Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 65, Number 2, Pages 23–36 (2016)
DOI: 10.1501/Commua1_0000000756
ISSN 1303–5991

INVERSE NODAL PROBLEM FOR $p-$LAPLACIAN DIFFUSION EQUATION WITH POLYNOMIALLY DEPENDENT SPECTRAL PARAMETER

TUBA GULSEN AND EMRAH YILMAZ

Abstract. In this study, solution of inverse nodal problem for one-dimensional p-Laplacian diffusion equation is extended to the case that boundary condition depends on polynomial eigenparameter. To find the spectral datas as eigenvalues and nodal parameters of this problem, we used a modified Prüfer substitution. Then, reconstruction formula of the potential function is also given by using nodal lenghts. Furthermore, this method is similar to used in [1], our results are more general.

(Dedicated to Prof. E. S. Panakhov on his 60-th birthday)

1. Introduction

Let us consider following $p-$Laplacian diffusion eigenvalue problem [1]

$$-\left(y'^{(p-1)}\right)' = (p-1)\left(\lambda^2 - q_m(x) - 2\lambda r_m(x)\right) y^{(p-1)},\ 0 \leq x \leq 1, \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} y(0) = 0, y'(0) = 1, \\ y'(1,\lambda) + f(\lambda) y(1,\lambda) = 0, \end{aligned} \tag{1.2}$$

where $p > 1$ is a constant, [2]

$$f(\lambda) = a_1\lambda + a_2\lambda^2 + ... + a_m\lambda^m, a_i \in \mathbb{R}, a_m \neq 0, m \in \mathbb{Z}^+, \tag{1.3}$$

λ is a spectral parameter and $y^{(p-1)} = |y|^{(p-2)} y$. Throughout this study, we suppose that $q_m(x) \in L^2(0,1)$ and $r_m(x) \in W_2^1(0,1)$ are real-valued functions defined in the interval $0 \leq x \leq 1$ for all $m \in \mathbb{Z}^+$. Equation (1.1) becomes following well-known diffusion equation (or quadratic pencil of differential pencil)

$$-y'' + [q_m + 2\lambda r_m] y = \lambda^2 y, \tag{1.4}$$

Received by the editors: February 09, 2016, Accepted: April 27, 2016.
2010 *Mathematics Subject Classification.* 34A55, 34L05, 34L20.
Key words and phrases. Inverse Nodal Problem, Prüfer Substitution, Diffusion equation.

for $p = 2$. Equation (1.4) is extremely important for both classical and quantum mechanics. For instance, such problems arise in solving the Klein-Gordon equations, which describe the motion of massless particles such as photons. Diffusion equations are also used for modelling vibrations of mechanical systems in viscous media (see [3]). We note that in this type of problem the spectral parameter λ is related to the energy of the system, and this motivates the terminology 'energy-dependent' used for the spectral problem of the form (1.4). Inverse problems of quadratic pencil have been studied by numerous authors (see [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]).

Inverse spectral problem consists in recovering differential equation from its spectral parameters like eigenvalues, norming constants and nodal points (zeros of eigenfunctions). These type problems divide into two parts as inverse eigenvalue problem and inverse nodal problem. They play important role and also have many applications in applied mathematics. Inverse nodal problem has been firstly studied by McLaughlin in 1988. She showed that the knowledge of a dense subset of nodal points is sufficient to determine the potential function of Sturm-Liouville problem up to a constant [21]. Also, some numerical results about this problem were given in [22]. Nowadays, many authors have given some interesting results about inverse nodal problems for different type operators (see [23], [24], [25], [26], [27]).

In this study, we concern ourselves with the inverse nodal problem for $p-$Laplacian diffusion equation with boundary condition polynomially dependent on spectral parameter. As far as we know, this problem has not been considered before. Furthermore, we give asymptotics of eigenparameters and reconstructing formula for potential function. Note that inverse eigenvalue problems for different $p-$Laplacian operators have been studied by several authors (see [28], [29], [30], [31], [32], [33], [34], [35], [36]).

The zero set $X_n = \left\{x_{j,m}^n\right\}_{j=1}^{n-1}$ of the eigenfunction $y_{n,m}(x,\lambda)$ corresponding to $\lambda_{n,m}$ is called the set of nodal points where $0 = x_{0,m}^{(n)} < x_{1,m}^{(n)} < ... < x_{n-1,m}^{(n)} < x_{n,m}^{(n)} = 1$ for all $m \in \mathbb{Z}^+$. And, $l_{j,m}^n = x_{j+1,m}^n - x_{j,m}^n$ is referred to the nodal length of $y_{n,m}$. The eigenfunction $y_{n,m}(x)$ has exactly $n-1$ nodal points in $(0,1)$.

Firstly, we need to introduce generalized sine function S_p which is the solution of the initial value problem

$$-\left(S_p'^{(p-1)}\right)' = (p-1)S_p^{(p-1)}, \tag{1.4}$$

$$S_p(0) = 0,\ S_p'(0) = 1.$$

S_p and S_p' are periodic functions satisfying the identity

$$|S_p(x)|^p + \left|S_p'(x)\right|^p = 1,$$

for arbitrary $x \in \mathbb{R}$. These functions are $p-$analogues of classical sine and cosine functions and are known as generalized sine and cosine functions. It is well known that

$$\hat{\pi} = \int_0^1 \frac{2}{(1-t^p)^{\frac{1}{p}}} dt = \frac{2\pi}{p \sin\left(\frac{\pi}{p}\right)},$$

is the first zero of S_p in positive axis (See [28], [29]). Note that following lemma is crucial in our results.

Lemma 1.1. [28]

a) For $S_p' \neq 0$,

$$\left(S_p'\right)' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p.$$

b)

$$\left(S_p S_p'^{(p-1)}\right)' = \left|S_p'\right|^p - (p-1)\left|S_p^p\right| = 1 - p\left|S_p\right|^p = (1-p) + p\left|S_p'\right|^p.$$

Using $S_p(x)$ and $S_p'(x)$, the generalized tangent function $T_p(x)$ can be defined by [28]

$$T_p(x) = \frac{S_p(x)}{S_p'(x)}, \text{ for } x \neq \left(k + \frac{1}{2}\right)\hat{\pi}.$$

This study is organized as follows: In Section 2, we give some asymptotic formulas for eigenvalues and nodal parameters for $p-$Laplacian diffusion eigenvalue problem (1.1)-(1.2) with polynomially dependent spectral parameter by using modified Prüfer substitution. In Section 3, we give a reconstruction formula of the potential functions for the problem (1.1)-(1.2). Finally, we expressed some conclusions in Section 4.

2. Asymptotics of Some Eigenparameters

In this section, we present some important results for the problem (1.1)-(1.2). To do this, we need to consider modified Prüfer's transformation which is one of the most powerful method for solution of inverse problem. Recalling that Prüfer's transformation for a nonzero solution y of (1.1) takes the form

$$y(x) = R(x) S_p\left(\lambda^{2/p}\ \theta(x,\lambda)\right), \tag{2.1}$$
$$y'(x) = \lambda^{2/p} R(x) S_p'\left(\lambda^{2/p}\ \theta(x,\lambda)\right),$$

or

$$\frac{y'(x)}{y(x)} = \lambda^{2/p} \frac{S_p'\left(\lambda^{2/p}\ \theta(x,\lambda)\right)}{S_p\left(\lambda^{2/p}\ \theta(x,\lambda)\right)}, \tag{2.2}$$

where $R(x)$ is amplitude and $\theta(x)$ is Prüfer variable [37]. After some straightforward computations, one can get easily [1]

$$\theta'(x,\lambda) = 1 - \frac{q_m(x)}{\lambda^2} S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right) - \frac{2}{\lambda} r_m(x) S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right). \tag{2.3}$$

Lemma 2.1. [30] *Define* $\theta(x,\lambda_{n,m})$ as in (2.1) and $\phi_n(x) = S_p^p\left(\lambda_{n,m}^{2/p}\ \theta(x,\lambda_{n,m})\right) - \frac{1}{p}$. Then, for any $g \in L^1(0,1)$

$$\int_0^1 \phi_n(x) g(x) dx = 0.$$

Theorem 2.1. *The eigenvalues* $\lambda_{n,m}$ *of the* $p-$ *Laplacian diffusion eigenvalue problem given in (1.1)-(1.2) have the form*

$$\lambda_{n,1}^{2/p} = n\hat{\pi} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p-2}{2}}} + \frac{1}{p(n\hat{\pi})^{p-1}} \int_0^1 q_1(x)dx + \frac{2}{p(n\hat{\pi})^{\frac{p-2}{2}}} \int_0^1 r_1(x)dx + O\left(\frac{1}{n^{p-2}}\right), \tag{2.4}$$

$$\begin{aligned} \lambda_{n,2}^{2/p} &= n\hat{\pi} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p-2}{2}} + a_2 (n\hat{\pi})^{p-1}} + \frac{1}{p(n\hat{\pi})^{p-1}} \int_0^1 q_2(x)dx \\ &\quad + \frac{2}{p(n\hat{\pi})^{\frac{p-2}{2}}} \int_0^1 r_2(x)dx + O\left(\frac{1}{n^{p-1}}\right), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \lambda_{n,m}^{2/p} &= n\hat{\pi} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p-2}{2}} + \ldots + a_m (n\hat{\pi})^{\frac{mp-2}{2}}} + \frac{1}{p(n\hat{\pi})^{p-1}} \int_0^1 q_m(x)dx \\ &\quad + \frac{2}{p(n\hat{\pi})^{\frac{p-2}{2}}} \int_0^1 r_m(x)dx + O\left(\frac{1}{n^{p-1}}\right), \end{aligned} \tag{2.6}$$

for $m = 1$, $m = 2$ and $m \geq 3$, respectively *as* $n \to \infty$.

Proof: Let $\theta(0,\lambda) = 0$ for the problem (1.1)-(1.2). Integrating both sides of (2.3) with respect to x from 0 to 1, we get

$$\theta(1,\lambda) = 1 - \frac{1}{\lambda^2} \int_0^1 q_m(x) S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right) dx - \frac{2}{\lambda} \int_0^1 r_m(x) S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right) dx.$$

By lemma 2.1, one can obtain

$$\int_0^1 q_m(x)\left\{S_p^p\left(\lambda_n^{2/p}\ \theta(x,\lambda)\right)-\frac{1}{p}\right\}dx=o(1),\ \text{as } n\to\infty.$$

Hence, we obtain

$$\lambda^{2/p}\theta(1,\lambda)=\lambda^{2/p}-\frac{1}{p\lambda^{2-\frac{2}{p}}}\int_0^1 q_m(x)dx-\frac{2}{p\lambda^{1-\frac{2}{p}}}\int_0^1 r_m(x)dx+O\left(\frac{1}{\lambda^{2-\frac{2}{p}}}\right). \quad (2.7)$$

Let $\lambda_{n,m}$ be an eigenvalue of the problem (1.1)-(1.2) for all m. Now, we will prove the lemma for $m=1$. By (1.2), we have

$$\lambda_{n,1}^{2/p}R(1)S_p'\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)+a_1\lambda_{n,1}R(1)S_p\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)=0,$$

or

$$-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}=\frac{S_p\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)}{S_p'\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)}=T_p\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right).$$

As n is sufficiently large, it follows

$$\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})=T_p^{-1}\left(-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}\right)=n\hat{\pi}-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}+o\left(\lambda_{n,1}^{\frac{4}{p}-2}\right). \quad (2.8)$$

By considering (2.7) and (2.8) together, we get

$$\lambda_{n,1}^{2/p}=n\hat{\pi}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}+\frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_1(x)dx+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_1(x)dx+O\left(\frac{1}{n^{p-2}}\right).$$

For the case $m=2$, by using the similar process as in $m=1$, we can easily obtain

$$\lambda_{n,2}^{2/p}R(1)S_p'\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)+\left(a_1\lambda_{n,2}+a_2\lambda_{n,2}^2\right)R(1)S_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)=0,$$

or

$$-\frac{\lambda_{n,2}^{\frac{2}{p}}}{a_1\lambda_{n,2}+a_2\lambda_{n,2}^2}=\frac{S_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)}{S_p'\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)}=T_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right),$$

and

$$\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})=n\hat{\pi}-\frac{\lambda_{n,2}^{\frac{2}{p}}}{a_1\lambda_{n,2}+a_2\lambda_{n,2}^2}+o\left(\frac{\lambda_{n,2}^{\frac{4}{p}}}{\left(a_1\lambda_{n,2}+a_2\lambda_{n,2}^2\right)^2}\right). \quad (2.9)$$

Therefore, we have

$$\begin{aligned}\lambda_{n,2}^{2/p} &= n\hat{\pi} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p-2}{2}} + a_2 (n\hat{\pi})^{p-1}} + \frac{1}{p (n\hat{\pi})^{p-1}} \int_0^1 q_2(x)dx \\ &\quad + \frac{2}{p (n\hat{\pi})^{\frac{p-2}{2}}} \int_0^1 r_2(x)dx + O\left(\frac{1}{n^{p-1}}\right),\end{aligned}$$

by using (2.7) and (2.9). Finally, let us find the asymptotic expansion of $\lambda_{n,m}$ for $m \geq 3$. Similarly, by using (1.2), we have

$$\begin{aligned}&\lambda_{n,m}^{2/p} R(1) S_p'\left(\lambda_{n,m}^{2/p}\ \theta(1, \lambda_{n,m})\right) \\ &\quad + \left(a_1 \lambda_{n,m} + ... + a_m \lambda_{n,m}^m\right) R(1) S_p\left(\lambda_{n,m}^{2/p}\ \theta(1, \lambda_{n,m})\right) = 0,\end{aligned}$$

or

$$-\frac{\lambda_{n,m}^{\frac{2}{p}}}{a_1 \lambda_{n,m} + ... + a_m \lambda_{n,m}^m} = \frac{S_p\left(\lambda_{n,m}^{2/p}\ \theta(1, \lambda_{n,m})\right)}{S_p'\left(\lambda_{n,m}^{2/p}\ \theta(1, \lambda_{n,m})\right)} = T_p\left(\lambda_{n,m}^{2/p}\ \theta(1, \lambda_{n,m})\right). \quad (2.10)$$

By considering (2.7) and (2.10) together and using similar procedure, we deduce that

$$\begin{aligned}\lambda_{n,m}^{2/p} &= n\hat{\pi} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p-2}{2}} + ... + a_m (n\hat{\pi})^{\frac{mp-2}{2}}} + \frac{1}{p (n\hat{\pi})^{p-1}} \int_0^1 q_m(x)dx \\ &\quad + \frac{2}{p (n\hat{\pi})^{\frac{p-2}{2}}} \int_0^1 r_m(x)dx + O\left(\frac{1}{n^{p-1}}\right).\end{aligned}$$

Theorem 2.2. *Asymptotic estimates* of *the nodal points for the problem (1.1)-(1.2) satisfies*

$$\begin{aligned}x_{j,1}^n &= \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}}} + \frac{j}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_1(t)dt + \frac{2j}{p n^{\frac{p}{2}+1} \hat{\pi}^{\frac{p}{2}}} \int_0^1 r_1(t)dt \\ &\quad + \frac{1}{(n\hat{\pi})^p} \int_0^{x_{j,1}^n} q_1(t) S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}} \int_0^{x_{j,1}^n} r_1(t) S_p^p dt + O\left(\frac{j}{n^p}\right), \quad (2.11)\end{aligned}$$

$$x_{j,2}^{n} = \frac{j}{n}-\frac{j}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+a_2 n^p\hat{\pi}^p}+\frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_2(t)dt+\frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_2(t)dt$$
$$+\frac{1}{(n\hat{\pi})^p}\int_0^{x_{j,2}^n} q_2(t)S_p^p dt+\frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_0^{x_{j,2}^n} r_2(t)S_p^p dt+O\left(\frac{j}{n^{p+1}}\right), \quad (2.12)$$

and

$$x_{j,m}^{n} = \frac{j}{n}-\frac{j}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+...+a_m n^{\frac{mp}{2}+1}\hat{\pi}^{\frac{mp}{2}}}+\frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_m(t)dt$$
$$+\frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_m(t)dt+\frac{1}{(n\hat{\pi})^p}\int_0^{x_{j,m}^n} q_m(t)S_p^p dt$$
$$+\frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_0^{x_{j,m}^n} r_m(t)S_p^p dt+O\left(\frac{j}{n^{p+1}}\right), \quad (2.13)$$

for $m=1$, $m=2$ and $m\geq 3$, respectively *as* $n\to\infty$.

Proof: Integrating (2.3) from 0 to $x_{j,m}^n$ and letting $\theta(x_{j,m}^n,\lambda)=\dfrac{j\hat{\pi}}{\lambda_{n,m}^{2/p}}$, we have

$$x_{j,m}^n=\frac{j\hat{\pi}}{\lambda_{n,m}^{2/p}}+\frac{1}{\lambda_{n,m}^2}\int_0^{x_{j,m}^n} q_m(t)S_p^p dt+\frac{2}{\lambda_{n,m}}\int_0^{x_{j,m}^n} r_m(t)S_p^p dt. \quad (2.14)$$

Now, we will find the asymptotic estimate of nodal points for $m=1$. From the formula (2.4), we deduce

$$\frac{1}{\lambda_{n,1}^{2/p}}=\frac{1}{n\hat{\pi}}-\frac{1}{a_1(n\hat{\pi})^{\frac{p+2}{2}}}+\frac{1}{p(n\hat{\pi})^{p+1}}\int_0^1 q_1(t)dt+\frac{2}{p(n\hat{\pi})^{\frac{p}{2}+1}}\int_0^1 r_1(t)dt+O\left(\frac{1}{n^p}\right), \quad (2.15)$$

and therefore we obtain the formula (2.11) by using (2.14) and (2.15).

In (2.11), if we take $\dfrac{j}{n}\to 1$ as $n\to\infty$, we obtain

$$x_{j,1}^n=\frac{j}{n}-\frac{j}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}}+\frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_1(t)dt+\frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_1(t)dt$$
$$+\frac{1}{p(n\hat{\pi})^p}\int_0^1 q_1(t)dt+\frac{2}{p(n\hat{\pi})^{\frac{p}{2}}}\int_0^1 r_1(t)dt+O\left(\frac{1}{n^{\frac{p}{2}+1}}\right). \quad (2.16)$$

By using (2.5), the asymptotic estimate of eigenvalues $1/\lambda_{n,2}^{2/p}$ for $m=2$ is considered as

$$\frac{1}{\lambda_{n,2}^{2/p}} = \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}} + a_2 (n\hat{\pi})^{p+1}} + \frac{1}{p (n\hat{\pi})^{p+1}} \int_0^1 q_2(t)dt + \frac{2}{p (n\hat{\pi})^{\frac{p}{2}+1}} \int_0^1 r_2(t)dt + O\left(\frac{1}{n^{p+1}}\right), \tag{2.17}$$

and, we conclude the formula (2.12) by using (2.14) and (2.17).

In the formula (2.12), if we take $\frac{j}{n} \to 1$ as $n \to \infty$, we have

$$x_{j,2}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + a_2 n^p \hat{\pi}^p} + \frac{j}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_2(t)dt + \frac{2j}{p n^{\frac{p}{2}+1} \hat{\pi}^{\frac{p}{2}}} \int_0^1 r_2(t)dt + \frac{1}{p (n\hat{\pi})^p} \int_0^1 q_2(t)dt + \frac{2}{p (n\hat{\pi})^{\frac{p}{2}}} \int_0^1 r_2(t)dt + O\left(\frac{1}{n^{\frac{p}{2}+1}}\right). \tag{2.18}$$

For $m \geq 3$, from the formula (2.6), it can be easily obtain that

$$\frac{1}{\lambda_{n,m}^{2/p}} = \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}} + ... + a_m (n\hat{\pi})^{\frac{mp+2}{2}}} + \frac{1}{p (n\hat{\pi})^{p+1}} \int_0^1 q_m(t)dt + \frac{2}{p (n\hat{\pi})^{\frac{p}{2}+1}} \int_0^1 r_m(t)dt + O\left(\frac{1}{n^{p+1}}\right), \tag{2.19}$$

and we get the formula (2.13) by using (2.14) and (2.19).

In (2.13), if we take $\frac{j}{n} \to 1$ as $n \to \infty$, we obtain

$$x_{j,m}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + ... + a_m n^{\frac{mp}{2}+1} \hat{\pi}^{\frac{mp}{2}}} + \frac{j}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_m(t)dt + \frac{2j}{p n^{\frac{p}{2}+1} \hat{\pi}^{\frac{p}{2}}} \int_0^1 r_m(t)dt + \frac{1}{p (n\hat{\pi})^p} \int_0^1 q_m(t)dt + \frac{2}{p (n\hat{\pi})^{\frac{p}{2}}} \int_0^1 r_m(t)dt + O\left(\frac{1}{n^{\frac{p}{2}+1}}\right). \tag{2.20}$$

Theorem 2.3. *Asymptotic estimate* of *the nodal lengths for the problem (1.1)-(1.2) satisfies*

$$l_{j,1}^n=\frac{1}{n}-\frac{1}{a_1n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}}+\frac{1}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_1(t)dt+\frac{2}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_1(t)dt \tag{2.21}$$

$$+\frac{1}{(n\hat{\pi})^p}\int_{x_{j,1}^n}^{x_{j+1,1}^n} q_1(t)S_p^pdt+\frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_{x_{j,1}^n}^{x_{j+1,1}^n} r_1(t)S_p^pdt+O\left(\frac{1}{n^p}\right),$$

$$l_{j,2}^n=\frac{1}{n}-\frac{1}{a_1n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+a_2n^{p+1}\hat{\pi}^p}+\frac{1}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_2(t)dt+\frac{2}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_2(t)dt \tag{2.22}$$

$$+\frac{1}{(n\hat{\pi})^p}\int_{x_{j,2}^n}^{x_{j+1,2}^n} q_2(t)S_p^pdt+\frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_{x_{j,2}^n}^{x_{j+1,2}^n} r_2(t)S_p^pdt+O\left(\frac{1}{n^{p+1}}\right),$$

and

$$l_{j,m}^n=\frac{1}{n}-\frac{1}{a_1n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+...+a_mn^{\frac{mp}{2}+1}\hat{\pi}^{\frac{mp}{2}}}+\frac{1}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_m(t)dt$$

$$+\frac{2}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_m(t)dt+\frac{1}{(n\hat{\pi})^p}\int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t)S_p^pdt$$

$$+\frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t)S_p^pdt+O\left(\frac{1}{n^{p+1}}\right), \tag{2.23}$$

for $m=1$, $m=2$ and $m\geq 3$, respectively *as* $n\to\infty$.

Proof: For large $n\in\mathbb{N}$, integrating (2.3) in $[x_{j,m}^n, x_{j+1,m}^n]$ and using the definition of nodal lengths, we have

$$l_{j,m}^n=\frac{\hat{\pi}}{\lambda_{n,m}^{2/p}}+\frac{1}{\lambda_{n,m}^2}\int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t)S_p^pdt+\frac{2}{\lambda_{n,m}}\int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t)S_p^pdt, \tag{2.24}$$

or

$$l_{j,m}^{n}=\frac{\hat{\pi}}{\lambda_{n,m}^{2/p}}+\frac{1}{p\lambda_{n,m}^{2}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}q_m(t)dt+\frac{2}{p\lambda_{n,m}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}r_m(t)dt+O\left(\frac{1}{n^{\frac{p}{2}+1}}\right).$$

For $m=1$, $m=2$ and $m\geq 3$, we can obtain easily (2.21), (2.22) and (2.23) by using the formulas (2.15), (2.17), (2.19), respectively.

3. Reconstruction of the potential function

In this section, we give an explicit formula for the potential functions of the diffusion equation (1.1) by using nodal lengths. The method used in the proof of the theorem is similar to classical problems; $p-$Laplacian Sturm-Liouville eigenvalue problem and $p-$Laplacian energy-dependent Sturm-Liouville eigenvalue problem (see [1], [29], [30], [31]).

Theorem 3.1. *Let $q_m(x)\in L^2(0,1)$ and $r_m(x)\in W_2^1(0,1)$ are real-valued functions defined in the interval* $0\leq x\leq 1$ for all m. *Then*

$$q_m(x)=\lim_{n\to\infty}\left(\frac{p\lambda_{n,m}^{\frac{2}{p}+2}l_{j,m}^{n}}{\hat{\pi}}-p\lambda_{n,m}^{2}\right), \tag{2.25}$$

and

$$r_m(x)=\lim_{n\to\infty}\left(\frac{p\lambda_{n,m}^{\frac{2}{p}+1}l_{j,m}^{n}}{2\hat{\pi}}-p\frac{\lambda_{n,m}}{2}\right),$$

for $j=j_{n,m}(x)=\max\left\{j:x_{j,m}^{n}<x\right\}$ and $m\in\mathbb{Z}^{+}$.

Proof: We need to consider Theorem 2.3 for proof. From (2.24), we have

$$\frac{p\lambda_{n,m}^{2/p+2}}{\hat{\pi}}l_{j,m}^{n}=p\lambda_{n,m}^{2}+\frac{p\lambda_{n,m}^{2/p}}{\hat{\pi}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}q_m(t)S_p^p dt+\frac{2p\lambda_{n,m}^{2/p+1}}{\hat{\pi}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}r_m(t)S_p^p dt.$$

Then, we can use similar procedure as those in [29] for $j=j_{n,m}(x)=\max\{j:x_{j,m}^{n}<x\}$ to show

$$\frac{\lambda_{n,m}^{2/p}}{\hat{\pi}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}q_m(t)dt\to q_m(x),$$

and

$$\frac{p\lambda_{n,m}^{2/p}}{\hat{\pi}}\int\limits_{x_{j,m}^{n}}^{x_{j+1,m}^{n}}q_m(t)\left(S_p^p-\frac{1}{p}\right)dt\to 0,$$

pointwise almost everywhere. Hence, we get

$$q_m(x) = \lim_{n\to\infty}\left(\frac{p\lambda_{n,m}^{\frac{2}{p}+2} l_{j,m}^{n}}{\hat{\pi}} - p\lambda_{n,m}^2\right).$$

By using similar way, we can easily get the asymptotic expansion of $r_m(x)$.

Theorem 3.2. *Let* $\left\{l_{j,m}^{(n)} : j = 1, 2, ..., n-1\right\}_{n=2}^{\infty}$ *be a set of nodal lengths of the problem (1.1)-(1.2) where* $q_m(x)$ *and* $r_m(x)$ *are real-valued functions on* $0 \le x \le 1$ for all m. *Let us define*

$$F_{n,1}(x) = p\,(n\hat{\pi})^p\left(nl_{j,1}^{(n)} - 1\right) - \frac{p}{a_1}\,(n\hat{\pi})^{p/2} + \int_0^1 q_1(t)dt + 2\,(n\hat{\pi})^{p/2}\int_0^1 r_1(t)dt, \quad (2.26)$$

$$F_{n,2}(x) = p\,(n\hat{\pi})^p\left(nl_{j,2}^{(n)} - 1\right) - \frac{p\,(n\hat{\pi})^{p/2}}{a_1 + a_2\,(n\hat{\pi})^{p/2}} + \int_0^1 q_2(t)dt + 2\,(n\hat{\pi})^{p/2}\int_0^1 r_2(t)dt, \quad (2.27)$$

$$\begin{aligned} F_{n,m}(x) &= p\,(n\hat{\pi})^p\left(nl_{j,m}^{(n)} - 1\right) - \frac{p\,(n\hat{\pi})^{p/2}}{a_1 + ... + a_m\,(n\hat{\pi})^{\frac{mp-p}{2}}} \\ &\quad + \int_0^1 q_m(t)dt + 2\,(n\hat{\pi})^{p/2}\int_0^1 r_m(t)dt. \qquad (2.28) \end{aligned}$$

and

$$G_{n,1}(x) = \frac{p\,(n\hat{\pi})^{\frac{p}{2}}}{2}\left(nl_{j,1}^{(n)} - 1\right) - \frac{p}{2a_1} + \frac{1}{2\,(n\hat{\pi})^{p/2}}\int_0^1 q_1(t)dt + \int_0^1 r_1(t)dt, \quad (2.29)$$

$$\begin{aligned} G_{n,2}(x) &= \frac{p\,(n\hat{\pi})^{\frac{p}{2}}}{2}\left(nl_{j,2}^{(n)} - 1\right) - \frac{p}{2\left(a_1 + a_2\,(n\hat{\pi})^{\frac{p}{2}}\right)} \\ &\quad + \frac{1}{2\,(n\hat{\pi})^{p/2}}\int_0^1 q_2(t)dt + \int_0^1 r_2(t)dt \qquad (2.30) \end{aligned}$$

$$G_{n,m}(x) = \frac{p\left(n\hat{\pi}\right)^{\frac{p}{2}}}{2}\left(nl_{j,m}^{(n)}-1\right) - \frac{p}{2\left(a_1+...+a_m\left(n\hat{\pi}\right)^{\frac{mp-p}{2}}\right)} + \frac{1}{2\left(n\hat{\pi}\right)^{\frac{p}{2}}}\int_0^1 q_m(t)dt + \int_0^1 r_m(t)dt \tag{2.31}$$

for $m = 1$, $m = 2$ *and* $m \geq 3$, *respectively. Then,* $\{F_{n,m}(x)\}$ *and* $\{G_{n,m}(x)\}$ *converge to* q_m *and* r_m *pointwise almost everywhere in* $L^1(0,1)$, *respectively, for all cases.*

Proof: We will prove this theorem only for $F_{n,1}$. Other cases can be shown similarly. By the asymptotic formulas of eigenvalues (2.4) and nodal lengths (2.21), we get

$$p\lambda_{n,1}^2\left(\frac{\lambda_{n,1}^{2/p}l_{j,1}^n}{\hat{\pi}}-1\right) = p\lambda_{n,1}^2\left(nl_{j,1}^{(n)}-1\right) - \frac{p}{a_1\pi}\left(n\hat{\pi}\right)^{p/2+1}l_{j,1}^{(n)} + nl_{j,1}^{(n)}\int_0^1 q_1(t)dt + 2n\left(n\hat{\pi}\right)^{p/2}l_{j,1}^{(n)}\int_0^1 r_1(t)dt + o(1).$$

Considering $nl_{j,1}^{(n)} = 1 + o(1)$, as $n \to \infty$, we have

$$p\left(n\hat{\pi}\right)^p\left(nl_{j,1}^{(n)}-1\right) - \frac{p}{a_1}\left(n\hat{\pi}\right)^{p/2} \to q_1(x) - \int_0^1 q_1(t)dt - 2\left(n\hat{\pi}\right)^{p/2}\int_0^1 r_1(t)dt$$

pointwise almost everywhere in $L^1(0,1)$. By using similar way, it is not difficult to show that $\{G_{n,m}(x)\}$ converges to r_m pointwise almost everywhere in $L^1(0,1)$, respectively, for all $m \in \mathbb{Z}^+$.

4. Conclusion

In this study, we give some asymptotic estimates for eigenvalues, nodal parameters and potential function of the $p-$Laplacian diffusion eigenvalue problem (1.1)-(1.2) with polynomially dependent spectral parameter. We show that the obtained results are generalizations of the classical problem.

References

[1] Koyunbakan, H. 2013. Inverse nodal problem for $p-$Laplacian energy-dependent Sturm-Liouville equation, *Boundary Value Problems* 2013:272 (Erratum: Inverse nodal problem for $p-$Laplacian energy-dependent Sturm-Liouville equation, *Boundary Value Problems*, 2014:222 (2014).

[2] Yang, C.F. and Yang, X. 2011. Ambarzumyan's theorem with eigenparameter in the boundary conditions, *Acta Mathematica Scientia* 31(4), 1561-1568.

[3] Jaulent, M. and Jean, C. 1972. The inverse s-wave scattering problem for a class of potentials depending on energy, *Commun.Math. Phys.* 28(3), 177-220.

[4] Gasymov, M. G. and Guseinov, G.S. 1981. Determination of a diffusion operator from spectral data, *Dokl. Akad. Nauk Azerb. SSR* 37(2), 19-23.

[5] Guseinov, G. S. 1985. On the spectral analysis of a quadratic pencil of Sturm-Liouville operators, *Sov. Math. Dokl.* 32, 1292-1296.

[6] Hryniv, R. and Pronska, N. 2012. Inverse spectral problems for energy dependent Sturm-Liouville equations, *Inverse Probl.* 28(8), 085008.

[7] Nabiev, I. M. 2007. The inverse quasiperiodic problem for a diffusion operator, *Dokl. Math.* 76(1), 527-529.

[8] Wang, Y.P., Yang, C.F. and Huang, Z.Y. 2011. Half inverse problem for a quadratic pencil of Schrödinger operators, *Acta Math. Sci.* 31(6), 1708-1717.

[9] Koyunbakan, H. and Yilmaz, E. 2008. Reconstruction of the potential function and its derivatives for the diffusion operator, *Z. Naturforch* A 63(3-4), 127-130.

[10] Buterin, S. A. and Shieh, C. T. 2008. Inverse nodal problem for differential pencils, *Applied Mathematics Letters* 22(8), 1240-1247.

[11] Koyunbakan, H. 2011. Inverse problem for a quadratic pencil of Sturm-Liouville operator, *J. Math. Anal. Appl.* 378(2), 549-554.

[12] Guseinov, I. M. and Nabiev, I. M. 2000. A class of inverse problems for a quadratic pencil of Sturm-Liouville operators, *Differential Equations* 36(3), 471-473.

[13] Maksudov, F. G. and Guseinov, G. Sh.1987. On the solution of the inverse scattering problem for a quadratic pencil of one dimensional Schrodinger operators on whole axis, *Soviet Mathematics Doklady* 34, 34-38.

[14] Bairamov, E., Cakar O. and Celebi, A. O. 1997. Quadratic pencil of Schrodinger operators with spectral singularities: discrete spectrum and principal functions, *Journal of Mathematical Analysis and Applications* 216(1), 303-320.

[15] Yang, C. F. and Zettl, A. 2012. Half inverse problems for quadratic pencils of Sturm-Liouville operators, *Taiwanese Journal of Mathematics* 16(5), 1829-1846.

[16] Amirov, Rauf Kh. and Nabiev, A. A. 2013. Inverse problems for the quadratic pencil of the Sturm-Liouville equations with Impulse, *Abstract and Applied Analysis* Volume 2013, Article ID 361989.

[17] Wang, Y. P. 2012. The inverse problem for differential pencils with eigenparameter dependent boundary conditions from interior spectral data, *Applied Mathematics Letters* 25(7) 1061-1067.

[18] Sat, M. and Panakhov, E. S. 2014. Spectral problem for diffusion operator, *Applicable Analysis* 93(6) 1178-1186.

[19] Chadan, K., Colton, D., Paivarinta, L. and Rundell, W. 1997. An introduction to inverse scattering and inverse spectral problems, S*ociety for Industrial and Applied Mathematics*.

[20] Aygar, Y. and Bohner M. 2015. On the spectrum of eigenparameter-dependent quantum difference equations, *Applied Mathematics &Information Sciences*, 9(4), 1725-1729.

[21] McLaughlin, J. R. 1988. Inverse spectral theory using nodal points as data-a uniqueness result, *J. Differential Equations* 73(2), 354-362.

[22] Hald, O. H. and McLaughlin, J.R. 1989. Solution of inverse nodal problems, *Inverse Probl.* 5(3), 307-347.

[23] Yurko, V. A. 2008. Inverse nodal problems for Sturm-Liouville operators on star-type graphs, *J. Inv. Ill Posed Probl.* 16(7), 715–722.

[24] Law, C. K. and Yang, C. F. 1999. Reconstruction of the potential function and its derivatives using nodal data, *Inverse Probl.* 14(2), 299-312.

[25] Yang, C. F. and Yang, X. P. 2011. Inverse nodal problem for the Sturm–Liouville equation with polynomially dependent on the eigenparameter, *Inverse Prob. Sci. Eng.* 19(7), 951–961 (2011).

[26] Browne, P. J. and Sleeman, B. D. 1996. Inverse nodal problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions, *Inverse Probl.* 12(4), 377–381.
[27] Ozkan, A. S. and Keskin, B. 2015. Inverse nodal problems for Sturm-Liouville equation with eigenparameter dependent boundary and jump conditions, *Inverse Problems in Science and Engineering* 23(8), 1306-1312.
[28] Chen, H. Y. 2009. On generalized trigonometric functions, *Master of Science*, National Sun Yat-sen University, Kaohsiung, Taiwan.
[29] Law, C. K., Lian, W. C. and Wang, W. C. 2009. The inverse nodal problem and the Ambarzumyan problem for the $p-$Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A Math.* 139(6), 1261-1273.
[30] Wang, W. C., Cheng, Y. H. and Lian, W. C. 2011. Inverse nodal problems for the $p-$Laplacian with eigenparameter dependent boundary conditions, Math. Comput. Modelling 54 (11-12), 2718-2724.
[31] Wang, W. C. 2010. Direct and inverse problems for one dimensional $p-$Laplacian operators, National Sun Yat-sen University, *PhD Thesis.*
[32] Elbert, A. 1987. On the half-linear second order differential equations, *Acta Math. Hungar.* 49(3-4), 487–508.
[33] Binding, P. and Drábek, P. 2003. Sturm–Liouville theory for the $p-$Laplacian, *Studia Sci. Math. Hungar.* 40(4), 375–396.
[34] Binding, P. A. and Rynne, B. P. 2008. Variational and non-variational eigenvalues of the $p-$Laplacian, *J. Differential Equations* 244(1), 24-39.
[35] Brown, B. M. and Reichel ,W. 2004. Eigenvalues of the radially symmetric $p-$Laplacian in R^n, *J. Lond. Math. Soc.* 69, 657-675.
[36] Walter, W. 1998. Sturm-Liouville theory for the radial Δ_p-operator. *Math. Z.* 227(1), 175-185.
[37] Yantır, A. 2004. Oscillation theory for second order differential equations and dynamic equations on time scales, *Master of Science*, Izmir institue of Technology, Izmir.

Current address: Tuba GULSEN : Firat University, Department of Mathematics, 23119, Elazıg TURKEY

E-mail address: tubagulsen@hotmail.com

Current address: Emrah YILMAZ : Firat University, Department of Mathematics, 23119, Elazıg TURKEY

E-mail address: emrah231983@gmail.com