Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 65, Number 1, Pages 105–126 (2016)
DOI: 10.1501/Commua1_0000000747
ISSN 1303–5991

BIPOLAR SOFT ROUGH RELATIONS

FARUK KARAASLAN

Abstract. In this study, Cartesian products of bipolar soft P-lower and P-upper approximations of two bipolar soft rough sets are defined and based on the these cartesian products, concepts of bipolar soft rough P-upper and P-lower relations are introduced, and some properties existing in the classical relations are obtained for bipolar soft rough relations. Also, some new concepts such as equivalence bipolar soft rough relation, inverse bipolar soft rough relation, bipolar equivalence class of an element in universal set and partition of bipolar soft rough set under equivalence bipolar soft rough relation are defined and supported by examples.

1. Introduction

To overcome complex problems containing uncertainty, vagueness and incomplete information in some areas such as economy, engineering, social sciences and environmental sciences many theories have been proposed up to now. Some of them are fuzzy set theory [49], intuitionistic fuzzy set theory [6], rough set theory [36], vague set theory [17], bipolar fuzzy set theory [52],[53]. However, each of these theories has inherent difficulties. Molodtsov pointed out these difficulties in [33] and suggested a novel approach called soft set theory in order to deal with these difficulties. Then Maji [29] defined set theoretical operations on soft sets such as union, intersection and complement. Ali et al. [4] proposed novel concepts as extended intersection, restricted intersection and restricted union based on idea which is not necessary having same approximations of two soft sets for common parameters. Çağman and Enginoğlu [8] redefined operations of soft sets so as to use effectively in decision making problems, and proposed a decision making method. Sezgin et al. [41] investigated basic soft set operations systematically. In 2014, Çağman [11] pointed out some gaps related to soft set definitions and operations given in [33], [29] and [8]. Soft group, which is first algebraic structure of soft sets, was first defined Aktaş and Çağman [2] in 2007. Recently, studies related to algebraic structures of soft sets have increased rapidly. For example, Sezgin and Atagün [40]

Received by the editors: January 20, 2016, Accepted: March 07, 2016.
2010 *Mathematics Subject Classification.* 03E20.
Key words and phrases. Rough set, soft set, bipolar soft set, bipolar soft rough set, cartesian product, relation.

©2016 Ankara University

pointed out some problematic cases in [2] and they corrected these problematic cases. They also defined concepts of normalistic soft groups and normalistic soft groups homomorphism, and investigated some properties of them. In 2016, Atagün and Aygün [5] introduced two new operations on soft sets and showed that the set of all soft sets over a universe is an abelian group under these novel soft set operations. Furthermore, many studies on set theoretical aspects of soft sets were made. Some of them can be seen in references [39],[45],[18] and [25]. Also concept of soft set combined with fuzzy sets (see [27], [31], [9]), intuitionistic fuzzy sets (see [28], [30], [10]), interval valued fuzzy sets (see [47]), vague set [46], interval-valued intuitionistic fuzzy sets [21] and bipolar fuzzy sets (see [1] [42]).

The rough set theory which is an important tool for imperfect data analysis was initiated by Pawlak [36] as an alternative approach to fuzzy set theory and tolerance theory. This theory has very important role in many applications such as machine learning, pattern recognition, image precessing and decision analysis. Aktaş and Çağman [2] discussed relations among concepts of fuzzy set, rough set and soft set. Dubois and Prade [12] extended notion of rough set to rough fuzzy set and fuzzy rough set by considering approximation of a fuzzy set in an approximation space and lover and upper approximations in Pawlak's approximation, respectively. Herawan and Deris [19] explained connection between rough set and soft set by using constructive and descriptive approaches of rough set theory. Feng and Liu [13] proposed some new concepts such as soft approximation spaces, soft rough approximations and soft rough sets by combining soft set approach with rough set theory, and gave an application of soft rough set in demand analysis. Soft sets were combined with fuzzy sets and rough sets by Feng et. al [14]. Feng et al.[15] gave a generalization of Pawlak's rough set model by using soft sets instead of relation in Pawlak's approach. They also presented basic properties of soft rough approximations based on soft approximation spaces and soft rough approximation and soft rough sets, and defined some new types of soft set such as full soft set, intersection complement soft set and partition soft set. Feng [16] developed a multi criteria group decision making method based on soft rough approximations, and gave an application of developed method in decision making. Meng et al. [32] proposed a new soft rough set model and derived its properties. They also established more general model called soft rough fuzzy set. Ali [3] discussed concept of approximation space associated with each parameter in a soft set and defined an approximation space associated with the soft sets, and established connection between soft set, fuzzy soft set and rough sets. Zhang [50] introduced concept of intuitionistic fuzzy soft rough sets by combining intuitionistic fuzzy soft set with rough set, and investigated its some fundamental properties. He also presented a decision making method for intuitionistic fuzzy soft sets based on this new rough set approach. Zhang [51] studied on parameter reduction of fuzzy soft sets based on soft fuzzy rough set and defined some new concepts such as lower soft fuzzy rough approximation operator and upper soft fuzzy rough approximation operator, etc. To find approximation of a set, Shabir et al. [43] proposed modified soft rough sets. Li et al. [26] investigated soft rough approximation operators and showed that Pawlak's rough set model are a special case of soft rough sets and every topological space on initial universe is a soft approximation spaces. Sun and Ma [44] proposed a new concept of soft fuzzy rough set by combining the fuzzy soft set with the traditional fuzzy rough set. They also defined concept of the pseudo fuzzy binary relation and defined the soft fuzzy rough lower and upper approximation operators of any fuzzy subset over the parameter set. Karaaslan [23] defined concept of soft class and based on this structure he proposed the notion of soft rough class, and developed a decision making method.

First study on soft set relations was made by Babitha and Sunil [7]. They defined some concepts such as relation, function on soft sets and investigated basic properties of them existing relations in classical set theory. In 2011, Yang and Guo [48] defined some notions such as anti-reflexive kernel, symmetric kernel, reflexive closure, and symmetric closure related to soft relations, and obtained some properties of these concepts. They also proposed concepts of inverse soft set relation and mapping and discussed some related properties. Ibrahim et al. [20] defined composition of soft set relation and gave matrix presentations of soft set relations, and showed that Warshall's algorithm satisfies construction of transitive closure in soft sets. Park et al. [35] investigated some properties of equivalence soft set relation defined by Babitha and Sunil [7]. Qin et al. [38] extended concepts of soft set relation and function defined in [7] by defining the Cartesian product of soft sets in different universes. They also investigated connections between soft set relations and fuzzy sets.

Concept of bipolar soft set and its operations such as union, intersection and complement were first defined by [42]. Karaaslan and Karataş [22] redefined bipolar soft sets with a new approximation providing opportunity to study on topological structures of bipolar soft sets.

In this paper, we define Cartesian products of bipolar soft P-lower and P-upper approximations and based on the these cartesian products we give definitions of concepts of bipolar soft rough P-upper and P lower relations on bipolar soft rough sets defined by Karaaslan and Çağman [24]. Also we introduced some new concepts such as equivalence bipolar soft rough relation, inverse bipolar soft rough relation, bipolar equivalence class of an element in universal set and partition of a bipolar soft rough set under bipolar soft rough relation. Furthermore we obtain some properties of bipolar soft rough relations. This paper is arranged in the following manner. In Section 2, some basic concepts related to soft set, bipolar soft set and soft relation are given. In Section 3, cartesian products of bipolar soft P-upper and P-lower approximations, bipolar soft rough P-upper and P-lower relations, inverse bipolar soft rough relations, equivalence bipolar soft rough relations and partition of bipolar soft rough set are defined and supported by examples, and some properties related to defined notions are obtained.

2. Preliminary

In this section, we present some definitions and properties required in next sections of the study.

Throughout the paper, U denotes an initial universe and E is parameter set and $P(U)$ is the power set of U.

Definition 1. [33] *Let U be a set of objects and E be a set of parameters. Then, a mapping $F : E \to P(U)$ is called a soft set over U and denoted by (F, E).*

From now on, set of all soft sets over U will be denoted by $\mathcal{S}(U)$.

Definition 2. [7] *Let $(F, E), (G, E) \in \mathcal{S}(U)$, then Cartesian product of (F, E) and (G, E) is defined by $(F, E) \times (G, E) = (H, E \times E)$, where $H : E \times E \to P(U \times U)$ and $H(a, b) = F(a) \times F(b)$, where $(a, b) \in E \times E$.*

Babitha and Sunil [7] defined soft set relation as a subset of $(F, E) \times (G, E)$ on same universe U. Then Qin et al. [38] pointed out that Babitha and Sunil's definition contradicts Cantor's set theory. Therefore they redefined concept of soft set relations on difference universe.

Definition 3. [38] *Let $(F, E) \in \mathcal{S}(U)$ and $(G, E) \in \mathcal{S}(V)$. Cartesian product of (F, E) and (G, E) is a soft set over $U \times V$ and is defined as $(F, E) \times (G, E) = (H, E \times E)$, where $H : E \times E \to P(U \times V)$ is given by $H(a, b) = F(a) \times G(b)$ for all $(a, b) \in E \times E$.*

Definition 4. [38] *Let $(F, E) \in \mathcal{S}(U)$ and $(G, E) \in \mathcal{S}(V)$.*

1. *If $(H, C) \subseteq (F, E) \times (G, E)$ i,e., $C \subseteq E \times E$ and $H(a, b) \subseteq F(a) \times G(b)$ such that $F(a) \neq \emptyset$ and $G(b) \neq \emptyset$ for each $(a, b) \in C$, then (H, C) is called a soft relation from (F, E) to (G, E).*
2. *A soft relation from (F, E) to (F, E) is called a soft relation on (F, E).*

2.1. **Bipolar soft sets.**

Definition 5. [42] *A triplet (F, G, E) is called a bipolar soft set (BS-set) over U, where F and G are mappings, given by $F : E \to P(U)$ and $G : \rceil E \to P(U)$ such that $F(e) \cap G(\neg e) = \emptyset$ for all $e \in E$.*

Here $\rceil E$ denotes the "NOT" set of a set of parameter set defined in [29]. For example, if $E = \{e_1 = \textit{good}, e_2 = \textit{cheap}, e_3 = \textit{modern}\}$, NOT set of parameter set E is $\rceil E = \{\neg e_1 = \textit{not good}, \neg e_2 = \textit{not cheap}, \neg e_3 = \textit{not modern}\}$.

Henceforward, set of all bipolar soft sets on initial universe U will be denoted by $\mathcal{BS}_U$.

Definition 6. [42] *Let $(F_1, G_1, E), (F_2, G_2, E) \in \mathcal{BS}_U$. (F_1, G_1, E) is a bipolar soft subset of (F_2, G_2, E), if $F_1(e) \subseteq F_2(e)$ and $G_2(\neg e) \subseteq G_1(\neg e)$ for all $e \in E$.*

This relationship is denoted by $(F_1, G_1, E)\tilde{\subseteq}(F_2, G_2, E)$. Similarly (F_1, G_1, E) is said to be a bipolar soft superset of (F_2, G_2, E), if (F_2, G_2, E) is a bipolar soft subset of (F_1, G_1, E). We denote it by $(F_1, G_1, E)\tilde{\supseteq}(F_2, G_2, E)$.

Definition 7. [42] *Let $(F_1, G_1, E), (F_2, G_2, E) \in \mathcal{BS}_U$. If (F_1, G_1, E) is a bipolar soft subset of (F_2, G_2, E) and (F_2, G_2, E) is a bipolar soft subset of (F_1, G_1, E), (F_1, G_1, E) and (F_2, G_2, E) are said to be equal.*

Definition 8. [42] *Let $(F, G, E) \in \mathcal{BS}_U$. The complement of a bipolar soft set (F, G, E) is denoted by $(F, G, E)^c$ and is defined by $(F, G, E)^c = (F^c, G^c, E)$ where F^c and G^c are mappings given by $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in E$.*

Definition 9. [42] *Let $(F, G, E) \in \mathcal{BS}_U$. If for all $e \in E$, $F(e) = \emptyset$ and $G(\neg e) = U$, bipolar soft set (F, G, E) is called a relative null bipolar soft set and denoted by $(\Phi, \mathfrak{U}, E)$.*

Definition 10. [42] *Let $(F, G, E) \in \mathcal{BS}_U$. If for all $e \in E$, $F(e) = U$ and $G(\neg e) = \emptyset$, bipolar soft set (F, G, E) is called relative absolute bipolar soft set and denoted by $(\mathfrak{U}, \Phi, E)$.*

Definition 11. [37] *Let U be a nonempty finite set of object, A be a nonempty finite set of attributes and a be a function from U to V_a for all $a \in A$ where V_a is the set of values of attributive a. $P = (U, A)$ is called an information system.*

Definition 12. [37] *Let A be a nonempty finite set of attributes and $B \subseteq A$. Then, B determines a binary relation $I(B)$, called indiscernibility relation, defined by*

$$xI(B)y \text{ if and only if } a(x) = a(y) \text{ for all } a \in B$$

where $a(x)$ denotes the value of attributive a for object $x \in U$.

Definition 13. [14, 37] *Let R be an equivalence relation over universe U. Then the pair (U, R) is called a Pawlak approximation space.*

Here R is considered as an indiscernibility relation obtained from an information system. Since R is an equivalence relation, relation R gives a partition of U due to the indiscernibility of objects in U. This partition is denoted by U/R. An equivalence class of R, i.e., the block of the partition U/R, containing x is denoted by $[x]_R$. These equivalence classes of R are referred to as R-elementary sets (or R-elementary granules). The elementary sets represent the basic building blocks (concepts) of our knowledge about reality.

Using indiscernibility relation R, for every subset $X \subseteq U$ lower and upper approximations of X with respect to (U, R) are defined, respectively as follows:

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$
$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Definition 14. [15] *Let (F, E) be a soft set over U and (U, R) be a Pawlak approximation space. If R is taken as (F, E), then the Pawlak approximation space is called a soft approximation space and denoted by $P = (U, (F, E))$.*

Definition 15. [15] *Let (F,E) be a soft set over U, $X \subseteq U$ and $P = (U,(F,E))$ be a soft approximation space. Then,*

$$\begin{aligned} \underline{S}_P(X) &= \{u \in U : \exists e \in E, [u \in F(e) \subseteq X]\} \\ \overline{S}_P(X) &= \{u \in U : \exists e \in E, [u \in F(e), F(e) \cap X \neq \emptyset]\} \end{aligned}$$

are called soft P-lower approximation and soft P-upper approximation of X, respectively.

In [24], Karaaslan and Çağman explain relations between bipolar soft sets and information systems.

Definition 16. [24]*Let (F,G,E) be a bipolar soft set over U and (U,R) be a Pawlak approximation spaces. If it is taken as $R = (F,G,E)$, Pawlak approximation space is called a bipolar soft approximation space and denoted by $P = (U,(F,G,E))$.*

Definition 17. [24] *Let $P = (U,(F,G,E))$ be a bipolar soft approximation space. Then soft approximation spaces denoted by $P^+ = (U,F)$ and $P^- = (U,G)$ are called positive soft approximation space and negative soft approximation space of bipolar soft set (F,G,E), respectively.*

Definition 18. [24] *Let $(F,G,E) \in \mathcal{BS}_U$. Then, the mappings $F : E \to P(U)$ and $G : \rceil E \to P(U)$ are called positive soft set and negative soft set of (F,G,E), respectively.*

From now onward positive and negative soft sets of bipolar soft set (F,G,E) will be denoted by F and G, respectively.

Definition 19. [24] *Let (F,G,E) be a bipolar soft set over U and $P = (U,(F,G,E))$ be a bipolar soft approximation space (BSA-space). Then,*

$$\begin{aligned} \underline{S}_{P^+}(X) &= \{u \in U : \exists e \in E, [u \in F(e) \subseteq X]\}, \\ \underline{S}_{P^-}(X) &= \{u \in U : \exists \neg e \in \rceil E, [u \in G(\neg e), G(\neg e) \cap \sim X \neq \emptyset]\}, \\ \overline{S}_{P^+}(X) &= \{u \in U : \exists e \in E, [u \in F(e), F(e) \cap X \neq \emptyset]\}, \\ \overline{S}_{P^-}(X) &= \{u \in U : \exists \neg e \in \rceil E, [u \in G(\neg e) \subseteq \sim X]\}. \end{aligned}$$

are called P-lower positive approximation ($SPL^+ -$ approximation), P-lower negative approximation ($SPL^- -$ approximation), soft P-upper positive approximation ($SPU^+ -$ approximation) and soft P-upper negative approximation ($SPU^- -$ approximation) of $X \subseteq U$, respectively.

Bipolar soft rough approximations of X with respect to BSA-space $P = (U,(F,G,E))$ can be written as two pairs as follows:

$$\underline{BS}_P(X) = \big(\underline{S}_{P^+}(X), \underline{S}_{P^-}(X)\big)$$

and

$$\overline{BS}_P(X) = \big(\overline{S}_{P^+}(X), \overline{S}_{P^-}(X)\big).$$

Note that, it need not to be $\underline{S}_{P^+}(X) \cap \underline{S}_{P^-}(X) = \emptyset$. Also $\underline{S}_{P^+}(X)$ and $\overline{S}_{P^+}(X)$ are identical to soft P-lover and P-upper approximation of X defined by Feng et al. in [15], respectively.

By Definition 19, we immediately have that $\underline{S}_{P^+}(X) \subseteq X$ and $\overline{S}_{P^-}(X) \subseteq \sim X$. However, $\underline{S}_{P^-}(X) \subseteq \sim X$ and $\overline{S}_{P^+}(X) \subseteq X$ don't hold in generally.

Definition 20. [24] *Let $(F, G, E) \in \mathcal{BS}_U$, $P = (U, (F, G, E))$ be a bipolar soft approximation space and let $\underline{BS}_P(X)$ and $\overline{BS}_P(X)$ be bipolar soft rough approximations of $X \subseteq U$ with respect to BSA-space $P = (U, (F, G, E))$. If $\underline{BS}_P(X) \neq \overline{BS}_P(X)$, X is called bipolar soft P-rough set.*

Example 1. *Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of cars under consideration, and E be the set of parameters, $E = \{e_1, e_2, e_3, e_4\}$ = {*cheap, comfort, high equipment, longtime warrant*}. Then $\rceil E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4\}$ = {*expensive, uncomfortable, low equipment, not longtime warrant *}. The bipolar soft sets (F, G, E) describe the "requirements of the cars' which Mr. X going to buy. Suppose that*

$$F(e_1) = \{u_2, u_3\}, F(e_2) = \{u_2, u_5\}, F(e_3) = \{u_3\}, F(e_4) = \{u_2, u_3, u_5\}$$

$$G(\neg e_1) = \{u_4, u_5\}, G(\neg e_2) = \{u_3, u_4\}, G(\neg e_3) = \{u_2, u_4\}, G(\neg e_4) = \{u_4\}.$$

For $X = \{u_1, u_2, u_3\} \subseteq U$, we have SPL^+ − approximation $\underline{S}_{P^+}(X) = \{u_2, u_3\}$ and SPL^- − approximation $\underline{S}_{P^-}(X) = \{u_2, u_3, u_4, u_5\}$. Thus

$$\underline{BS}_P(X) = \big(\{u_2, u_3\}, \{u_2, u_3, u_4, u_5\}\big). \tag{2.1}$$

Also we have SPU^+-approximation $\overline{S}_{P^+}(X) = \{u_2, u_3, u_5\}$ and SPU^--approximation $\overline{S}_{P^-}(X) = \{u_4, u_5\}$. Thus

$$\overline{BS}_P(X) = \big(\{u_2, u_3, u_5\}, \{u_4, u_5\}\big). \tag{2.2}$$

From (2.1) and (2.2)

$$\underline{BS}_P(X) \neq \overline{BS}_P(X)$$

and so X is a bipolar soft P-rough set.

Definition 21. [24] *Let $(F, G, E) \in \mathcal{BS}_U$, $P = (U, (F, G, E))$ be a BSA-space and $X, Y \subseteq U$. Then,*

(1) $\underline{BS}_P(X) \sqsubseteq \underline{BS}_P(Y) \Leftrightarrow \underline{S}_{P^+}(X) \subseteq \underline{S}_{P^+}(Y)$ *and* $\underline{S}_{P^-}(X) \supseteq \underline{S}_{P^-}(Y)$
(2) $\overline{BS}_P(X) \sqsubseteq \overline{BS}_P(Y) \Leftrightarrow \overline{S}_{P^+}(X) \subseteq \overline{S}_{P^+}(Y)$ *and* $\overline{S}_{P^-}(X) \supseteq \overline{S}_{P^-}(Y)$

Corollary 1. [24] *if $X = Y$, then $\underline{BS}_P(X) \sqsubseteq \overline{BS}_P(X)$.*

Definition 22. [24] *Let $(F, G, E) \in \mathcal{BS}_U$ and $P = (U, (F, G, E))$ be a BSA-space. If $\underline{BS}_P(X) \sqsubseteq \overline{BS}_P(X)$ and $\overline{BS}_P(X) \sqsubseteq \underline{BS}_P(X)$, then*

$$\underline{BS}_P(X) = \overline{BS}_P(X).$$

Definition 23. [24] *Let* $(F,G,E) \in \mathcal{BS}_U$, $P = (U,(F,G,E))$ *be a* BSA*-space and* $X, Y \subseteq U$. *Then, union of bipolar soft* P*-lower approximations and bipolar soft* P*-upper approximations of sets* X *and* Y *are defined, respectively, as follows:*

$$\underline{BS}_P(X) \sqcup \underline{BS}_P(Y) = \big(\underline{S}_{P^+}(X) \cup \underline{S}_{P^+}(Y), \underline{S}_{P^-}(X) \cap \underline{S}_{P^-}(Y)\big)$$

$$\overline{BS}_P(X) \sqcup \overline{BS}_P(Y) = \big(\overline{S}_{P^+}(X) \cup \overline{S}_{P^+}(Y), \overline{S}_{P^-}(X) \cap \overline{S}_{P^-}(Y)\big).$$

Definition 24. [24] *Let* $(F,G,E) \in \mathcal{BS}_U$, $P = (U,(F,G,E))$ *be a* BSA*-space and* $X, Y \subseteq U$. *and* $X, Y \subseteq U$. *Then, intersection of bipolar soft* P*-lower approximations and bipolar soft* P*-upper approximations of sets* X *and* Y *are defined, respectively, as follows:*

$$\underline{BS}_P(X) \sqcap \underline{BS}_P(Y) = \big(\underline{S}_{P^+}(X) \cap \underline{S}_{P^+}(Y), \underline{S}_{P^-}(X) \cup \underline{S}_{P^-}(Y)\big),$$

$$\overline{BS}_P(X) \sqcap \overline{BS}_P(Y) = \big(\overline{S}_{P^+}(X) \cap \overline{S}_{P^+}(Y), \overline{S}_{P^-}(X) \cup \overline{S}_{P^-}(Y)\big).$$

Example 2. *Let us consider BS-set given in Example 1 and subsets* $X = \{u_1, u_2, u_3\}$ *and* $Y = \{u_2, u_3, u_4\}$ *of* U. *From Example 1, we know that* $\underline{BS}_P(X) = \big(\{u_2, u_3\}, \{u_2, u_3, u_4, u_5\}\big)$ *and* $\overline{BS}_P(X) = \big(\{u_2, u_3, u_5\}, \{u_4, u_5\}\big)$. *Since* $Y = \{u_2, u_3, u_4\}$, $\underline{BS}_P(Y) = \big(\{u_2, u_3\}, \{u_3, u_4, u_5\}\big)$ *and* $\overline{BS}_P(Y) = \big(\{u_2, u_3, u_5\}, \{\}\big)$. *Here, since* $\underline{S}_{P^+}(X) \subseteq \underline{S}_{P^+}(Y)$ *and* $\underline{S}_{P^-}(X) \supseteq \underline{S}_{P^-}(Y)$, $\underline{BS}_P(X) \sqsubseteq \underline{BS}_P(Y)$. *Also* $\overline{BS}_P(X) \sqsubseteq \overline{BS}_P(Y)$ *due to the fact that* $\overline{S}_{P^+}(X) \subseteq \overline{S}_{P^+}(Y)$ *and* $\overline{S}_{P^-}(X) \supseteq \overline{S}_{P^-}(Y)$. *Union and intersection of bipolar soft rough approximations are as follows:*

$$\underline{BS}_P(X) \sqcup \underline{BS}_P(Y) = \big(\{u_2, u_3\}, \{u_3, u_4, u_5\}\big),$$
$$\overline{BS}_P(X) \sqcup \overline{BS}_P(Y) = \big(\{u_2, u_3, u_5\}, \{\}\big)$$

and

$$\underline{BS}_P(X) \sqcap \underline{BS}_P(Y) = \big(\{u_2, u_3\}, \{u_2, u_3, u_4, u_5\}\big),$$
$$\overline{BS}_P(X) \sqcap \overline{BS}_P(Y) = \big(\{u_2, u_3, u_5\}, \{u_4, u_5\}\big)$$

3. Bipolar soft rough relations

In this section, Cartesian product of two bipolar soft rough sets and concept of bipolar soft rough relation are defined and investigated some properties of them.

Definition 25. *Let* $(F,G,E) \in \mathcal{BS}_U$, $P = (U,(F,G,E))$ *be a BSA-space,* $X, Y \subseteq U$ *and all of soft* P*-approximations (positive lower, upper and negative lower, upper) be nonempty sets. Then, cartesian product of bipolar soft* P*-lower approximations and cartesian product of bipolar soft* P*-upper approximations of sets* X *and* Y *are defined, respectively, as follows:*

$$\underline{BS}_P(X) \times \underline{BS}_P(Y) = \Big(\underline{S}_{P^+}(X) \times \underline{S}_{P^+}(Y), \underline{S}_{P^-}(X) \times \underline{S}_{P^-}(Y)\Big)$$

$$\overline{BS}_P(X) \times \overline{BS}_P(Y) = \Big(\overline{S}_{P^+}(X) \times \overline{S}_{P^+}(Y), \overline{S}_{P^-}(X) \times \overline{S}_{P^-}(Y)\Big)$$

Example 3. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ *be the initial universe and* $E = \{e_1, e_2, e_3\}$ *be a parameter set. Consider positive and negative soft sets* $F(e_1) = \{u_1, u_3\}, F(e_2) = \{u_2, u_4\}, F(e_3) = \{u_1, u_4\}$ *and* $G(e_1) = \{u_5\}, G(e_2) = \{u_7, u_8\}, G(e_3) = \{u_6, u_8\}$*. If we take* $X = \{u_1, u_4, u_5, u_7\}, Y = \{u_1, u_3, u_5\} \subset U$*, then*

$$\begin{aligned}\underline{S}_{P^+}(X) &= \{u_1, u_4\}, \quad \underline{S}_{P^-}(X) = \{u_6, u_7, u_8\}\\ \overline{S}_{P^+}(X) &= \{u_1, u_2, u_3, u_4\}, \ \overline{S}_{P^-}(X) = \{u_6, u_8\}.\end{aligned}$$

and

$$\begin{aligned}\underline{S}_{P^+}(Y) &= \{u_1, u_3\}, \quad \underline{S}_{P^-}(Y) = \{u_6, u_7, u_8\}\\ \overline{S}_{P^+}(Y) &= \{u_1, u_3, u_4\}, \ \overline{S}_{P^-}(Y) = \{u_6, u_7, u_8\}.\end{aligned}$$

Cartesian products of bipolar soft P*-lower approximations and bipolar soft* P*-upper approximations of sets* X *and* Y *are as follows:*

$$\begin{aligned}\underline{BS}_P(X) \times \underline{BS}_P(Y) = \Big(&\Big\{(u_1, u_1), (u_1, u_3), (u_4, u_1), (u_4, u_3)\Big\}, \Big\{(u_6, u_6), (u_6, u_7),\\ &(u_6, u_8), (u_7, u_6), (u_7, u_7), (u_7, u_8), (u_8, u_6), (u_8, u_7),\\ &(u_8, u_8)\Big\}\Big)\end{aligned}$$

and

$$\begin{aligned}\overline{BS}_P(X) \times \overline{BS}_P(Y) = \Big(&\Big\{(u_1, u_1), (u_1, u_3), (u_1, u_4), (u_2, u_1), (u_2, u_3), (u_2, u_4),\\ &(u_3, u_1), (u_3, u_3), (u_3, u_4), (u_4, u_1), (u_4, u_3), (u_4, u_4)\Big\},\\ &\Big\{(u_6, u_6), (u_6, u_7), (u_6, u_8), (u_8, u_6), (u_8, u_7), (u_8, u_8)\Big\}\Big),\end{aligned}$$

respectively.

It is clear that $\underline{BS}_P(X) \times \underline{BS}_P(Y) \sqsubseteq \overline{BS}_P(X) \times \overline{BS}_P(Y)$.

Definition 26. *Let* $(F, G, E) \in \mathcal{BS}_U$*,* $P = (U, (F, G, E))$ *be a BSA-space and* $X, Y \subseteq U$*. Bipolar soft rough* P*-lower relation (*$BSRP_L$*-relation) denoted by* $\underline{R}_P(X, Y)$ *and bipolar soft rough* P*-upper relation (*$BSRP_U$*-relation) denoted by* $\overline{R}_P(X, Y)$ *are defined, respectively, as follows;*

$$\underline{R}_P(X, Y) = \Big(\underline{R}_{P^+}(X, Y), \underline{R}_{P^-}(X, Y)\Big),$$

$$\overline{R}_P(X, Y) = \Big(\overline{R}_{P^+}(X, Y), \overline{R}_{P^-}(X, Y)\Big).$$

Here $\underline{R}_{P^+}(X,Y) \subseteq \underline{S}_{P^+}(X) \times \underline{S}_{P^+}(Y)$, $\underline{R}_{P^-}(X,Y) \subseteq \underline{S}_{P^-}(X) \times \underline{S}_{P^-}(Y)$, $\overline{R}_{P^+}(X,Y) \subseteq \overline{S}_{P^+}(X) \times \overline{S}_{P^+}(Y)$ and $\overline{R}_{P^-}(X,Y) \subseteq \overline{S}_{P^-}(X) \times \overline{S}_{P^-}(Y)$.

Note that, $\underline{R}_{P^+}(X,Y)$, $\underline{R}_{P^-}(X,Y)$, $\overline{R}_{P^+}(X,Y)$ and $\overline{R}_{P^-}(X,Y)$ are classical relations.

Definition 27. *Let $\underline{R}_P(X,Y)$ and $\overline{R}_P(X,Y)$ be bipolar soft rough $P-$lover and bipolar soft rough $P-$upper relations from $BS_P(X)$ to $BS_P(Y)$, respectively. If $\overline{R}_P(X,Y) \setminus \underline{R}_P(X,Y) \neq \emptyset$ and $\underline{R}_P(X,Y) \subset \overline{R}_P(X,Y)$, then $R_P(X,Y)$ is called bipolar soft rough P-relation (BSRP-relation) from $BS_P(X)$ to $BS_P(Y)$. If $\underline{R}_P(X,Y) = \overline{R}_P(X,Y)$, $R_P(X,Y)$ is called bipolar soft $P-$definable relation.*

Example 4. *Let us consider Cartesian products of bipolar soft P-lower approximations and bipolar soft P-upper approximations of sets X and Y in Example 3. If we get*

$$\underline{R}_P(X,Y) = \Big(\Big\{(u_1,u_1),(u_4,u_3)\Big\},\Big\{(u_6,u_6),(u_6,u_8),(u_7,u_6),(u_8,u_6)\Big\}\Big)$$

$$\overline{R}_P(X,Y) = \Big(\Big\{(u_1,u_1),(u_4,u_1),(u_4,u_3)\Big\},\Big\{(u_6,u_6),(u_6,u_8)\Big\}\Big),$$

then $R_P(X,Y) = \Big(\underline{R}_P(X,Y),\overline{R}_P(X,Y)\Big)$ is a $BSRP-$relation from $BS_P(X)$ to $BS_P(Y)$.

In an equivalent way, we can define the $BSRP_L$-relation and $BSRP_U$-relation on the $X \subseteq U$ as follows:

$$\underline{R}_P(X,X) = \Big(\underline{R}_{P^+}(X,X),\underline{R}_{P^-}(X,X)\Big)$$

$$\overline{R}_P(X,X) = \Big(\overline{R}_{P^+}(X,X),\overline{R}_{P^-}(X,X)\Big).$$

For convenience, a bipolar soft rough P-relation over $BS_P(X)$ is denoted by $R_P(X)$.

From now on, set of all bipolar soft rough P-relation over $BS_P(X) \in \mathcal{BS}_U$ and $P = (U,(F,G,E))$ will be denoted by $\mathcal{BSR}_P(X)$.

Example 5. *Let us consider SPL^+-approximation (resp, SPL^--approximation) and SPU^+-approximation (resp, SPU^--approximation) of X in Example 4 given as follows:*

$$\begin{aligned} \underline{S}_{P^+}(X) &= \{u_1,u_4\}, \quad \underline{S}_{P^-}(X) = \{u_6,u_7,u_8\} \\ \overline{S}_{P^+}(X) &= \{u_1,u_3,u_4\}, \ \overline{S}_{P^-}(X) = \{u_6,u_8\}. \end{aligned}$$

Then,

$$\underline{R}_P(X) = \Big(\Big\{(u_1,u_1),(u_4,u_1)\Big\},\Big\{(u_6,u_6),(u_7,u_8),(u_8,u_8)\Big\}\Big)$$

and

$$\overline{R}_P(X) = \Big(\Big\{(u_1,u_3),(u_1,u_4),(u_3,u_4),(u_4,u_1)\Big\},\Big\{(u_6,u_8),(u_8,u_8)\Big\}\Big)$$

are $BSRP_L$-relation and $BSRP_U$-relation over $BS_P(X)$, respectively.

Definition 28. *Let* $(F,G,E) \in \mathcal{BS}_U$, $P=(U,(F,G,E))$ *be a BSA-space and* $X,Y \subseteq U$. *Inverse of BSRP-relation* $R_P(X)$ *is defined as follows:*

$$R_P^{-1}(X,Y)=\Big(\underline{R}_P^{-1}(X,Y),\overline{R}_P^{-1}(X,Y)\Big).$$

Here,

$$\underline{R}_P^{-1}(X,Y)=\Big(\underline{R}_{P^+}^{-1}(X,Y),\underline{R}_{P^-}^{-1}(X,Y)\Big)$$

and

$$\overline{R}_P^{-1}(X,Y)=\Big(\overline{R}_{P^+}^{-1}(X,Y),\overline{R}_{P^-}^{-1}(X,Y)\Big),$$

and

$$\begin{aligned}
\underline{R}_{P^+}^{-1}(X,Y) &= \{(y,x):(x,y)\in \underline{R}_{P^+}\}\\
\underline{R}_{P^-}^{-1}(X,Y) &= \{(y,x):(x,y)\in \underline{R}_{P^-}\}\\
\overline{R}_{P^+}^{-1}(X,Y) &= \{(y,x):(x,y)\in \overline{R}_{P^+}\}\\
\overline{R}_{P^-}^{-1}(X,Y) &= \{(y,x):(x,y)\in \overline{R}_{P^-}\}.
\end{aligned}$$

Example 6. *Let us consider* $BSRP_L$*-relation and* $BSRP_U$*-relation in Example 4. Then,*

$$\underline{R}_P^{-1}(X,Y)=\Big(\Big\{(u_1,u_1),(u_3,u_4)\Big\},\Big\{(u_6,u_6),(u_8,u_6),(u_6,u_7),(u_6,u_8)\Big\}\Big)$$

$$\overline{R}_P^{-1}(X,Y)=\Big(\Big\{(u_1,u_1),(u_1,u_4),(u_3,u_4)\Big\},\Big\{(u_6,u_6),(u_8,u_6)\Big\}\Big),$$

and $R_P^{-1}(X,Y)=\Big(\underline{R}_P^{-1}(X,Y),\overline{R}_P^{-1}(X,Y)\Big)$.

Definition 29. *Let* $R_{1_P}(X),R_{2_P}(X)\in \mathcal{BSR}_P(X)$. *If*

(1) $\underline{R_1}_{P^+}(X)\subseteq \underline{R_2}_{P^+}(X)$, $\underline{R_1}_{P^-}(X)\supseteq \underline{R_2}_{P^-}(X)$

(2) $\overline{R_1}_{P^+}(X)\subseteq \overline{R_2}_{P^+}(X)$, $\overline{R_1}_{P^-}(X)\supseteq \overline{R_2}_{P^-}(X)$

then, it is said that $R_{1_P}(X)$ *is a subset of* $R_{2_P}(X)$, *and denoted by* $R_{1_P}(X)\ll R_{2_P}(X)$.

Definition 30. *Let* $R_{1_P}(X),R_{2_P}(X)\in \mathcal{BSR}_P(X)$. *Then, union and intersection of BSR-relations* $R_{1_P}(X)$*and* $R_{2_P}(X)$ *are defined, respectively, as follows:*

$$R_{1_P}(X)\Cup R_{2_P}(X) = \Big(\underline{R_1}_P(X)\sqcup \underline{R_2}_P(X),\overline{R_1}_P(X)\sqcup \overline{R_2}_P(X)\Big)$$

and

$$R_{1_P}(X)\Cap R_{2_P}(X) = \Big(\underline{R_1}_P(X)\sqcap \underline{R_2}_P(X),\overline{R_1}_P(X)\sqcap \overline{R_2}_P(X)\Big).$$

Here,

$$\begin{aligned}
\underline{R_1}_P(X) \sqcup \underline{R_2}_P(X) &= \big(\underline{R_1}_{P^+}(X) \cup \underline{R_2}_{P^+}(X), \underline{R_1}_{P^-}(X) \cap \underline{R_2}_{P^-}(X)\big)\\
\overline{R_1}_P(X) \sqcup \overline{R_2}_P(X) &= \big(\overline{R_1}_{P^+}(X) \cup \overline{R_2}_{P^+}(X), \overline{R_1}_{P^-}(X) \cap \overline{R_2}_{P^-}(X)\big)\\
\underline{R_1}_P(X) \sqcap \underline{R_2}_P(X) &= \big(\underline{R_1}_{P^+}(X) \cap \underline{R_2}_{P^+}(X), \underline{R_1}_{P^-}(X) \cup \underline{R_2}_{P^-}(X)\big)\\
\overline{R_1}_P(X) \sqcap \overline{R_2}_P(X) &= \big(\overline{R_1}_{P^+}(X) \cap \overline{R_2}_{P^+}(X), \overline{R_1}_{P^-}(X) \cup \overline{R_2}_{P^-}(X)\big).
\end{aligned}$$

Proposition 1. *Let $R_{1_P}(X), R_{2_P}(X) \in \mathcal{BSR}_P(X)$. Then,*

1. $(R_{1_P}^{-1})^{-1}(X) = R_{1_P}(X)$.
2. $(R_{1_P}(X) \Cup R_{2_P}(X))^{-1} = R_{1_P}^{-1}(X) \Cup R_{2_P}^{-1}(X)$.
3. $(R_{1_P}(X) \Cap R_{2_P}(X))^{-1} = R_{1_P}^{-1}(X) \Cap R_{2_P}^{-1}(X)$.

Proof. (1) Since $\underline{R}_{P^+}(X)$, $\underline{R}_{P^-}(X), \overline{R}_{P^+}(X)$ and $\overline{R}_{P^-}(X)$ are classical relations, the proof is clear.

(2)

$$\begin{aligned}
(R_{1_P}(X) \Cup R_{2_P}(X))^{-1} &= \Big(\big(\underline{R_1}_P(X) \sqcup \underline{R_2}_P(X), \overline{R_1}_P(X) \sqcup \overline{R_2}_P(X)\big)\Big)^{-1}\\
&= \Big(((\underline{R_1}_{P^+}(X) \cup \underline{R_2}_{P^+}(X), \underline{R_1}_{P^-}(X) \cap \underline{R_2}_{P^-}(X)),\\
&\quad (\overline{R_1}_{P^+}(X) \cup \overline{R_2}_{P^+}(X), \overline{R_1}_{P^-}(X) \cap \overline{R_2}_{P^-}(X)))\Big)^{-1}\\
&\quad \text{(properties of classical relation)}\\
&= \Big(((\underline{R_1}_{P^+}^{-1}(X) \cup \underline{R_2}_{P^+}^{-1}(X), \underline{R_1}_{P^-}^{-1}(X) \cap \underline{R_2}_{P^-}^{-1}(X)),\\
&\quad (\overline{R_1}_{P^+}^{-1}(X) \cup \overline{R_2}_{P^+}^{-1}(X), \overline{R_1}_{P^-}^{-1}(X) \cap \overline{R_2}_{P^-}^{-1}(X)))\Big)\\
&= \Big(\underline{R_1}_P^{-1}(X) \sqcup \underline{R_2}_P^{-1}(X), \overline{R_1}_P^{-1}(X) \sqcup \overline{R_2}_P^{-1}(X)\Big)\\
&= R_{1P}^{-1}(X) \cup R_{2P}^{-1}(X).
\end{aligned}$$

(3) The proof can be made in similar way to proof of (2)

□

Definition 31. *Let $R_P(X) \in \mathcal{BSR}_P(X)$. Then, BSRP-relation $R_P(X)$ is reflexive, if it is satisfied following conditions:*

1. *$\underline{R}_{P^+}(X)$ and $\underline{R}_{P^-}(X)$ are reflexive relations,*
2. *$\overline{R}_{P^+}(X)$ and $\overline{R}_{P^-}(X)$ are reflexive relations.*

Example 7. *Let us consider positive(negative) soft sets and initial universe U given in Example 3 and $X = \{u_1, u_4, u_5, u_7\} \subseteq U$. Let SPL^+-approximation (resp, SPL^--approximation) and SPU^+-approximation (resp, SPU^--approximation) of*

X are given as follows:

$$\begin{aligned}\underline{S}_{P^+}(X) &= \{u_1, u_4\}, \quad \underline{S}_{P^-}(X) = \{u_6, u_7, u_8\} \\ \overline{S}_{P^+}(X) &= \{u_1, u_2, u_3, u_4\}, \ \overline{S}_{P^-}(X) = \{u_6, u_8\}.\end{aligned}$$

Then

$$\underline{R}_P(X) = \Big(\Big\{(u_1,u_1),(u_1,u_4),(u_4,u_4)\Big\},\Big\{(u_6,u_6),(u_7,u_7),(u_7,u_8),(u_8,u_8)\Big\}\Big)$$

$$\overline{R}_P(X) = \Big(\Big\{(u_1,u_1),(u_2,u_2),(u_1,u_3),(u_3,u_3),(u_4,u_4)\Big\},\Big\{(u_6,u_6),(u_6,u_8),(u_8,u_8)\Big\}\Big).$$

Thus,

$$R_P(X) = \Big(\underline{R}_P(X), \overline{R}_P(X)\Big).$$

is reflexive bipolar soft rough relation.

Definition 32. *Let $R_P(X) \in \mathcal{BSR}_P(X)$. Then, BSR-relation $R_P(X)$ is symmetric, if it is satisfied following conditions:*

(1) $\underline{R}_{P^+}(X)$ *and* $\underline{R}_{P^-}(X)$ *are symmetric relations,*
(2) $\overline{R}_{P^+}(X)$ *and* $\overline{R}_{P^-}(X)$ *are symmetric relations.*

Definition 33. *Let $R_P(X) \in \mathcal{BSR}_P(X)$. Then, BSR-relation $R_P(X)$ is transitive, if it is satisfied following conditions;*

(1) $\underline{R}_{P^+}(X)$ *and* $\underline{R}_{P^-}(X)$ *are transitive relations,*
(2) $\overline{R}_{P^+}(X)$ *and* $\overline{R}_{P^-}(X)$ *are transitive relations.*

Definition 34. *Let $R_P(X) \in \mathcal{BSR}_P(X)$. A BSR-relation $R_P(X)$ is called an equivalence bipolar soft rough relation (EBSR-relation) if it is reflexive, symmetric and transitive.*

Example 8. *Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be an initial universe and $E = \{e_1, e_2, e_3\}$ be a set of parameters. Suppose that positive and negative soft sets are given as follows:*

$$F(e_1) = \{u_1, u_2\}, F(e_2) = \{u_2, u_4, u_5\}, F(e_3) = \{u_3, u_4\}$$

$$G(\neg e_1) = \{u_3, u_5\}, G(\neg e_2) = \{u_6, u_7\}, G(\neg e_3) = \{u_1, u_5\}$$

for $X = \{u_1, u_2, u_7\} \subseteq U$, SPL^+, SPL^-, SPU^+ and SPU^--approximations are as follows:

$$\begin{aligned}\underline{S}_{P^+}(X) &= \{u_1, u_2\}, \underline{S}_{P^-}(X) = \{u_1, u_3, u_5, u_6, u_7\}, \\ \overline{S}_{P^+}(X) &= \{u_1, u_2, u_4, u_5\}, \overline{S}_{P^-}(X) = \{u_3, u_5\}.\end{aligned}$$

If we consider $BSRP_L$-relation $\underline{R}_{P^+}(X)$ and $BSRP_U$-relation $\overline{R}_{P^+}(X)$ given as follows:

$$\begin{aligned}
\underline{R}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2),(u_2,u_1),(u_2,u_2)\big\},\big\{(u_1,u_1),(u_1,u_6),(u_3,u_3),\\
&\quad (u_3,u_5),(u_3,u_7),(u_5,u_3),(u_5,u_5),(u_5,u_7),(u_6,u_1),(u_6,u_6),\\
&\quad (u_7,u_3),(u_7,u_5),(u_7,u_7)\big\}\Big)\\
\overline{R}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2),(u_2,u_1),(u_2,u_2),(u_4,u_4),(u_5,u_5)\big\},\\
&\quad \big\{(u_3,u_3),(u_5,u_5),(u_5,u_3),(u_3,u_5)\big\}\Big),
\end{aligned}$$

then $R_P(X) = (\underline{R}_P(X), \overline{R}_P(X))$ is equivalence BSR-relation.

Definition 35. *Let $R_{1P}(X), R_{2P}(X) \in \mathcal{BSR}_P(X)$. Composition of BSR-relations $R_{1P}(X)$ and $R_{2P}(X)$, denoted by $R_{1P}(X) \circ R_{2P}(X)$, defined as follows:*

$$\begin{aligned}
R_{1P}(X) \circ R_{2P}(X) &= \Big(\big(\underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X), \underline{R_1}_{P^-}(X) \circ \underline{R_2}_{P^-}(X)\big),\\
&\quad \big(\overline{R_1}_{P^+}(X) \circ \overline{R_2}_{P^+}(X), \overline{R_1}_{P^-}(X) \circ \overline{R_2}_{P^-}(X)\big)\Big)
\end{aligned}$$

where

$$\underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X) = \Big\{(x_1,y_1) : (z_1,y_1) \in \underline{R_1}_{P^+}(X) \wedge (x_1,z_1) \in \underline{R_2}_{P^+}(X)\Big\},$$

$$\underline{R_1}_{P^-}(X) \circ \underline{R_2}_{P^-}(X) = \Big\{(x_2,y_2) : (x_2,z_2) \in \underline{R_1}_{P^-}(X) \wedge (z_2,y_2) \in \underline{R_2}_{P^-}(X)\Big\},$$

$$\overline{R_1}_{P^+}(X) \circ \overline{R_2}_{P^+}(X) = \Big\{(x_3,y_3) : (x_3,z_3) \in \overline{R_1}_{P^+}(X) \wedge (z_3,y_3) \in \overline{R_2}_{P^+}(X)\Big\},$$

$$\overline{R_1}_{P^-}(X) \circ \overline{R_2}_{P^-}(X) = \Big\{(x_4,y_4) : (x_4,z_4) \in \overline{R_1}_{P^-}(X) \wedge (z_4,y_4) \in \overline{R_2}_{P^-}(X)\Big\}.$$

Example 9. *Let us consider $R_P(X)$ in Example 8 as $R_{1P}(X)$*

$$\begin{aligned}
\underline{R}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2),(u_2,u_1),(u_2,u_2)\big\},\big\{(u_1,u_1),(u_1,u_6),(u_3,u_3),\\
&\quad (u_3,u_5),(u_3,u_7),(u_5,u_3),(u_5,u_5),(u_5,u_6),(u_5,u_7),(u_6,u_1),(u_6,u_6),\\
&\quad (u_7,u_3),(u_7,u_5),(u_7,u_7)\big\}\Big)\\
\overline{R}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2),(u_2,u_1),(u_2,u_2),(u_4,u_4),(u_5,u_5)\big\},\\
&\quad \big\{(u_3,u_3),(u_5,u_5),(u_5,u_3),(u_3,u_5)\big\}\Big),
\end{aligned}$$

Let us consider $R_{2P}(X)$ given as follow:

$$\begin{aligned}\underline{R_2}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2)\big\},\big\{(u_1,u_6),(u_3,u_5),(u_3,u_7),(u_5,u_3),(u_5,u_5),\\&\quad (u_5,u_7),(u_6,u_6),(u_7,u_3)\big\}\Big)\\ \overline{R_2}_P(X) &= \Big(\big\{(u_1,u_1),(u_1,u_2),(u_1,u_4),(u_2,u_1),(u_2,u_2),(u_2,u_4),(u_4,u_1),\\&\quad (u_4,u_2),(u_4,u_4),(u_5,u_5)\big\},\big\{(u_3,u_5),(u_5,u_3),(u_5,u_5)\big\}\Big).\end{aligned}$$

Then,

$$\begin{aligned}R_{1P}(X)\circ R_{2P}(X) &= \Big((\{(u_1,u_1),(u_1,u_2)\},\{(u_1,u_1),(u_1,u_6),(u_3,u_3),(u_3,u_5),\\&\quad (u_3,u_7),(u_5,u_3),(u_5,u_5),(u_5,u_7),(u_6,u_1),(u_6,u_6),(u_7,u_3),\\&\quad (u_7,u_5),(u_7,u_7)\}\big),\big(\{(u_1,u_1),(u_1,u_2),(u_1,u_4),(u_2,u_1),\\&\quad (u_2,u_2),(u_2,u_4),(u_4,u_1),(u_4,u_2),(u_4,u_4),(u_5,u_5)\},\\&\quad \{(u_3,u_3),(u_3,u_5),(u_5,u_3),(u_5,u_5)\})\Big)\end{aligned}$$

Proposition 2. *Let $R_P(X)$, $R_{1P}(X)$, $R_{2P}(X)$, $R_{3P}(X)$, $K_P(X)$, $K_{1P}(X)$ and $K_{2P}(X) \in \mathcal{BSR}_P(X)$. Then,*

1. $R_{1P}(X)\circ(R_{2P}(X)\circ R_{3P}(X)) = (R_{1P}(X)\circ R_{2P}(X))\circ R_{3P}(X)$.
2. *If* $R_{1P}(X) \ll K_{1P}(X)$ *and* $R_{2P}(X) \ll K_{2P}(X)$, *then* $R_{1P}(X)\circ R_{2P}(X) \ll K_{1P}(X)\circ K_{2P}(X)$.
3. $R_{1P}(X)\circ(R_{2P}(X)\Cup R_{3P}(X)) = (R_{1P}(X)\circ R_{2P}(X))\Cup(R_{1P}(X)\circ R_{3P}(X))$, $R_{1P}(X)\circ(R_{2P}(X)\Cap R_{3P}(X)) = (R_{1P}(X)\circ R_{2P}(X))\Cap(R_{1P}(X)\circ R_{3P}(X))$.
4. $R_{1P}(X) \ll R_{2P}(X)$, *then* $R_{1P}^{-1}(X) \ll R_{2P}^{-1}(X)$.
5. $(R_{1P}(X)\circ R_{2P}(X))^{-1} = (R_{1P}(X))^{-1}\circ(R_{2P}(X))^{-1}$.
6. $R_P(X) \ll R_P(X)\Cup K_P(X)$, $K_P(X) \ll R_P(X)\Cup K_P(X)$.
7. $R_P(X)\Cap K_P(X) \ll R_P(X)$, $R_P(X)\Cap K_P(X) \ll R_P(X)$.

Proof. (1) Let $(x,q) \in \big(\underline{R_1}_{P^+}(X)\circ(\underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X))\big)$. Then, for some $z \in \underline{S}_{P^+}(X)$ $(x,z) \in \underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X)$ and $(z,q) \in \underline{R_1}_{P^+}(X)$. Since $(x,z) \in \underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X)$, for some $y \in \underline{S}_{P^+}(X)$ $(x,y) \in \underline{R_3}_{P^+}(X)$ and $(y,z) \in \underline{R_2}_{P^+}(X)$. Thus $(y,q) \in \underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)$ and $(x,y) \in \underline{R_3}_{P^+}(X)$. Therefore $(x,q) \in \big(\underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)\big)\circ\underline{R_3}_{P^+}(X)$ and

$$\big(\underline{R_1}_{P^+}(X)\circ(\underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X))\big) \subseteq \big(\underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)\big)\circ\underline{R_3}_{P^+}(X). \quad (3.1)$$

Conversely, Let $(x,q) \in \big((\underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X))\circ\underline{R_3}_{P^+}(X)\big)$.

Then, for some $y \in \underline{S}_{P^+}(X)$ $(x,y) \in \underline{R_3}_{P^+}(X)$ and $(y,q) \in \underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)$. Since $(y,q) \in \underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)$, for some $z \in \underline{S}_{P^+}(X)$ $(y,z) \in \underline{R_2}_{P^+}(X)$ and $(z,q) \in \underline{R_1}_{P^+}(X)$. Thus $(x,z) \in \underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X)$ and $(z,q) \in \underline{R_1}_{P^+}(X)$. Therefore $(x,q) \in \underline{R_1}_{P^+}(X)\circ(\underline{R_2}_{P^+}(X)\circ\underline{R_3}_{P^+}(X))$

and

$$(\underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X)) \circ \underline{R_3}_{P^+}(X) \subseteq \underline{R_1}_{P^+}(X) \circ (\underline{R_2}_{P^+}(X) \circ \underline{R_3}_{P^+}(X)) \quad (3.2)$$

From Eqs. (**??**) and (3.2) $R_{1P^+}(X) \circ (R_{2P^+}(X) \circ R_{3P^+}(X)) = (R_{1P^+}(X) \circ R_{2P^+}(X)) \circ R_{3P^+}(X)$.

Similarly, we can shown that

$$\underline{R_1}_{P^-}(X) \circ (\underline{R_2}_{P^-}(X) \circ \underline{R_3}_{P^-}(X)) = (\underline{R_1}_{P^-}(X) \circ \underline{R_2}_{P^-}(X)) \circ \underline{R_3}_{P^-}(X)$$

$$\overline{R_1}_{P^+}(X) \circ (\overline{R_2}_{P^+}(X) \circ \overline{R_3}_{P^+}(X)) = (\overline{R_1}_{P^+}(X) \circ \overline{R_2}_{P^+}(X)) \circ \overline{R_3}_{P^+}(X)$$

and

$$\overline{R_1}_{P^-}(X) \circ (\overline{R_2}_{P^-}(X) \circ \overline{R_3}_{P^-}(X)) = (\overline{R_1}_{P^-}(X) \circ \overline{R_2}_{P^-}(X)) \circ \overline{R_3}_{P^-}(X).$$

(2) Let $R_{1P}(X) \subseteq K_{1P}(X)$ and $R_{2P}(X) \subseteq K_{2P}(X)$. Then, from Definition 29, $\underline{R_1}_{P^+}(X) \subseteq \underline{K_1}_{P^+}(X)$, $\underline{R_1}_{P^-}(X) \supseteq \underline{K_1}_{P^-}(X)$, $\overline{R_1}_{P^+}(X) \subseteq \overline{K_1}_{P^+}(X)$, $\overline{R_1}_{P^-}(X) \supseteq \overline{K_1}_{P^-}(X)$ and $\underline{R_2}_{P^+}(X) \subseteq \underline{K_2}_{P^+}(X)$, $\underline{R_2}_{P^-}(X) \supseteq \underline{K_2}_{P^-}(X)$, $\overline{R_2}_{P^+}(X) \subseteq \overline{K_2}_{P^+}(X)$, $\overline{R_2}_{P^-}(X) \supseteq \overline{K_2}_{P^-}(X)$. Let $(x,z) \in \underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X)$. Then, for some (x,y), (x,z) $(x,y) \in \underline{R_2}_{P^+}(X)$ and $(y,z) \in \underline{R_1}_{P^+}(X)$. Since $\underline{R_1}_{P^+}(X) \subseteq \underline{K_1}_{P^+}(X)$ and $\underline{R_2}_{P^+}(X) \subseteq \underline{K_2}_{P^+}(X)$, $(x,y) \in \underline{K_2}_{P^+}(X)$ and $(y,z) \in \underline{K_1}_{P^+}(X)$. Therefore $(x,z) \in \underline{K_1}_{P^+}(X) \circ \underline{K_2}_{P^+}(X)$ and so $\underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X) \subseteq \underline{K_1}_{P^+}(X) \circ \underline{K_2}_{P^+}(X)$. Let $(x,z) \in \underline{K_1}_{P^-}(X) \circ \underline{K_2}_{P^-}(X)$. Then, for some (x,y), (x,z) $(x,y) \in \underline{K_2}_{P^-}(X)$ and $(y,z) \in \underline{K_1}_{P^-}(X)$. Since $\underline{K_1}_{P^-}(X) \subseteq \underline{R_1}_{P^-}(X)$ and $\underline{K_2}_{P^-}(X) \subseteq \underline{R_2}_{P^-}(X)$, $(x,y) \in \underline{R_2}_{P^-}(X)$ and $(y,z) \in \underline{R_1}_{P^-}(X)$. Therefore $(x,z) \in \underline{R_1}_{P^-}(X) \circ \underline{R_2}_{P^-}(X)$ and so $\underline{K_1}_{P^-}(X) \circ \underline{K_2}_{P^-}(X) \subseteq \underline{K_1}_{P^-}(X) \circ \underline{K_2}_{P^-}(X)$.

Similarly, it can can be shown that $\overline{R_1}_{P^+}(X) \circ \overline{R_2}_{P^+}(X) \subseteq \overline{K_1}_{P^+}(X) \circ \overline{K_2}_{P^+}(X)$. and $\overline{K_1}_{P^-}(X) \circ \overline{K_2}_{P^-}(X) \subseteq \overline{K_1}_{P^-}(X) \circ \overline{K_2}_{P^-}(X)$. Then,

$$R_{1P}(X) \circ R_{2P}(X) \ll K_{1P}(X) \circ K_{2P}(X).$$

(3) From Definition 30 and Definition 35, we known that

$$\begin{aligned} R_{1P}(X) \circ (R_{2P}(X) \Cup R_{3P}(X)) \quad &= \quad \Big(\underline{R_1}_P(X) \circ (\underline{R_2}_P(X) \sqcup \underline{R_3}_P(X)), \underline{R_1}_P(X) \circ \\ &\qquad (\overline{R_2}_P(X) \sqcup \overline{R_3}_P(X))\Big) \end{aligned}$$

and

$$\begin{aligned}
\underline{R_1}_P(X)\circ(\underline{R_2}_P(X)\sqcup\underline{R_3}_P(X)) &= \Big((\underline{R_1}_{P^+}(X),\underline{R_1}_{P^-}(X))\circ[(\underline{R_2}_{P^+}(X)\cup\underline{R_3}_{P^+}(X)),\\
&\quad (\underline{R_2}_{P^-}(X)\cap\underline{R_3}_{P^-}(X))]\\
\overline{R_1}_P(X)\circ(\overline{R_2}_P(X)\sqcup\overline{R_3}_P(X)) &= \Big((\overline{R_1}_{P^+}(X),\overline{R_1}_{P^-}(X))\circ[(\overline{R_2}_{P^+}(X)\cup\overline{R_3}_{P^+}(X)),\\
&\quad (\overline{R_2}_{P^-}(X)\cap\overline{R_3}_{P^-}(X))]
\end{aligned}$$

Since $\underline{R_i}_{P^+}(X)$, $\underline{R_i}_{P^-}(X)$, $\overline{R_i}_{P^+}(X)$ and $\overline{R_i}_{P^-}(X)$ $(i=1,2,3)$ are classical relations,

$$\begin{aligned}
&(\underline{R_1}_{P^+}(X),\underline{R_1}_{P^-}(X))\circ[(\underline{R_2}_{P^+}(X)\cup\underline{R_3}_{P^+}(X)),(\underline{R_2}_{P^-}(X)\cap\underline{R_3}_{P^-}(X))] \qquad (3.3)\\
=\;&\Big((\underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X))\cup(\underline{R_1}_{P^+}(X)\circ\underline{R_3}_{P^+}(X)),\\
&(\overline{R_1}_{P^+}(X)\circ\overline{R_2}_{P^+}(X))\cap(\overline{R_1}_{P^+}(X)\circ\overline{R_3}_{P^+}(X))\Big)\\
=\;&\Big(\big((\underline{R_1}_{P^+}(X)\circ\underline{R_2}_{P^+}(X)),(\underline{R_1}_{P^-}(X)\circ\underline{R_2}_{P^-}(X))\big)\sqcup(\underline{R_1}_{P^+}(X)\circ\underline{R_3}_{P^+}(X)),\\
&(\underline{R_1}_{P^-}(X)\circ\underline{R_3}_{P^-}(X)))\Big)\\
=\;&\Big((\underline{R_1}_P(X)\circ\underline{R_2}_P(X))\sqcup(\underline{R_1}_P(X)\circ\underline{R_3}_P(X))\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(\overline{R_1}_{P^+}(X),\overline{R_1}_{P^-}(X))\circ[(\overline{R_2}_{P^+}(X)\cup\overline{R_3}_{P^+}(X)),(\overline{R_2}_{P^-}(X)\cap\overline{R_3}_{P^-}(X))] \qquad (3.4)\\
=\;&\Big((\overline{R_1}_{P^+}(X)\circ\overline{R_2}_{P^+}(X))\cup(\overline{R_1}_{P^+}(X)\circ\overline{R_3}_{P^+}(X)),\\
&(\overline{R_1}_{P^+}(X)\circ\overline{R_2}_{P^+}(X))\cap(\overline{R_1}_{P^+}(X)\circ\overline{R_3}_{P^+}(X))\Big)\\
=\;&\Big(\big((\overline{R_1}_{P^+}(X)\circ\overline{R_2}_{P^+}(X)),(\overline{R_1}_{P^-}(X)\circ\overline{R_2}_{P^-}(X))\big)\sqcup(\overline{R_1}_{P^+}(X)\circ\overline{R_3}_{P^+}(X)),\\
&(\overline{R_1}_{P^-}(X)\circ\underline{R_3}_{P^-}(X)))\Big)\\
=\;&\Big((\overline{R_1}_P(X)\circ\overline{R_2}_P(X))\sqcup(\overline{R_1}_P(X)\circ\overline{R_3}_P(X))\Big).
\end{aligned}$$

Using Eqs. 3.3 and 3.4, we can write following equations

$$\begin{aligned}
&\Big(\underline{R_1}_P(X)\circ(\underline{R_2}_P(X)\sqcup\underline{R_3}_P(X)),\underline{R_1}_P(X)\circ(\overline{R_2}_P(X)\sqcup\overline{R_3}_P(X))\Big)\\
=\;&\big((\underline{R_1}_P(X)\circ(\underline{R_2}_P(X)),(\overline{R_1}_P(X)\circ(\overline{R_2}_P(X))\big)\\
\Cup\;&\big((\underline{R_1}_P(X)\circ(\underline{R_3}_P(X)),(\overline{R_1}_P(X)\circ(\overline{R_3}_P(X))\big)\\
=\;&(R_{1P}(X)\circ R_{2P}(X))\Cup(R_{1P}(X)\circ R_{3P}(X))
\end{aligned}$$

and so we get that $R_{1P}(X) \circ (R_{2P}(X) \Cup R_{3P}(X)) = (R_{1P}(X) \circ R_{2P}(X)) \Cup (R_{1P}(X) \circ R_{3P}(X))$.

Proof of $R_{1P}(X) \circ (R_{2P}(X) \Cap R_{3P}(X)) = (R_{1P}(X) \circ R_{2P}(X)) \Cap (R_{1P}(X) \circ R_{3P}(X))$ can be made in similar way.

(4) Let $R_{1P}(X) \ll R_{2P}(X)$. From Definition 29, we know that $\underline{R_1}_{P^+}(X) \subseteq \underline{R_2}_{P^+}(X)$, $\underline{R_1}_{P^-}(X) \supseteq \underline{R_2}_{P^-}(X)$, $\overline{R_1}_{P^+}(X) \subseteq \overline{R_2}_{P^+}(X)$ and $\overline{R_1}_{P^+}(X) \supseteq \overline{R_2}_{P^-}(X)$. Since definitions of $\underline{R_1}_{P^+}(X)$ and $\underline{R_2}_{P^+}(X)$, $\underline{R_1}_{P^+}^{-1}(X) \subseteq \underline{R_2}_{P^+}^{-1}(X)$. Similarly $\underline{R_1}_{P^-}^{-1}(X) \supseteq \underline{R_2}_{P^-}^{-1}(X)$, $\overline{R_1}_{P^+}^{-1}(X) \subseteq \overline{R_2}_{P^+}^{-1}(X)$ and $\overline{R_1}_{P^+}^{-1}(X) \supseteq \overline{R_2}_{P^-}^{-1}(X)$. We conclude that $R_{1P}^{-1}(X) \ll R_{2P}^{-1}(X)$.

(5)

$$\begin{aligned}
(x,y) \in (\underline{R_2}_{P^+}(X))^{-1} \circ (\underline{R_1}_{P^+}(X))^{-1} \quad &\Leftrightarrow \quad (y,z) \in (\underline{R_2}_{P^+}(X))^{-1} \\
&\qquad \text{and } (x,y) \in (\underline{R_1}_{P^+}(X))^{-1} \\
&\qquad \text{for some } (y,z) \in \underline{S}_{P^+}(X) \times \underline{S}_{P^+}(X) \\
&\Leftrightarrow \quad (z,y) \in \underline{R_2}_{P^+}(X) \text{ and } (y,x) \in \underline{R_1}_{P^+}(X) \\
&\Leftrightarrow \quad (z,x) \in \underline{R_2}_{P^+}(X) \circ \underline{R_1}_{P^+}(X) \\
&\Leftrightarrow \quad (x,z) \in (\underline{R_1}_{P^+}(X) \circ \underline{R_2}_{P^+}(X))^{-1}
\end{aligned}$$

$$\begin{aligned}
(x,y) \in (\underline{R_2}_{P^-}(X))^{-1} \circ (\underline{R_1}_{P^-}(X))^{-1} \quad &\Leftrightarrow \quad (y,z) \in (\underline{R_2}_{P^-}(X))^{-1} \\
&\qquad \text{and } (x,y) \in (\underline{R_1}_{P^-}(X))^{-1} \\
&\qquad \text{for some } (y,z) \in \underline{S}_{P^-}(X) \times \underline{S}_{P^-}(X) \\
&\Leftrightarrow \quad (z,y) \in \underline{R_2}_{P^-}(X) \text{ and } (y,x) \in \underline{R_1}_{P^-}(X) \\
&\Leftrightarrow \quad (z,x) \in \underline{R_2}_{P^-}(X) \circ \underline{R_1}_{P^-}(X) \\
&\Leftrightarrow \quad (x,z) \in (\underline{R_1}_{P^-}(X) \circ \underline{R_2}_{P^-}(X))^{-1}.
\end{aligned}$$

Similarly, it can be shown that

$$(\overline{R_1}_{P^+}(X) \circ \overline{R_2}_{P^+}(X))^{-1} = (\overline{R_2}_{P^+}(X))^{-1} \circ (\overline{R_1}_{P^+}(X))^{-1}$$

and

$$(\overline{R_2}_{P^-}(X))^{-1} \circ (\overline{R_1}_{P^-}(X))^{-1} = (\overline{R_2}_{P^-}(X))^{-1} \circ (\overline{R_1}_{P^-}(X))^{-1}.$$

The proofs of (6) and (7) are obvious. □

Definition 36. *Let $R_P(X)$ be a EBSR-relation, then lower and upper bipolar equivalence class of $u_i \in U$ are defined as follows:*

$$[\underline{u}_i] = \Big(\{u_j : u_i \underline{R}_{P^+}(X) u_j\}, \{u_k : u_i \underline{R}_{P^-}(X) u_k\}\Big)$$
$$[\overline{u}_i] = \Big(\{u_j : u_i \overline{R}_{P^+}(X) u_j\}, \{u_k : u_i \overline{R}_{P^-}(X) u_k\}\Big)$$

Here, for the sake of shortness, $\{u_j : u_i\underline{R}_{P^+}(X)u_j\}, \{u_k : u_i\underline{R}_{P^-}(X)u_k\}, \{u_j : u_i\overline{R}_{P^+}(X)u_j\}$ and $\{u_k : u_i\overline{R}_{P^-}(X)u_k\}$ will be indicated $[\underline{u}_i]^+, [\underline{u}_i]^-, [\overline{u}_i]^+$ and $[\overline{u}_i]^-$, respectively.

Remark 1. *$\Big\{x_i : x_i = \{u_j : [\underline{u}_i] = [\underline{u}_j]\}, i = 1, 2, ..., |U|\Big\}$ and $\Big\{y_i : y_i = \{u_j : [\overline{u}_i] = [\overline{u}_j]\}, i = 1, 2, ..., |U|\Big\}$ are called partitions of U under EBSR-relation $R_P(X)$ and denoted by $U/\underline{R}_P(X)$ and $U/\overline{R}_P(X)$, respectively.*

Example 10. *Let us consider EBSR-relation in Example 8. Then, for all $u_i \in U$, bipolar equivalence classes;*

$$\begin{array}{lcllcl}
[\underline{u}_1] & = & \Big(\{u_1, u_2\}, \{u_1, u_6\}\Big), & [\overline{u}_1] & = & \Big(\{u_1, u_2\}, \emptyset\Big) \\
[\underline{u}_2] & = & \Big(\{u_1, u_2\}, \emptyset\Big), & [\overline{u}_2] & = & \Big(\{u_1, u_2\}, \emptyset\Big) \\
[\underline{u}_3] & = & \Big(\emptyset, \{u_3, u_5, u_7\}\Big), & [\overline{u}_3] & = & \Big(\emptyset, \{u_3, u_5\}\Big) \\
[\underline{u}_4] & = & \Big(\emptyset, \emptyset\Big) & [\overline{u}_4] & = & \Big(\{u_4\}, \emptyset\Big) \\
[\underline{u}_5] & = & \Big(\emptyset, \{u_3, u_5, u_7\}\Big), & [\overline{u}_5] & = & \Big(\{u_5\}, \{u_3, u_5\}\Big) \\
[\underline{u}_6] & = & \Big(\emptyset, \{u_1, u_6\}\Big), & [\overline{u}_6] & = & \Big(\emptyset, \emptyset\Big) \\
[\underline{u}_7] & = & \Big(\emptyset, \{u_3, u_5, u_7\}\Big) & [\overline{u}_7] & = & \Big(\emptyset, \emptyset\Big)
\end{array}$$

Note that, $\bigcup_{u_i \in U}[\underline{u}_i]^+ = \underline{S}_{P^+}(X), \bigcup_{u_i \in U}[\underline{u}_i]^- = \underline{S}_{P^-}(X), \bigcup_{u_i \in U}[\overline{u}_i]^+ = \overline{S}_{P^+}(X)$ and $\bigcup_{u_i \in U}[\overline{u}_i]^- = \overline{S}_{P^-}(X)$.

Also,

$$\begin{array}{lcl}
U/\underline{R}_P(X) & = & \Big\{\{u_1\}, \{u_2\}, \{u_3, u_5, u_7\}, \{u_4\}, \{u_6\}\Big\} \\
U/\overline{R}_P(X) & = & \Big\{\{u_1, u_2\}, \{u_3\}, \{u_4\}, \{u_5\}, \{u_6, u_7\}\Big\}.
\end{array}$$

Proposition 3. *Let $R_P(X)$ be EBSR-relation such that $\underline{R}_P(X) = \overline{R}_P(X)$. Then, $U/\underline{R}_P(X) = U/\overline{R}_P(X)$.*

Proof. The proof is clear. □

Definition 37. *Let $(F, G, E) \in \mathcal{BS}_U$, $P = (U, (F, G, E))$ be a BSA-space, $X \subseteq U$ and $K = (\underline{BS}_P(X), \overline{BS}_P(X))$ be a bipolar soft rough set. Partition of bipolar soft rough set K is defined by partitions soft P-lower(upper) positive and soft P-lower(upper) negative, respectively, as follows:*

$$\underline{BS}_P(X) = \Big(\mathfrak{P}(\underline{S}_{P^+}(X)), \mathfrak{P}(\underline{S}_{P^-}(X))\Big)$$

$$\overline{BS}_P(X) = \Big(\mathfrak{P}(\overline{S}_{P^+}(X)), \mathfrak{P}(\overline{S}_{P^-}(X))\Big)$$

where $\mathfrak{P}(\underline{S}_{P^+}(X)), \mathfrak{P}(\underline{S}_{P^-}(X))$, $\mathfrak{P}(\overline{S}_{P^+}(X))$ *and* $\mathfrak{P}(\overline{S}_{P^-}(X))$ *are partitions of* $\underline{S}_{P^+}(X)$, $\underline{S}_{P^-}(X)$, $\overline{S}_{P^+}(X)$ *and* $\overline{S}_{P^-}(X)$, *respectively.*

Corollary 2. *Let* $K = (\underline{BS}_P(X), \overline{BS}_P(X))$ *be a bipolar soft rough set and* $R_P(X)$ *be a* $EBSR-$*relation over bipolar soft rough set* K. *Then, collection of* $[\underline{u}_i]^+$, *such that,* $[\underline{u}_i]^+ \neq \emptyset$ *is a partition of* $\underline{S}_{P^+}(X)$. *Similarly, collection of each of* $[\underline{u}_i]^-, [\overline{u}_i]^+$ *and* $[\overline{u}_i]^-$ *such that* $[\underline{u}_i]^- \neq \emptyset, [\overline{u}_i]^+ \neq \emptyset$ *and* $[\overline{u}_i]^- \neq \emptyset$, *are partition of* $\underline{S}_{P^-}(X), \overline{S}_{P^+}(X)$ *and* $\overline{S}_{P^-}(X)$, *respectively.*

Example 11. *Let us consider bipolar equivalence classes in Example 10. Then, a partition of bipolar soft rough set given in Example 8 by EBSR-relation* $R_P(X)$ *is as follows:*

$$\Big(\big(\{\{u_1,u_2\}\},\{\{u_1,u_6\},\{u_3,u_5,u_7\}\}\big),\big(\{\{u_1,u_2\},\{u_4\},\{u_5\}\},\{\{u_3,u_5\}\}\big)\Big)$$

4. Conclusion

Throughout the paper we introduced some new concepts such as bipolar soft $P-$lower(upper) relations, bipolar soft rough relation, equivalence bipolar soft rough relation, and gave examples related to the concepts. Also we obtained some properties related to these concepts. Next, we will define concept of the bipolar soft rough functions as a generalization of soft functions and investigate their properties.

References

[1] Abdullah, S., Aslam, M. and Ullah, K., Bipolar fuzzy soft sets and its applications in decision making problem, *Journal of Intelligent and Fuzzy Systems* (2014), 27(2), 729-742.

[2] Aktaş H. and Çağman, N., Soft sets and soft groups, *Information Sciences* (2007), 177, 2726-2735.

[3] Ali, M. I., A note on soft sets, rough soft sets and fuzzy soft sets, *Applied Soft Computing Journal* (2011), 11(4), 3329-3332.

[4] Ali, M.I., Feng, F., Liu, X., Min, W.K. and Shabir, M., On some new operations in soft set theory, *Computers and Mathematics with Application* (2009), 57, 1547-1553.

[5] Atagün, A.O. and Aygün, E., Groups of soft sets, *Journal of Intelligent and Fuzzy Systems* (2016), 30(2), 729-733.

[6] Atanassov, K.T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* (1986), 20, 87-96.

[7] Babitha, K.V. and Sunil, J.J., Soft set relations and functions, *Computers and Mathematics with Applications* (2010), 60, 1840-1849.

[8] Çağman, N. and Enginoğlu, S., Soft set theory and uni-int decision making, *European Journal of Operational Research* (2010), 207, 848-855.

[9] Çağman, N. and Enginoğlu, S., Fuzzy soft set theory and its applications, *Iranian journal of fuzzy systems* (2011), 8(3), 137-147.

[10] Çağman, N. and Karataş, S., Intuitionistic fuzzy soft set theory and its decision making, *Journal of Intelligent and Fuzzy Systems* (2013), 24, 829-836.

[11] Çağman, N., Contributions to the theory of soft sets, *Journal of New Results in Science* (2014), 4, 33-41.

[12] Dubois, D., Prade, H., Rough fuzzy sets and fuzzy rough sets, *International Journal of General Systems* (1990), 17, 191-209.

[13] Feng F. and Liu, X., Soft rough sets with applications to demand analysis, *Intelligent Systems and Applications* (2009), DOI:10.1109/IWISA.2009.5073114.

[14] Feng, F., Li, C., Davvaz, B. and Ali, M. I., Soft sets combined with fuzzy sets and rough sets: a tentative approach, *Soft Computing* (2010), 14(9), 899-911.

[15] Feng, F., Liu, X., Leoreanu-Fotea, V. and Jun, Y. B., Soft sets and soft rough sets, *Information Sciences* (2011), 181(6), 1125-1137.

[16] Feng, F., Soft rough sets applied to multicriteria group decision making, *Annals of Fuzzy Mathematics and Informatics* (2011), 2(1), 69-80.

[17] Gau, W.L., Buehrer, D.J., Vague sets, *IEEE Transactions on Systems, Man and Cybernetics* (1993), 23(2), 610-614.

[18] Gong, K., Xiao, Z. and Zhang, X., The bijective soft set with its operations, *Computers and Mathematics with Applications* (2010), 60, 2270-2278.

[19] Herawan, T. and Deris, M.M., A Direct Proof of Every Rough Set is a Soft Set, *Third Asia International Conference on Modelling and Simulation* (2009), DOI 10.1109/AMS.2009.148.

[20] Ibrahim, A. M., Dauda, M. K. and Singh, D., Composition of Soft Set Relations and Construction of Transitive, *Mathematical Theory and Modeling* (2012), 2(7), 98-107.

[21] Jiang, Y., Tang, Y., Chen, Q., Liu, H. and Tang, J., Intervalvalued intuitionistic fuzzy soft sets and their properties, *Computers and Mathematics with Applications* (2010), 60, 906-918.

[22] Karaaslan, F. and Karataş, S., A new approach bipolar soft sets and its applications, *Discrete Mathematics, Algorithm and Applications* (2015), 7(4), 1550054.

[23] Karaaslan, F., Soft classes and soft rough classes with application in decision making, *Mathematical Problem in Engineering*, In press.

[24] Karaaslan, F. and Çağman, N., Bipolar soft rough sets and their applications in decision making, Submitted.

[25] Kima, Y. K. and Minb, W. K., Full soft sets and full soft decision systems, *Journal of Intelligent and Fuzzy Systems*, DOI:10.3233/IFS-130783.

[26] Li, Z., Qin, B. and Cai, Z., Soft Rough Approximation Operators and Related Results, *Journal of Applied Mathematics*, doi.org/10.1155/2013/241485.

[27] Maji, P.K., Biswas, R. and Roy, A.R., Fuzzy soft sets, *Journal of Fuzzy Mathematics* (2001), 9(3),589-602.

[28] Maji, P.K., Biswas, R. and Roy, A.R., Intuitionistic fuzzy soft sets, *Journal of Fuzzy Mathematics* (2001), 9(3), 677-692.

[29] Maji, P.K., Biswas, R. and Roy, A.R., Soft set theory, *Computers and Mathematics with Applications* (2003), 45, 555-562.

[30] Maji, P.K., Roy, A.R. and Biswas, R., On intuitionistic fuzzy soft sets, *Computers and Mathematics with Applications*(2004), 12(3), 669-683.

[31] Majumdar, P. and Samanta, S.K., Generalised fuzzy soft sets, *Computers and Mathematics with Applications* (2010), 59, 1425-1432.

[32] Meng, D., Zhang, X. H. and Qin, K. Y., Soft rough fuzzy sets and soft fuzzy rough sets, *Computers and Mathematics With Applications* (2011), 62(12), 4635-4645.

[33] Molodtsov, D., Soft set theory first results, Computers and Mathematics with Applications (1999), 37, 19-31.

[34] Naz, M. and Shabir, M., On bipolar fuzzy soft sets, their algebaraic structures and applications, *Journal of Intelligent and Fuzzy Systems* (2014), 26(4) 1645-1656.

[35] Park, J.H., Kima, O.H. and Kwun, Y. C., Some properties of equivalence soft set relations, *Computers and Mathematics with Applications* (2012), 63, 1079-1088.

[36] Pawlak, Z., Rough sets, *International Journal of Computer and Information Sciences* (1982), 11(5), 341-356.

[37] Pawlak, Z., Skowron, A. Rudiments of rough sets, Information Sciences (2007), 177, 3–27.

[38] Qin, K., Liu, Q. and Xu, Y., Redefined soft relations and soft functions, *International journal of Computational Intelligence Systems* (2015), 8(5), 819-828.

[39] Qin, K.Y. and Hong, Z.Y. On soft equality, *Journal of Computational and Applied Mathematics* (2010), 234, 1347-1355.

[40] Sezgin, A. and Atagün, A.O., Soft groups and normalistic soft groups, *Computers and Mathematics with Applications* (2011), 62(2), 685-698.
[41] Sezgin, A. and Atagün, A.O., On operations of soft sets, *Computers and Mathematics with Applications* (2011), 61, 1457-1467.
[42] Shabir, M. and Naz, M., On bipolar soft sets, arXiv:1303.1344v1 [math.LO] (2013)
[43] Shabir, M., Ali, M.I. and Shaheen, T., Another approach to soft rough sets, *Knowledge-Based Systems* (2013), 61, 72-80.
[44] Sun, B. and Ma, W., Soft fuzzy rough sets and its application in decision making, *Artificial Intelligence Review* (2014), 41(1), 67-68.
[45] Xiao, Z., Gong, K., Xia, S. and Zou, Y., Exclusive disjunctive soft sets, *Computers and Mathematics with Applications* (2010), 59, 2128-2137.
[46] Xu, W., Ma, J., Wang, S. and Hao, G., Vague soft sets and their properties, *Computers and Mathematics with Applications* (2010), 59, 787-794.
[47] Yang, X.B., Lin, T.Y., Yang, J.Y., Li, Y. and Yu, D.J., Combination of interval-valued fuzzy set and soft set, *Computers and Mathematics with Applications* (2009), 58, 521-527.
[48] Yang, H.L, and Guoa, Z.L., Kernels and closures of soft set relations, and soft set relation mappings, *Computers and Mathematics with Applications* (2011), 61 651-662.
[49] Zadeh, L. A., Fuzzy sets, *Information and Control* (1965), 8, 338-353.
[50] Zhang, Z., A rough set approach to intuitionistic fuzzy soft set based decision making, *Applied Mathematical Modelling*, (2011), DOI: 10.1016/j.apm.2011.11.071.
[51] Zhang, Z., The Parameter Reduction of Fuzzy Soft Sets Based on Soft Fuzzy Rough Sets, *Advances in Fuzzy Systems* (2013), doi.org/10.1155/2013/197435, .
[52] Zhang, W.-R., Bipolar fuzzy set and relations: a computational framework for cognitive modeling and multiagent decision analysis, *Proceeding of IEEE Conference* (1994), 305-309.
[53] Zhang, W.-R., Bipolar fuzzy sets, *Proceeding of FUZZY-IEEE* (1994), 835-840.

Current address: Department of Mathematics, Faculty of Sciences, Çankırı Karatekin University, 18100, Çankırı, Turkey
E-mail address: fkaraaslan@karatekin.edu.tr