\right]. \qquad (20)\end{aligned}$$

From (18) and (19), we have for all sufficiently large values of $r$ that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q]}\mu_k^{-1}\mu_g(r)} \leq \frac{\left(\rho_h^{[p]}(f\circ g)+\varepsilon\right)\log r}{\left(\lambda_k^{[q]}(g)-\varepsilon\right)\log r}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q]}\mu_k^{-1}\mu_g(r)} \leq \frac{\rho_h^{[p]}(f\circ g)}{\lambda_k^{[q]}(g)}. \qquad (21)$$

Again from (18) and (20), we get for all sufficiently large values of $r$ that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_g(\exp(r))} \leq \frac{\left(\rho_h^{[p]}(f\circ g)+\varepsilon\right)\log r}{\exp\left[\left(\lambda_k^{[q]}(g)-\varepsilon\right)r\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_g(\exp(r))} &= 0 \\ i.e.,\ \lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[q-1]}\mu_k^{-1}\mu_g(\exp(r))} &= 0. \qquad (22)\end{aligned}$$

Thus the first part of the theorem follows from (21) and (22).
By similar reasoning as above the second part of the theorem can also be deduced and therefore its proof is omitted. □

In view of Theorem 8, the following theorem can be carried out:

**Theorem 9.** *Let $f$, $g$, $h$ and $k$ be any four entire functions such that (i) $\rho_h^{[p]}(f\circ g)<\infty$ and (ii) $\lambda_k^{[q]}(f)>0$ where $p,q$ are any two positive integers. Then*

$$(i)\ \lim_{r\to\infty}\frac{\left[\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\right]^2}{\log^{[q-1]}\mu_k^{-1}\mu_f(\exp(r))\cdot\log^{[q]}\mu_k^{-1}\mu_f(r)}=0$$

*and*

$$(ii)\ \lim_{r\to\infty}\frac{\left[\log^{[p]}M_h^{-1}M_{f\circ g}(r)\right]^2}{\log^{[q-1]}M_k^{-1}M_f(\exp(r))\cdot\log^{[q]}M_k^{-1}M_f(r)}=0.$$

The proof is omitted.

**Theorem 10.** *Let $f$, $g$, $h$, $k$ and $l$ be any five entire functions such that (i) $\rho_k^{[m]}(g) < \infty$ (ii) $\lambda_l^{[n]}(f \circ g) > 0$, and (ii) $\lambda_h^{[p]}(f) > 0$ where $m, n, p$ are any three positive integers. Then for every positive constant $\delta$ with $\delta < \rho_g^{[q]}$ where $q$ is any positive integer $\geq 2$,*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[n]} \mu_l^{-1}\mu_{f\circ g}(r) \cdot \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]} \mu_k^{-1}\mu_g\left(\exp^{[q-1]} r^\delta\right) \cdot \log^{[m]} \mu_k^{-1}\mu_g(r)} = \infty$$

*and*

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[n]} M_l^{-1}M_{f\circ g}(r) \cdot \log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[m]} M_k^{-1}M_g\left(\exp^{[q-1]} r^\delta\right) \cdot \log^{[m]} M_k^{-1}M_g(r)} = \infty.$$

*Proof.* Since $\mu_h^{-1}(r)$ is an increasing function of $r$, it follows from Lemma 2 and Lemma 5 for a sequence of values of $r$ that

$$\begin{aligned} \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[p]} \mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right) \\ i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\log\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right) - \frac{|g(0)|}{3}\right) \\ i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) &\geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\log\mu_g\left(\frac{r}{4}\right) + O(1) \qquad (23) \end{aligned}$$

$$i.e.,\ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\rho_g^{[q]}-\varepsilon} + O(1)\ . \qquad (24)$$

Again for any arbitrary positive $\varepsilon$, we have for all sufficiently large values of $r$ that

$$\log^{[m]} \mu_k^{-1}\mu_g\left(\exp^{[q-1]} r^\delta\right) \leq \left(\rho_k^{[m]}(g) + \varepsilon\right)\exp^{[q-2]} r^\delta\ . \qquad (25)$$

Now from (24) and (25), it follows for a sequence of values of $r$ that

$$\frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]} \mu_k^{-1}\mu_g\left(\exp^{[q-1]} r^\delta\right)} \geq \frac{\left(\lambda_h^{[p]}(f) - \varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\rho_g^{[q]}-\varepsilon} + O(1)}{\left(\rho_k^{[m]}(g) + \varepsilon\right)\exp^{[q-2]} r^\delta}\ . \qquad (26)$$

Again for all sufficiently large values of $r$ we get that

$$\log^{[n]} \mu_l^{-1}\mu_{f\circ g}(r) \geq \left(\lambda_l^{[n]}(f\circ g) - \varepsilon\right)\log r$$

and

$$\log^{[m]} \mu_k^{-1}\mu_g(r) \leq \left(\rho_k^{[m]}(g) + \varepsilon\right)\log r\ .$$

Therefore from the above two inequalities, we obtain for all sufficiently large values of $r$ that

$$\frac{\log^{[n]}\mu_l^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g(r)} \geq \frac{\left(\lambda_l^{[n]}(f\circ g)-\varepsilon\right)\log r}{\left(\rho_k^{[m]}(g)+\varepsilon\right)\log r}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[n]}\mu_l^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g(r)} \geq \frac{\lambda_l^{[n]}(f\circ g)}{\rho_k^{[m]}(g)+\varepsilon}\ . \tag{27}$$

Since $\delta<\rho_g^{[q]}$, therefore from (26) it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}r^{\delta}\right)}=\infty\ . \tag{28}$$

Thus the first part of the theorem follows from (27) and (28).
In a like manner the second part of the theorem can be established. □

**Theorem 11.** *Let $f$, $g$, $h$ and $l$ be any four entire functions such that (i) $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ (ii) $\lambda_l^{[n]}(f\circ g)>0$ where $n,p$ are any two positive integers. Then for every positive constant $\delta$ with $\delta<\rho_g^{[q]}$ where $q$ is any positive integer $\geq 2$,*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[n]}\mu_l^{-1}\mu_{f\circ g}(r)\cdot\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^{\delta}\right)\cdot\log^{[p]}\mu_h^{-1}\mu_f(r)}=\infty$$

*and*

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[n]}M_l^{-1}M_{f\circ g}(r)\cdot\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp^{[q-1]}r^{\delta}\right)\cdot\log^{[p]}M_h^{-1}M_f(r)}=\infty.$$

We omit the proof of Theorem 11 as it can be carried out in the line of Theorem 10.

**Theorem 12.** *Let $f$, $g$ and $h$ be any three entire functions such that $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ and $0<\lambda^{[q]}g\leq\rho^{[q]}g<\infty$ where $p,q$ are any two positive integers such that $p\geq 1$ and $q\geq 2$. Then for every positive constant $A$,*

$$(i)\ \frac{\lambda_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\leq\liminf_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)}$$

$$\leq\min\left\{\frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)},\frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\right\}\leq\max\left\{\frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)},\frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\right\}$$

$$\leq\limsup_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)}\leq\frac{\rho_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)}$$

*and*

$$(ii)\ \frac{\lambda_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\le\liminf_{r\to\infty}\frac{\log^{[p+q-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f(r^A)}$$
$$\le\min\left\{\frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)},\frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\right\}\le\max\left\{\frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)},\frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\right\}$$
$$\le\limsup_{r\to\infty}\frac{\log^{[p+q-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f(r^A)}\le\frac{\rho_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)}.$$

*Proof.* For any $\beta>1$, it follows from (7) for all sufficiently large values of $r$ that

$$\begin{aligned}\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)&\le\left(\rho_h^{[p]}(f)+\varepsilon\right)\log\mu_g(\beta r)+O(1)\\ i.e.,\ \log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)&\le\log^{[q]}\mu_g(\beta r)+O(1)\\ i.e.,\ \log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)&\le\left(\rho_g^{[q]}+\varepsilon\right)\log r+O(1)\end{aligned}\tag{29}$$

and for a sequence of values of $r$ that

$$\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)\le\left(\lambda_g^{[q]}+\varepsilon\right)\log r+O(1)\ .\tag{30}$$

Further from (23), it follows for a sequence of values of $r$ that

$$\begin{aligned}\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)&\ge\log^{[q]}\mu_g\left(\frac{r}{4}\right)+O(1)\\ i.e.,\ \log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)&\ge\left(\rho_g^{[q]}-\varepsilon\right)\log r+O(1)\end{aligned}\tag{31}$$

and for all sufficiently large values of $r$ that

$$\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)\ge\left(\lambda_g^{[q]}-\varepsilon\right)\log r+O(1)\ .\tag{32}$$

Again from the definition of generalized order and generalized lower order, we have for arbitrary positive $\varepsilon$ and for all sufficiently large values of $r$ that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right)\geqslant A\cdot\left(\lambda_h^{[p]}(f)-\varepsilon\right)\log r\tag{33}$$

and

$$\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right)\le A\cdot\left(\rho_h^{[p]}(f)+\varepsilon\right)\log r\ .\tag{34}$$

Again we get for a sequence of values of $r$ tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right)\le A\cdot\left(\lambda_h^{[p]}(f)+\varepsilon\right)\log r\tag{35}$$

and

$$\log^{[p]}\mu_h^{-1}\mu_f\left(r^A\right)\geqslant A\cdot\left(\rho_h^{[p]}(f)-\varepsilon\right)\log r\ .\tag{36}$$

Therefore from (29) and (33), we obtain for all sufficiently large values of $r$ that

$$\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\left(\rho_g^{[q]}+\varepsilon\right)\log r + O(1)}{A\cdot\left(\lambda_h^{[p]}(f)-\varepsilon\right)\log r}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\rho_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)}\ . \tag{37}$$

Similarly, from (29) and (36) we have for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\left(\rho_g^{[q]}+\varepsilon\right)\log r + O(1)}{A\cdot\left(\rho_h^{[p]}(f)-\varepsilon\right)\log r}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\ . \tag{38}$$

Analogously we get from (30) and (33) for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\left(\lambda_g^{[q]}+\varepsilon\right)\log r + O(1)}{A\cdot\left(\lambda_h^{[p]}(f)-\varepsilon\right)\log r}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)}\ . \tag{39}$$

Now from (38) and (39), it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \leq \min\left\{\frac{\lambda^{[q]}g}{A\cdot\lambda_h^{[p]}(f)}, \frac{\rho^{[q]}g}{A\cdot\rho_h^{[p]}(f)}\right\}\ . \tag{40}$$

Further from (31) and (34), we get for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \geq \frac{\left(\rho_g^{[q]}-\varepsilon\right)\log r + O(1)}{A\cdot\left(\rho_h^{[p]}(f)+\varepsilon\right)\log r}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p+q-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f(r^A)} \geq \frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\ . \tag{41}$$

Likewise from (32) and (35), we obtain for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \frac{\left(\lambda_g^{[q]}-\varepsilon\right)\log r+O(1)}{A\cdot\left(\lambda_h^{[p]}(f)+\varepsilon\right)\log r}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)}\ . \tag{42}$$

Thus from (41) and (42), it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \max\left\{\frac{\lambda_g^{[q]}}{A\cdot\lambda_h^{[p]}(f)},\frac{\rho_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\right\}\ . \tag{43}$$

Also from (32) and (34), we obtain for all sufficiently large values of $r$ that

$$\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \frac{\left(\lambda_g^{[q]}-\varepsilon\right)\log r+O(1)}{A\cdot\left(\rho_h^{[p]}(f)+\varepsilon\right)\log r}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f(r^A)} \geq \frac{\lambda_g^{[q]}}{A\cdot\rho_h^{[p]}(f)}\ . \tag{44}$$

Therefore the first part of the theorem follows from (37), (40), (43) and (44). Using the similar technique as above, the second part of the theorem follows from Lemma 3 and therefore its proof is omitted. □

**Theorem 13.** *Let $f$, $g$, $h$ and $k$ be any four entire functions such that $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$, $0<\lambda_k^{[m]}(g)\leq\rho_k^{[m]}(g)<\infty$ and $0<\lambda^{[q]}g\leq\rho^{[q]}g<\infty$ where $p,q,m$ are any three positive integers such that $p\geq 1$, $m\geq 1$ and $q\geq 2$. Then for every positive constant $B$,*

$$(i)\ \frac{\lambda_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\leq\liminf_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g(r^B)}$$
$$\leq\min\left\{\frac{\lambda_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)},\frac{\rho_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\right\}\leq\max\left\{\frac{\lambda_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)},\frac{\rho_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\right\}$$
$$\leq\limsup_{r\to\infty}\frac{\log^{[p+q-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g(r^B)}\leq\frac{\rho_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)}$$

*and*

$$(ii)\ \frac{\lambda_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\leq\liminf_{r\to\infty}\frac{\log^{[p+q-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[m]}M_k^{-1}M_g(r^B)}$$

$$\leq\min\left\{\frac{\lambda_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)},\frac{\rho_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\right\}\leq\max\left\{\frac{\lambda_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)},\frac{\rho_g^{[q]}}{B\cdot\rho_k^{[m]}(g)}\right\}$$

$$\leq\limsup_{r\to\infty}\frac{\log^{[p+q-1]}M_h^{-1}M_{f\circ g}(r)}{\log^{[m]}M_k^{-1}M_g(r^B)}\leq\frac{\rho_g^{[q]}}{B\cdot\lambda_k^{[m]}(g)}.$$

The proof of Theorem 13 is omitted as it can be carried out in the line of Theorem 12.

**Theorem 14.** *Let $f$ and $h$ be any two entire functions such that $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ for any positive integer $p>1$. Then for any entire $g$ with $0<A<\lambda_g^{[q]}$ where $q$ is any positive integer $>2$*

$$(i)\ \lim_{r\to\infty}\frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right)}=\infty$$

*and*

$$(ii)\ \lim_{r\to\infty}\frac{M_h^{-1}M_{f\circ g}(r)}{M_h^{-1}M_f\left(\exp^{[q-1]}r^A\right)}=\infty\ .$$

*Proof.* We have from $(23)$, for all sufficiently large values of $r$ that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\geq\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_g^{[q]}-\varepsilon}+O(1)\ . \tag{45}$$

Again from the definition of the generalized relative order, we obtain for all sufficiently large values of $r$ that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right)\leq\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^A\ . \tag{46}$$

So combining $(45)$ and $(46)$, we obtain for all sufficiently large values of $r$ that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right)}\geqslant\frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]}\left(\frac{r}{4}\right)^{\lambda_g^{[q]}-\varepsilon}+O(1)}{\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^A}. \tag{47}$$

Since $0<A<\lambda_g^{[q]}$, we can choose $\varepsilon\,(\varepsilon>0)$ in such a way that

$$A<\lambda_g^{[q]}-\varepsilon\ . \tag{48}$$

Thus from $(47)$ and $(48)$, we get that

$$\lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right)}=\infty\ .$$

So from above it follows for all sufficiently large values of $r$ that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r) \geqslant K\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right), \quad \text{for } K>1$$

$$i.e., \ \log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r) \geqslant \left\{\log^{[p-1]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r^A\right)\right\}^K,$$

from which the first part of the theorem follows.
Accordingly the second part of the theorem can be deduced with the help of the first part of Lemma 3 and therefore its proof is omitted. □

Analogously the following theorem can be carried out in the line of Theorem 15:

**Theorem 15.** *Let $f$, $g$, $h$ and $k$ be any four entire functions with $\lambda_h^{[p]}(f)>0$ and $\rho_k^{[m]}(g)<\infty$ where $p, m$ are any positive integers. Then for every positive constant $A$ such that $0<A<\lambda_g^{[q]}$ for any positive integer $q>1$,*

$$(i)\ \lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}r^A\right)}=\infty$$

*and*

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[m]}M_k^{-1}M_g\left(\exp^{[q-1]}r^A\right)}=\infty\ .$$

The proof is omitted.

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[0]Başlık: Bileşik tam fonksiyonların büyümesi üzerinde genelleştirilmiş bağıl basamağın etkisi
Anahtar Kelimeler: Tam fonksiyon, genelleştirilmiş bağıl alt basamak, bileşke, büyüme.