Commun.Fac.Sci.Univ.Ank.Series A1
Volume 64, Number 1, Pages 87–97 (2015)
DOI: 10.1501/Commua1_0000000729
ISSN 1303–5991

EIGENVALUES FOR ITERATIVE SYSTEMS OF DYNAMIC EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we establish criterion which determine the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, for the existence of at least one positive solution for the iterative systems of dynamic equations with integral boundary conditions. By this purpose, we use a fixed point theorem in a cone. Examples are also given to show applicability of our main result.

1. Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger in his PhD thesis in 1988 [7]. The time scales approach, not only unifies differential and difference equations, but also provides accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. We refer the books [5, 6] which include basic definitions and theorems on time scales.

There has been much interest shown in obtaining optimal eigenvalue intervals for the existence of positive solutions of the boundary value problems on time scales, often using Guo-Krasnosel'skii fixed point theorem. To mention a few papers along these lines, see [1, 2, 3]. On the other hand, there is not much work concerning the eigenvalues for iterative system of nonlinear boundary value problems on time scales, see [4, 8].

In [4], Benchohra et al. studied the eigenvalues for iterative system of nonlinear boundary value problems on time scales,

$$u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t)) = 0,\ 1 \le i \le n,\ t \in [0,1]_{\mathbb{T}},$$
$$u_{n+1}(t) = u_1(t),\ t \in [0,1]_{\mathbb{T}},$$

satisfying the boundary conditions,

$$u_i(0) = 0 = u_i\left(\sigma^2(1)\right),\ 1 \le i \le n.$$

The method involves application of Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space.

Received by the editors: November 07, 2015; Accepted: May 01, 2015.
2010 *Mathematics Subject Classification.* 34B18; 34N05.
Key words and phrases. Green's function, iterative system, eigenvalue interval, time scales, boundary value problem, fixed-point theorem, positive solution.

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In [8], Prasad et al. investigated the eigenvalues for the iterative system of nonlinear boundary value problems on time scales,

$$y_i^{\Delta\Delta}(t)+\lambda_i p_i(t) f_i(y_{i+1}(t))=0,\ 1\le i\le n,\ t\in[t_1,t_m]_{\mathbb{T}},$$
$$y_{n+1}(t)=y_1(t),\ t\in[t_1,t_m]_{\mathbb{T}},$$

satisfying the m-point boundary conditions,

$$y_i(t_1)=0,$$
$$\alpha y_i(\sigma(t_m))+\beta y_i^{\Delta}(\sigma(t_m))=\sum_{k=2}^{m-1} y_i^{\Delta}(t_k),\ 1\le i\le n.$$

They used the Guo-Krasnosel'skii fixed point theorem.

In [9], Karaca and Tokmak studied the eigenvalues for the iterative system of nonlinear m-point boundary value problems on time scales,

$$\begin{cases} u_i^{\Delta\Delta}(t)+\lambda_i q_i(t) f_i(u_{i+1}(t))=0,\ t\in[0,1]_{\mathbb{T}},\ 1\le i\le n,\\ u_{n+1}(t)=u_1(t),\ t\in[0,1]_{\mathbb{T}}, \end{cases}$$

satisfying the m-point boundary conditions,

$$\begin{cases} au_i(0)-bu_i^{\Delta}(0)=\displaystyle\sum_{j=1}^{m-2}\alpha_j u_i(\xi_j),\\ cu_i(1)+du_i^{\Delta}(1)=\displaystyle\sum_{j=1}^{m-2}\beta_j u_i(\xi_j),\ 1\le i\le n. \end{cases}$$

They used the Guo-Krasnosel'skii fixed point theorem.

Motivated by the above results, in this study, we are concerned with determining the eigenvalue intervals of $\lambda_i,\ 1\le i\le n$, for which there exist positive solutions for the iterative system of nonlinear boundary value problems with integral boundary conditions on time scales,

$$\begin{cases} u_i^{\Delta\Delta}(t)+\lambda_i q_i(t) f_i(u_{i+1}(t))=0,\ t\in[0,1]_{\mathbb{T}},\ 1\le i\le n,\\ u_{n+1}(t)=u_1(t),\ t\in[0,1]_{\mathbb{T}}, \end{cases}\tag{1.1}$$

satisfying the integral boundary conditions,

$$\begin{cases} au_i(0)-bu_i^{\Delta}(0)=\displaystyle\int_0^1 g_1(t)u_i(t)\Delta t,\\ cu_i(1)+du_i^{\Delta}(1)=\displaystyle\int_0^1 g_2(t)u_i(t)\Delta t,\ 1\le i\le n, \end{cases}\tag{1.2}$$

where $\mathbb{T}$ is a time scale, $0,1\in\mathbb{T},\ [0,1]_{\mathbb{T}}=[0,1]\cap\mathbb{T}$.

Throughout this study we assume that following conditions hold:

- $(C1)$**:** $a,b,c,d\in[0,\infty)$ with $ac+ad+bc>0$, $g_1,\ g_2\in\mathcal{C}([0,1]_{\mathbb{T}},\mathbb{R}^+)$,
- $(C2)$**:** $f_i:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, for $1\le i\le n$,
- $(C3)$**:** $q_i\in\mathcal{C}([0,1]_{\mathbb{T}},\mathbb{R}^+)$ and q_i does not vanish identically on any closed subinterval of $[0,1]_{\mathbb{T}}$, for $1\le i\le n$,

$(C4)$**:** Each of $\lim\limits_{x\to 0^+} \dfrac{f_i(x)}{x} = 0$, for $1 \le i \le n$.

This paper is organized as follows. In Section 2, we provide preliminary lemmas which are key tools for our main result. In Section 3, we determine the eigenvalue intervals for which there exist positive solutions of the boundary value problem (1.1) − (1.2) by using a fixed point theorem for operators on a cone in a Banach space. Finally, in Section 4, we give two examples to demonstrate our main result.

2. Preliminaries

In this section, we present auxiliary lemmas which will be used later.

We define $\mathbb{B} = \mathcal{C}[0,1]$, which is a Banach space with the norm

$$\|u\| = \sup_{t\in[0,1]_{\mathbb{T}}} |u(t)|.$$

Let $h \in \mathcal{C}[0,1]$, then we consider the following boundary value problem

$$\begin{cases} -u_1^{\Delta\Delta}(t) = h(t), \ t \in [0,1]_{\mathbb{T}}, \\ au_1(0) - bu_1^{\Delta}(0) = \displaystyle\int_0^1 g_1(t)u_1(t)\Delta t, \\ cu_1(1) + du_1^{\Delta}(1) = \displaystyle\int_0^1 g_2(t)u_1(t)\Delta t. \end{cases} \tag{2.1}$$

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$-u_1^{\Delta\Delta}(t) = 0, \ t \in [0,1]_{\mathbb{T}}, \tag{2.2}$$

under the initial conditions

$$\begin{cases} \theta(0) = b, \quad \theta^{\Delta}(0) = a, \\ \varphi(1) = d, \quad \varphi^{\Delta}(1) = -c. \end{cases} \tag{2.3}$$

Using the initial conditions (2.3), we can deduce from equation (2.2) for θ and φ the following equations:

$$\theta(t) = b + at, \quad \varphi(t) = d + c(1-t).$$

Set

$$\mathcal{D} := \begin{vmatrix} -\displaystyle\int_0^1 g_1(t)\,(b+at)\,\Delta t & \rho - \displaystyle\int_0^1 g_1(t)\,(d+c(1-t))\,\Delta t \\ \rho - \displaystyle\int_0^1 g_2(t)\,(b+at)\,\Delta t & -\displaystyle\int_0^1 g_2(t)\,(d+c(1-t))\,\Delta t \end{vmatrix},$$

and

$$\rho := ad + ac + bc.$$

Lemma 2.1. *Let* $(C1)$ *hold. Assume that*

$(C5)$**:** $\mathcal{D} \neq 0$.

If $u_1 \in \mathcal{C}[0,1]$ *is a solution of the equation*

$$u_1(t) = \int_0^1 G(t,s)\,h(s)\Delta s + A(h)(b+at) + B(h)(d+c(1-t)), \tag{2.4}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b+a\sigma(s))\,(d+c(1-t)), & \sigma(s)\le t, \\ (b+at)\,(d+c(1-\sigma(s))), & t\le s, \end{cases} \tag{2.5}$$

$$A(h) := \frac{1}{\mathcal{D}}\begin{vmatrix} \int_0^1 g_1(t)\left(\int_0^1 G(t,s)\,h(s)\Delta s\right)\Delta t & \rho - \int_0^1 g_1(t)\,(d+c(1-t))\,\Delta t \\ \int_0^1 g_2(t)\left(\int_0^1 G(t,s)\,h(s)\Delta s\right)\Delta t & -\int_0^1 g_2(t)\,(d+c(1-t))\,\Delta t \end{vmatrix}, \tag{2.6}$$

and

$$B(h) := \frac{1}{\mathcal{D}}\begin{vmatrix} -\int_0^1 g_1(t)\,(b+at)\,\Delta t & \int_0^1 g_1(t)\left(\int_0^1 G(t,s)\,h(s)\Delta s\right)\Delta t \\ \rho - \int_0^1 g_2(t)\,(b+at)\,\Delta t & \int_0^1 g_2(t)\left(\int_0^1 G(t,s)\,h(s)\Delta s\right)\Delta t \end{vmatrix}, \tag{2.7}$$

then u_1 *is a solution of the boundary value problem* (2.1).

Proof. Let u_1 satisfy the integral equation (2.4), then we have

$$\begin{aligned} u_1(t) \;&=\; \int_0^t \frac{1}{\rho}(b+a(\sigma(s))(d+c(1-t))h(s)\Delta s + \int_t^1 \frac{1}{\rho}(b+at)(d+c(1-\sigma(s)))h(s)\Delta s \\ &\quad + A(h)(b+at) + B(h)(d+c(1-t)), \end{aligned}$$

$$u_1^{\Delta}(t) = -\int_0^t \frac{c}{\rho}(b+a(\sigma(s)))h(s)\Delta s + \int_t^1 \frac{a}{\rho}(d+c(1-\sigma(s)))h(s)\Delta s + A(h)a - B(h)c.$$

So that

$$\begin{aligned} u_1^{\Delta\Delta}(t) &= \frac{1}{\rho}\left(-c(b+a(\sigma(t))) - a(d+c(1-\sigma(t)))\right)h(t) \\ &= \frac{1}{\rho}\left(-(ad+ac+bc)\right)h(t) = -h(t), \\ -u_1^{\Delta\Delta}(t) &= h(t). \end{aligned}$$

Since

$$\begin{aligned} u_1(0) &= \int_0^1 \frac{b}{\rho}(d+c(1-\sigma(s)))h(s)\Delta s + A(h)b + B(h)(d+c), \\ u_1^{\Delta}(0) &= \int_0^1 \frac{a}{\rho}(d+c(1-\sigma(s)))h(s)\Delta s + A(h)a - B(h)c, \end{aligned}$$

we have

$$\begin{aligned} au_1(0) - bu_1^{\Delta}(0) &= B(h)\,(ad+ac+bc) \\ &= \int_0^1 g_1(s)\Big[\int_0^1 G(s,r)\,q(r)f(r,u(r))\,\Delta r \end{aligned}$$

$$+A(h)(b+as)+B(h)(d+c(1-s))\Big]\Delta s. \tag{2.8}$$

Since

$$\begin{aligned} u_1(1) &= \int_0^1 \frac{d}{\rho}(b+a(\sigma(s))h(s)\Delta s + A(h)(b+a) + B(h)d, \\ u_1^{\Delta}(1) &= -\int_0^1 \frac{c}{\rho}(b+a(\sigma(s)))h(s)\Delta s + A(h)a - B(h)c, \end{aligned}$$

we have

$$\begin{aligned} cu_1(1) + du_1^{\Delta}(1) &= A(f)\,(ad+ac+bc) \\ &= \int_0^1 g_2(s)\Big[\int_0^1 G\,(s,r)\,q(r)f\,(r,u(r))\,\Delta r \\ &\quad +A(h)(b+as)+B(h)(d+c(1-s))\Big]\Delta s. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we get that

$$\begin{cases} \left[-\int_0^1 g_1(s)(b+as)\Delta s\right]A(h) + \left[\rho - \int_0^1 g_1(s)(d+c(1-s))\Delta s\right]B(h) \\ = \int_0^1 g_1(s)[\int_0^1 G(s,r)q_1(r)f(r,u_1(r))\Delta r]\Delta s \\ \left[\rho-\int_0^1 g_2(s)(b+as)\Delta s\right]A(h) + \left[-\int_0^1 g_2(s)(d+c(1-s))\Delta s\right]B(h) \\ = \int_0^1 g_2(s)[\int_0^1 G(s,r)q_1(r)f(r,u_1(r))\Delta r]\Delta s \end{cases},$$

which implies that $A(h)$ and $B(h)$ satisfy (2.6) and (2.7), respectively. So, it is proved that if u_1 satisfies the integral equation (2.4), then u_1 is a solution of the problem (2.1). □

Lemma 2.2. *Let* $(C1)$ *hold. Assume*

$$(C6)\textbf{:}\ \mathcal{D}<0,\ \rho-\int_0^1 g_2(t)(b+at)\Delta t>0,\ \rho-\int_0^1 g_1(t)(d+c(1-t))\Delta t>0.$$

Then for $u_1 \in \mathcal{C}[0,1]$ *with* $h \geq 0$, *the solution* u_1 *of the problem* (2.1) *satisfies*

$$u_1(t) \geq 0 \ \textit{for}\ t \in [0,1]_{\mathbb{T}}.$$

Proof. The proof directly follows the facts $G \geq 0$ on $[0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}$ and $A(h) \geq 0,\ B(h) \geq 0$. □

We note that an n-tuple $(u_1(t), u_2(t), ..., u_n(t))$ is a solution of the boundary value problem (1.1)-(1.2) if and only if

$$\begin{aligned} u_1(t) &= \lambda_1 \int_0^1 G(t,s_1)\, q_1(s_1) f_1 \Bigg(\lambda_2 \int_0^1 G(s_1,s_2)\, q_2(s_2) \cdots \\ &\quad f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1},s_n)\, q_n(s_n) f_n(u_1(s_n))\, \Delta s_n \right) \cdots \Delta s_2 \Bigg) \Delta s_1 \\ &\quad + A\left(\lambda_1 q_1(\cdot) f_1(u_2(\cdot))\right)(b+at) \\ &\quad + B\left(\lambda_1 q_1(\cdot) f_1(u_2(\cdot))\right)(d+c(1-t)),\ t\in[0,1]_{\mathbb{T}}, \\ u_i(t) &= \lambda_i \int_0^1 G(t,s)\, q_i(s) f_i(u_{i+1}(s))\, \Delta s + A\left(\lambda_i q_i(\cdot) f_i(u_{i+1}(\cdot))\right)(b+at) \\ &\quad + B\left(\lambda_i q_i(\cdot) f_i(u_{i+1}(\cdot))\right)(d+c(1-t)),\ 2\le i\le n,\ t\in[0,1]_{\mathbb{T}}, \end{aligned}$$

and

$$u_{n+1}(t) = u_1(t),\ t\in[0,1]_{\mathbb{T}},$$

where
$A\left(\lambda_i q_i(\cdot) f_i(u_{i+1}(\cdot)\right) :=$

$$\frac{1}{\mathcal{D}} \begin{vmatrix} \int_0^1 g_1(t) \left(\lambda_i \int_0^1 G(t,s)\, q_i(s) f_i(u_{i+1}(s))\, \Delta s \right) \Delta t & \rho - \int_0^1 g_1(t)(d+c(1-t))\Delta t \\ \int_0^1 g_2(t) \left(\lambda_i \int_0^1 G(t,s)\, q_i(s) f_i(u_{i+1}(s))\, \Delta s \right) \Delta t & -\int_0^1 g_2(t)(d+c(1-t))\Delta t \end{vmatrix},$$

$B\left(\lambda_i q_i(\cdot) f_i(u_{i+1}(\cdot))\right) :=$

$$\frac{1}{\mathcal{D}} \begin{vmatrix} -\int_0^1 g_1(t)(b+at)\Delta t & \int_0^1 g_1(t) \left(\lambda_i \int_0^1 G(t,s)\, q_i(s) f_i(u_{i+1}(s))\, \Delta s \right) \Delta t \\ \rho - \int_0^1 g_2(t)(b+at)\Delta t & \int_0^1 g_2(t) \left(\lambda_i \int_0^1 G(t,s)\, q_i(s) f_i(u_{i+1}(s))\, \Delta s \right) \Delta t \end{vmatrix}.$$

We need the following known result to prove the existence of solutions for (1.1)$-$(1.2).

Theorem 2.1. [10] *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \bar{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|,\ \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

3. Main Result

In this section, we establish criterion to determine the eigenvalue intervals for which the boundary value problem (1.1)-(1.2) has at least one positive solution in a cone.

Now, we define an integral operator $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$, for $u_1 \in \mathcal{C}[0,1]$, by

$$\begin{aligned} Tu_1(t) &= \lambda_1 \int_0^1 G(t,s_1)\, q_1(s_1) f_1\Bigg(\lambda_2 \int_0^1 G(s_1,s_2)\, q_2(s_2) \cdots \\ &\quad f_{n-1}\left(\lambda_n \int_0^1 G(s_{n-1},s_n)\, q_n(s_n) f_n(u_1(s_n))\,\Delta s_n\right) \cdots \Delta s_2\Bigg)\Delta s_1 \\ &\quad + A(\lambda_1 q_1(\cdot) f_1(u_2(\cdot)))\,(b+at) + B(\lambda_1 q_1(\cdot) f_1(u_2(\cdot)))\,(d+c(1-t)). \end{aligned}$$

The operator T is completely continuous by an application of the Arzela-Ascoli Theorem.

Let

$$M = \min_{1 \le i \le n} \left\{ \left[\left(\int_0^1 G(\sigma(s),s)\, q_i(s)\Delta s + A(q_i(\cdot))(b+a) + B(q_i(\cdot))(d+c)\right)\right]^{-1} \right\}.$$

Theorem 3.1. *Suppose conditions* $(C1)-(C6)$ *are satisfied. Then, for each* $\lambda_1, \lambda_2, ..., \lambda_n$ *satisfying*

$$\lambda_i \le M, \quad 1 \le i \le n,$$

there exist an n*-tuple* $(u_1, u_2, ..., u_n)$ *satisfying* $(1.1)-(1.2)$ *such that* $u_i(t) > 0,\ 1 \le i \le n,$ *on* $[0,1]_{\mathbb{T}}$.

Proof. From $(C4)$, we have $\lim_{u \to 0^+} \dfrac{f_i(u)}{u} = 0$ for $1 \le i \le n$. Therefore there exists a constant $r > 0$ such that, for each $1 \le i \le n$,

$$|f_i(u)| \le \delta |u|$$

for $0 < |u| < r$, where $0 < \delta < 1$. Let us set $\Omega = \{u \in \mathcal{C}[0,1] : \|u\| < r\}$ and take $u_1 \in \mathcal{C}[0,1]$ such that $\|u_1\| = r$, that is, $u_1 \in \partial\Omega$. We have from (2.5) for $0 \le s_{n-1} \le 1$,

$$\begin{aligned} &\lambda_n \int_0^1 G(s_{n-1},s_n)\, q_n(s_n) f_n(u_1(s_n))\,\Delta s_n \\ \le\ &\lambda_n \int_0^1 G(\sigma(s_n),s_n)\, q_n(s_n) f_n(u_1(s_n))\,\Delta s_n \\ \le\ &\delta\lambda_n \int_0^1 G(\sigma(s_n),s_n)\, q_n(s_n) u_1(s_n)\Delta s_n \\ \le\ &\delta\lambda_n \int_0^1 G(\sigma(s_n),s_n)\, q_n(s_n)\Delta s_n\, \|u_1\| \\ <\ &\|u_1\| = r. \end{aligned}$$

It follows in a similar manner, for $0 \le s_{n-2} \le 1$,

$$\begin{aligned}
& \lambda_{n-1} \int_0^1 G\left(s_{n-2}, s_{n-1}\right) q_{n-1}(s_{n-1}) f_{n-1} \\
& \times \left(\lambda_n \int_0^1 G\left(s_{n-1}, s_n\right) q_n(s_n) f_n\left(u_1(s_n)\right) \Delta s_n \right) \Delta s_{n-1} \\
\leq \; & \delta \lambda_{n-1} \int_0^1 G\left(\sigma(s_{n-1}), s_{n-1}\right) q_{n-1}(s_{n-1}) \Delta s_{n-1} \left\| u_1 \right\| \\
< \; & \left\| u_1 \right\| = r.
\end{aligned}$$

Continuing with this bootstrapping argument, we have for $0 \leq t \leq 1$,

$$\begin{aligned}
& \lambda_1 \int_0^1 G\left(t, s_1\right) q_1(s_1) f_1 \left(\lambda_2 \int_0^1 G\left(s_1, s_2\right) q_2(s_2) \cdots f_n\left(u_1(s_n)\right) \Delta s_n \cdots \Delta s_2 \right) \Delta s_1 \\
\leq \; & \delta r \lambda_1 \int_0^1 G\left(\sigma(s_1), s_1\right) q_1(s_1) \Delta s_1,
\end{aligned}$$

$$\begin{aligned}
& A\left(\lambda_1 q_1(.) f_1(u_2(.))\right) \\
& \leq \tfrac{\lambda_1}{\mathcal{D}} \begin{vmatrix} \int_0^1 g_1(t)(\int_0^1 G(t,s) q_1(s) \Delta s) \Delta t, & \rho - \int_0^1 g_1(s)(d + c(1-s)) \Delta s \\ \int_0^1 g_2(t)(\int_0^1 G(t,s) q_1(s) \Delta s) \Delta t & - \int_0^1 g_2(s)(d + c(1-s)) \Delta s \end{vmatrix} \left\| f_1(u_2) \right\| \\
& \leq \lambda_1 A(q_1(.)) \left\| f_1(u_2) \right\|,
\end{aligned}$$

$$\begin{aligned}
& B\left(\lambda_1 q_1(.) f_1(u_2(.))\right) \\
& \leq \tfrac{\lambda_1}{\mathcal{D}} \begin{vmatrix} - \int_0^1 g_1(s)(b + as) \Delta s & \int_0^1 g_1(t)(\int_0^1 G(t,s) q_1(s) \Delta s) \Delta t \\ \rho - \int_0^1 g_2(s)(b + as) \Delta s & \int_0^1 g_2(t)(\int_0^1 G(t,s) q_1(s) \Delta s) \Delta t \end{vmatrix} \left\| f_1(u_2) \right\| \\
& \leq \lambda_1 B(q_1(.)) \left\| f_1(u_2) \right\|,
\end{aligned}$$

so that

$$\begin{aligned}
Tu_1(t) \; \leq \; & \lambda_1 (\int_0^1 G\left(\sigma(s_1), s_1\right) q_1(s_1) \Delta s_1 + A(q_1(.)) \left\| f_1(u_2) \right\| (b+a) \\
& + B(q_1(.)) \left\| f_1(u_2) \right\| (d+c)) \\
\leq \; & \delta r \lambda_1 \left(\int_0^1 G\left(\sigma(s_1), s_1\right) q_1(s_1) \Delta s_1 + A(q_1(.))(b+a) + B(q_1(.))(d+c) \right) \\
< \; & r = \left\| u_1 \right\|.
\end{aligned}$$

Thus, it follows that $\left\| Tu_1 \right\| < \left\| u_1 \right\|, \; u_1 \in \partial\Omega$. Therefore, by Theorem 2.1, the operator T has at least one fixed point, which in turn implies that the problem

(2.1) has at least one positive solution $u_1 \in \bar{\Omega}$. Therefore, setting $u_{n+1} = u_1$, we obtain a positive solution $(u_1, u_2, ..., u_n)$ of $(1.1) - (1.2)$ given iteratively by

$$\begin{aligned} u_k(t) &= \lambda_k \int_0^1 G(t,s)\, q_k(s) f_k(u_{k+1}(s))\, \Delta s + A\left(\lambda_k q_k(.) f_k(u_{k+1}(.))\right)(b+at) \\ &\quad + B\left(\lambda_k q_k(.) f_k(u_{k+1}(.))\right)(d+c(1-t)),\ k = n, n-1, ..., 1. \end{aligned}$$

The proof is completed. □

4. Examples

Example 4.1 In BVP $(1.1) - (1.2)$, suppose that $\mathbb{T} = [0,1],\ n = 3,\ q_1(t) = q_2(t) = q_3(t) = 1,\ a = 1,\ b = 2,\ c = d = \frac{1}{4},\ g_1(t) = t,\ g_2(t) = t^3$ i.e.,

$$\begin{cases} u_i''(t) + \lambda_i f_i(u_{i+1}(t)) = 0,\ t \in [0,1],\ 1 \le i \le 3, \\ u_4(t) = u_1(t),\ t \in [0,1], \end{cases} \tag{4.1}$$

satisfying the following boundary conditions,

$$\begin{cases} u_i(0) - 2u_i'(0) = \displaystyle\int_0^1 t u_i(t) dt, \\ \dfrac{1}{4} u_i(1) + \dfrac{1}{4} u_i'(1) = \displaystyle\int_0^1 t^3 u_i(t) dt,\ 1 \le i \le 3, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} f_1(u_2) &= \sin^2 u_2, \\ f_2(u_3) &= u_3^2 e^{u_3}, \\ f_3(u_1) &= 1 - \cos u_1. \end{aligned}$$

It is easy to see that $(C1) - (C6)$ are satisfied. By simple calculation, we get $\rho = 1,\ \theta(t) = 2 + t,\ \varphi(t) = \frac{1}{4}(2-t),\ \mathcal{D} = -\frac{3}{20},\ A(1) = \frac{235}{432},\ B(1) = \frac{121}{108}$,

$$G(t,s) = \frac{1}{4} \begin{cases} (2+s)(2-t), \quad s \le t, \quad (2+t)(2-s), \quad t \le s. \\ \\ (2+t)(2-s), \quad t \le s. \end{cases}$$

We can obtain $M = \frac{432}{1343}$. Applying Theorem 3.1, we get the optimal eigenvalue interval $\lambda_i \le \frac{432}{1343},\ i = 1, 2, 3$, for which the boundary value problem $(4.1) - (4.2)$ has a positive solution.

Example 4.2 In BVP $(1.1) - (1.2)$, suppose that $\mathbb{T} = \left[0, \frac{1}{3}\right] \cup \left\{\frac{1}{2}\right\} \cup \left[\frac{2}{3}, 1\right],\ n = 3,\ q_1(t) = q_2(t) = q_3(t) = 1,\ a = 2,\ b = 1,\ c = 4,\ d = 3,\ g_1(t) = 2,\ g_2(t) = 3$, i.e.,

$$\begin{cases} u_i^{\Delta\Delta}(t) + \lambda_i f_i(u_{i+1}(t)) = 0, \ t \in [0,1]_{\mathbb{T}}, \ 1 \le i \le 3, \\ u_4(t) = u_1(t), \ t \in [0,1]_{\mathbb{T}}, \end{cases} \tag{4.3}$$

satisfying the following boundary conditions,

$$\begin{cases} 2u_i(0) - u_i^{\Delta}(0) = \int_0^1 2u_i(t)\Delta t, \\ 4u_i(1) + 3u_i^{\Delta}(1) = \int_0^1 3u_i(t)\Delta t, \ 1 \le i \le 3, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} f_1(u_2) &= (1-\cos u_2)e^{u_2}, \\ f_2(u_3) &= (\sin u_3)^2 - u_3^2 e^{u_3}, \\ f_3(u_1) &= \cos(2u_1^2) - e^{u_1^2}. \end{aligned}$$

It is easy to see that $(C1)-(C6)$ are satisfied. By simple calculation, we get $\rho = 18$,

$$G(t,s) = \frac{1}{18}\begin{cases} (1+2\sigma(s))(7-4t), & \sigma(s) \le t, \\ (1+2t)(7-4\sigma(s)), & t \le s, \end{cases}$$

$\mathcal{D} = -35$, $A(1) = \frac{563}{945}$, $B(1) = \frac{1126}{2835}$ and $M = \frac{102060}{519329}$. Applying Theorem 3.1, we get the optimal eigenvalue interval $\lambda_i \le \frac{102060}{519329}$, $i = 1,2,3$, for which the boundary value problem $(4.3)-(4.4)$ has a positive solution.

Acknowledgements: The authors would like to thank the referee for her/his valuable suggestions and comments.

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[0]Başlık: İntegral sınır koşullu dinamik iteratif sistemlerin özdeğerleri

Anahtar Kelimeler: Green fonksiyonu, iteratif sistem, özdeğer aralığı, zaman skalası, sınır değer problemi, sabit nokta teoremi, pozitif çözüm