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ON AN EXTENSION OF THE POLAR TAXICAB DISTANCE IN SPACE

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Abstract. The aim of this paper is to provide an alternative distance function instead of Euclidean distance, which is very much used in navigation and spherical trigonometry will contribute to advancement of logistics and optimal facility location on spherical surfaces [8]. In this sense, we extend the polar taxicab distance function defined in [7] to three dimesional analytical space.

1. Introduction

We live on a spherical Earth rather than on a Euclidean 3- space $\mathbb{R}^3$. We must think of the distance as though a car would drive in the urban geography where physical obstacles have to be avoided. So, one had to travel through horizontal and vertical streets to get from one location to another. In this sense, the taxicab geometry was first introduced by K. Menger [4] and has developed by E. F. Krause [2]. Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in the $\mathbb{R}^3$, Z. Akca and R. Kaya [14] define the taxicab distance in $\mathbb{R}^3$ as follow $d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$. Also, the paths of taxicab distance d_T from P_1 to P_2 as shown in Figure 1.

Although Euclidean geometry is convenient, taxicab geometry is a better model than Euclidean geometry for urban world.

Researchers give alternative distance functions of which paths are different from path of Euclidean metric in the two or three dimensional analytic space. For example, G. Chen developed Chinese checker distance in the $\mathbb{R}^2$ of which paths are similar to the movement made by Chinese checker [3]. Afterwards, Ö. Gelişgen et. al. [12] defined Chinese checker distance in the $\mathbb{R}^3$ of which paths from P_1 to P_2 as shown in Figure 2. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be any two points in the $\mathbb{R}^3$, then Chinese checker distance is defined by

$$d_{CC}(P_1, P_2) = d_L(P_1, P_2) + \left(\sqrt{2} - 1\right) d_S(P_1, P_2)$$

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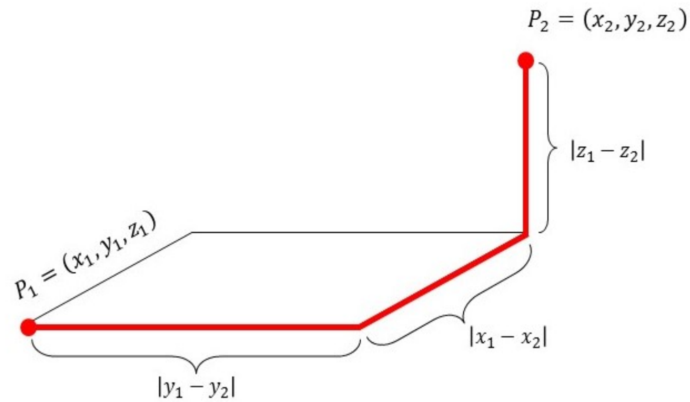

FIGURE 1. The Paths of Taxicab Distance d_T

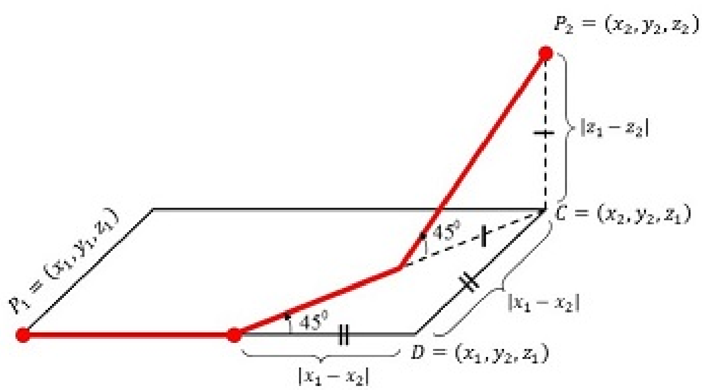

FIGURE 2. The Paths of Chinese Checker Distance d_{CC}.

where

$d_S(P_1, P_2) = \min\{|x_1 - x_2| + |y_1 - y_2|, |y_1 - y_2| + |z_1 - z_2|, |z_1 - z_2| + |x_1 - x_2|\}$

and $d_L(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$.

S. Tian [13] gave a family of metrics, $\alpha-$metric (alpha metric) for $\alpha \in [0, \pi/4]$, which includes the taxicab and Chinese checker metrics as special cases. Then, Ö. Gelişgen and R. Kaya extended the $\alpha-$distance to three and n dimensional spaces in [11, 10], respectively. Afterwards, H. B. Çolakoğlu [6] extended the $\alpha-$metric for $\alpha \in [0, \pi/2)$. For $\lambda(\alpha) = (\sec\alpha\text{-}\tan\alpha)$, $d_\alpha(P_1, P_2) = d_L(P_1, P_2) +$

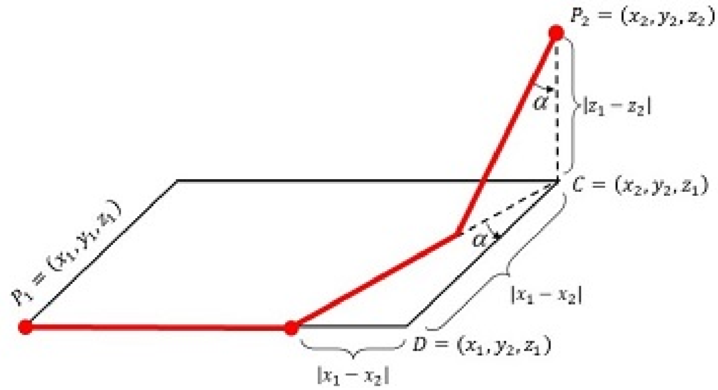

FIGURE 3. The Paths of Alpha Distance d_α.

$\left(\sqrt{2}-1\right) d_S\left(P_1, P_2\right)$ the paths of alpha metric d_α from P_1 to P_2 as shown in Figure 3.

Later, H. B. Çolakoğlu and R. Kaya [5] give the generalized $m-$metric $\mathbb{R}^n$ which includes the taxicab, Chinese checker, maximum, and alpha metrics. It is the most important property of generalized $m-$metric that its paths are not parallel to the coordinate axes in n-dimensional analytical space. Finally, H. G. Park et. al. [7] define the polar taxicab distance d_{PT} in the $\mathbb{R}^2$ of which paths composed of arc in circle and line segments. The polar taxicab metric has very important applications in urban geography beacuse cities formed not only linear streets but also curvilinear streets (Figure 4).

FIGURE 4. (a) Sun city in Arizona (b) Square of the Star in Paris

When we examine the common features of the metrics d_M, d_T, d_{CC}, d_α and d_{PT}, we see that these metrics were first defined in a planar surface. Considering distance of air travel or travel over water in terms of Euclidean distance, these

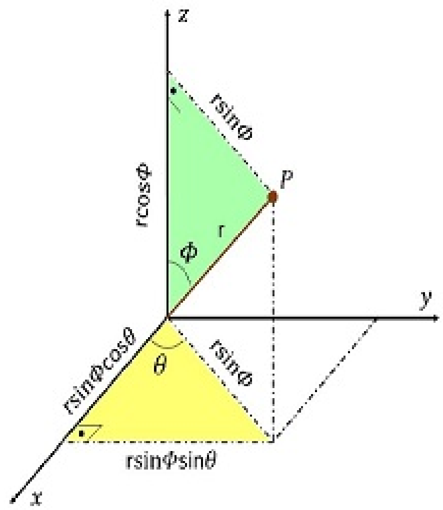

FIGURE 5

travels are made through the interior of spherical Earth which is impossible [8]. Using the idea given in [7], we have defined a new alternative metric on spherical surfaces due to disadvantage and disharmony of Euclidean distance on earth' s surface. This metric composed of arc of circle on sphere and line segments will be denoted d_{CL}. Also another alternative metric on sphere was defined by A. Bayar and R. Kaya [1].

2. An Alternative Metric In The $\mathbb{R}^3$

Let's remember spherical coordinates, before definition of alternative metric is given. The Cartesian coordinate of x, y, z of a point can be expressed in terms of r, ϕ, θ as shown in the Figure 5 ($x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi$).

Now, we define the distance function d_{CL} in the three dimensional analytic space as follows.

Definition 1. *Let $P_1 = (r_1, \phi_1, \theta_1)$ and $P_2 = (r_2, \phi_2, \theta_2)$ be two any points in the spherical coordinates and the angle $\angle P_1OP_2$ is denoted $\varphi_{P_1P_2}$. The distance function d_{CL} is defined by*

$$d_{CL}(P_1, P_2) = \begin{cases} \varphi_{P_1P_2} \times \min\{r_1, r_2\} + |r_1 - r_2| & ,\ 0 \leq \varphi_{P_1P_2} \leq 2 \\ r_1 + r_2 & ,\ 2 < \varphi_{P_1P_2} \leq \pi \end{cases}$$

where

$$\varphi_{P_1P_2} = \arcsin\left(\sqrt{(2 - \lambda_{P_1P_2})\,\lambda_{P_1P_2}}\right)$$

such that

$$\lambda_{P_1P_2} = (\sin\phi_1 - \sin\phi_2)^2 - \sin\phi_1 \sin\phi_2 \left[1 - \cos(\theta_1 - \theta_2)\right].$$

The following theorem show that d_{CL} is a metric.

Theorem 2. *d_{CL} distance function is a metric in the $\mathbb{R}^3$.*

Proof. Let $A = (r_1, \phi_1, \theta_1)$, $B = (r_2, \phi_2, \theta_2)$ and $C = (r_3, \phi_3, \theta_3)$ be any three points in the spherical coordinates. Without lose of generality, we can take $r_3 \geq r_2 \geq r_1 \geq 0$. For the sake of simple, the angles $\angle AOB$, $\angle BOC$ and $\angle AOC$ are denoted φ_{AB}, φ_{BC} and φ_{AC} , respectively. Consider the sphere with center the origin and radius r_i for $i = 1, 2, 3$, we write B_{r_i} and C_{r_i} to mean that the intersection points of this sphere and the vectors $\overrightarrow{OB}$ and $\overrightarrow{OC}$, respectively. Also the points A, B_{r_1} and C_{r_1} are on the sphere with center the origin $(0,0,0)$ and radius r_1. The shortest arc length joining these points can be denoted by $d_{CL}(A, B_{r_1})$, $d_{CL}(B_{r_1}, C_{r_1})$ and $d_{CL}(A, C_{r_1})$ in terms of Definition 1. Using the fact that the triangle inequality is valid for the spherical triangles, we exactly write $d_{CL}(A, B_{r_1}) + d_{CL}(B_{r_1}, C_{r_1}) \geq d_{CL}(A, C_{r_1})$.

To show distance function d_{CL} is the metric, we have proved following axioms for d_{CL} holds such that for all A, B and $C \in \mathbb{R}^3$

i) $d_{CL}(A, B) \geq 0$, $(d_{CL}(A, B) = 0 \Longleftrightarrow A = B)$

ii) $d_{CL}(A, B) = d_{CL}(B, A)$

iii) $d_{CL}(A, B) + d_{CL}(B, C) \geq d_{CL}(A, C)$

Note that $d_{CL}(A, B) \geq 0$ since absolute values, each of r_1 and r_2 and φ_{AB} are non-negative. Thus (i) for distance d_{CL} holds. If $A = B$, then $\varphi_{AB} = 0$ and $r_1 = r_2$, so this means $d_{CL}(A, B) = 0$. On the other hand, if $d_{CL}(A, B) = 0$, then there are two cases;

Case 1: For $0 \leq \varphi_{AB} \leq 2$;
$d_{CL}(A, B) = \varphi_{AB} \times \min\{r_1, r_2\} + |r_1 - r_2| = 0$, each of two terms $\varphi_{AB} \times \min\{r_1, r_2\}$ and $|r_1 - r_2|$ must be zero; $\varphi_{AB} \times \min\{r_1, r_2\} = 0$ and $|r_1 - r_2| = 0$. So, $|r_1 - r_2| = 0 \Rightarrow r_1 = r_2$ and $\varphi_{AB} \times \min\{r_1, r_2\} = 0 \Rightarrow \varphi_{AB} = 0$ since $\min\{r_1, r_2\} \geq 0$. So, $A = B$ is obtained.

Case 2: For $2 < \varphi_{AB} \leq \pi$;
$d_{CL}(P_1, P_2) = r_1 + r_2 = 0$, since $\min\{r_1, r_2\} \geq 0$, $r_1 = r_2 = 0$. Thus, $A = B$.

It is clearly that $d_{CL}(A, B) = d_{CL}(B, A)$. That is d_{CL} is symmetric.

As for final axiom (iii), is known as *Triangle Inequality,* we have to show that $d_{CL}(A, B) + d_{CL}(B, C) \geq d_{CL}(A, C)$ for all A, B and $C \in \mathbb{R}^3$.

Case 1: Let the angles φ_{AB}, φ_{BC}, φ_{AC} be in $[0, 2]$, then

$$d_{CL}(A, B) = d_{CL}(A, B_{r_1}) + r_2 - r_1,$$
$$d_{CL}(B, C) = d_{CL}(B, C_{r_2}) + r_3 - r_2.$$

Also, we obtain that

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= d_{CL}(A,B_{r_1})+d_{CL}(B,C_{r_2})+r_3-r_1 \\ &= d_{CL}(A,B_{r_1})+d_{CL}(B,C_{r_2})+d_{CL}(A,C) \\ &\quad -d_{CL}(A,C_{r_1}). \end{aligned}$$

Therefore,

$d_{CL}(A,B)+d_{CL}(B,C)-d_{CL}(A,C)$

$$\begin{aligned} &= d_{CL}(A,B_{r_1})+d_{CL}(B,C_{r_2})-d_{CL}(A,C_{r_1}) \\ &\geq d_{CL}(A,B_{r_1})+d_{CL}(B_{r_1},C_{r_1})-d_{CL}(A,C_{r_1}) \\ &\geq d_{CL}(A,C_{r_1})-d_{CL}(A,C_{r_1}) \\ &= 0. \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 2: Let the angles φ_{AB}, φ_{BC} be in $[0,2]$ and φ_{AC} be in $(2,\pi]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= d_{CL}(A,B_{r_1})+d_{CL}(B,C_{r_2})+r_3-r_1 \\ &\geq d_{CL}(A,B_{r_1})+d_{CL}(B_{r_1},C_{r_1})+r_3-r_1 \\ &\geq d_{CL}(A,C_{r_1})+r_3-r_1 \\ &\geq r_3+r_1 \\ &= d_{CL}(A,C). \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 3: Let the angles φ_{AB} and φ_{AC} be in $[0,2]$, φ_{BC} be in $(2,\pi]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= d_{CL}(A,B_{r_1})+r_2-r_1+r_2+r_3 \\ &\geq d_{CL}(A,B_{r_1})+2r_2+d_{CL}(A,C)-d_{CL}(A,C_{r_1}). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C)-d_{CL}(A,C) &= d_{CL}(A,B_{r_1})-d_{CL}(A,C_{r_1})+2r_2 \\ &\geq d_{CL}(A,B_{r_1})+2r_2 \\ &\geq 0. \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 4: Let the angles φ_{AB} be in $[0,2]$, φ_{BC} and φ_{AC} be in $(2,\pi]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= d_{CL}(A,B_{r_1})+r_2-r_1+r_2+r_3 \\ &\geq d_{CL}(A,B_{r_1})+2r_2+r_3 \\ &\geq d_{CL}(A,B_{r_1})+r_1+r_3 \\ &= d_{CL}(A,B_{r_1})+d_{CL}(A,C) \\ &\geq d_{CL}(A,C). \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 5: Let the angles φ_{AB} be in $(2,\pi]$, φ_{BC} and φ_{AC} be in $[0,2]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= r_1+r_2+d_{CL}(B,C_{r_2})+r_3-r_2 \\ &= d_{CL}(B,C_{r_2})+r_3+r_1 \\ &\geq d_{CL}(B,C_{r_2})+r_3-r_1 \\ &= d_{CL}(B,C_{r_2})+d_{CL}(A,C)-d_{CL}(A,C_{r_1}). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C)-d_{CL}(A,C) &= d_{CL}(B,C_{r_2})-d_{CL}(A,C_{r_1}) \\ &\geq 0. \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 6: Let the angles φ_{AB} and φ_{AC} be in $(2,\pi]$, φ_{BC} be in $[0,2]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= r_1+r_2+d_{CL}(B,C_{r_2})+r_3-r_2 \\ &= d_{CL}(B,C_{r_2})+r_3+r_1 \\ &\geq r_3+r_1 \\ &= d_{CL}(A,C). \end{aligned}$$

Namely, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 7: Let the angles φ_{AB} and φ_{BC} be in $(2,\pi]$, φ_{AC} be in $[0,2]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= r_1+r_2+r_2+r_3 \\ &= 2r_2+r_3+r_1 \\ &\geq r_3+r_1 \\ &\geq d_{CL}(A,C). \end{aligned}$$

Thus, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$.

Case 8: Let the angles φ_{AB}, φ_{BC}, φ_{AC} be in $(2,\pi]$, then

$$\begin{aligned} d_{CL}(A,B)+d_{CL}(B,C) &= r_1+r_2+r_2+r_3 \\ &= 2r_2+r_3+r_1 \\ &\geq r_3+r_1 \\ &= d_{CL}(A,C). \end{aligned}$$

Thus, $d_{CL}(A,B)+d_{CL}(B,C)\geq d_{CL}(A,C)$. Therefore d_{CL} holds the triangle inequality for all cases. Consequently d_{CL} is a metric. □

3. Isometries of $\mathbb{R}^3_{CL}$

For the sake of simplicity, $\mathbb{R}^3$ furnished by the metric d_{CL} is denoted $\mathbb{R}^3_{CL}$ in the rest of the article.

A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is called orthogonal if it preserves the length of vectors. Also, we know that an orthogonal transformation preserves angles between vectors. For example, the reflection σ_Δ about the plane Δ that passing the origin is a example of orthogonal transformations.

Suppose $A=(r_1,\phi_1,\theta_1)$ and $B=(r_2,\phi_2,\theta_2)$ are two any points in the spherical coordinates and let image of the points A and B under transformation σ_Δ are

$\sigma_\Delta(A) = A_\Delta$ and $\sigma_\Delta(B) = B_\Delta$, respectively. Since the reflection σ_Δ is a orthogonal transformations, the distance from the point A_Δ to origin is r_1 (similarly, the distance from the point B_Δ to origin is r_2) and $\varphi_{AB} = \varphi_{A_\Delta B_\Delta}$. If $0 \leq \varphi_{AB} \leq 2$, then $0 \leq \varphi_{\sigma_\Delta(A)\sigma_\Delta(B)} \leq 2$. Thus

$$\begin{aligned} d_{CL}(A,B) &= \varphi_{AB} \times \min\{r_1, r_2\} + |r_1 - r_2| \\ &= \varphi_{\sigma_\Delta(A)\sigma_\Delta(B)} \times \min\{r_1, r_2\} + |r_1 - r_2| \\ &= \varphi_{A_\Delta B_\Delta} \times \min\{r_1, r_2\} + |r_1 - r_2| \\ &= d_{CL}(A_\Delta, B_\Delta). \end{aligned}$$

Namely, we obtain that the equality $d_{CL}(A,B) = d_{CL}(A_\Delta, B_\Delta)$ for $0 \leq \varphi_{AB} \leq 2$. Similarly, the equality $d_{CL}(A,B) = d_{CL}(A_\Delta, B_\Delta)$ can be easily shown for $2 < \varphi_{AB} \leq \pi$. Consequently, we have proved the following theorem.

Theorem 3. *The reflection σ_Δ about the plane Δ passing through the origin is an isometry in the $\mathbb{R}^3_{CL}$.*

Orthogonal transformations in two or three-dimensional Euclidean space are rigid rotations, reflections, or combinations of rotations and reflections (also known as rotary reflection, rotary inversion and inversion). A rotation can be written as the composition of two distinct reflections about intersecting planes. That is, a rotation R_ϕ about axis l is defined by $\sigma_\Delta\sigma_\Gamma$ where l is line of intersection between planes Γ and Δ. It is known that the rotation $R_\phi = \sigma_\Delta\sigma_\Gamma$ is an orthogonal transformation such that two planes Γ and Δ pass through the origin. Therefore, following Theorem 4 can be given similar to Theorem 3. A rotary reflection is an transformation which is the combination of a rotation about an axis and a reflection in a plane. That is, a rotary reflection ρ is defined by $\sigma_\Pi\sigma_\Delta\sigma_\Gamma$ such that Γ and Δ are two intersecting planes each perpendicular to plane Π. Also, a rotary reflection $\rho = \sigma_\Pi\sigma_\Delta\sigma_\Gamma$ is an orthogonal transformation if the planes Π, Δ and Γ pass through the origin. A inversion according to the origin O can be written as the $\sigma_O(X) = Y$ such that O is the midpoint of X and Y for X , $Y \in \mathbb{R}^3$. Also the inversion σ_O is an orthogonal transformation. Finally, rotary inversion is the combination of a rotation and an inversion in a point. That is, a rotary inversion φ is defined by $\sigma_O R_\phi$ where R_ϕ is a rotation transformation and σ_O is a inversion according to the origin O. Also a rotary inversion $\varphi = \sigma_O R_\phi$ is a example of orthogonal transformations. In the light of above explanation, the following theorems can be proven similar to Theorem 3.

Theorem 4. *A rotation R_ϕ with axis l through the origin is an isometry in the $\mathbb{R}^3_{CL}$.*

Theorem 5. *Let the planes Π, Δ and Γ pass through the origin. A rotary reflection $\rho = \sigma_\Pi\sigma_\Delta\sigma_\Gamma$ where Γ and Δ are two intersecting planes each perpendicular to plane Π is an isometry in the $\mathbb{R}^3_{CL}$.*

Theorem 6. *A inversion σ_O according to the origin O is an isometry in the $\mathbb{R}^3_{CL}$.*

Theorem 7. *Let R_ϕ be rotation with axis l through the origin and σ_O be inversion according to the origin. A rotary inversion $\varphi = \sigma_O R_\phi$ is an isometry in the $\mathbb{R}^3_{CL}$.*

Thus, if we again consider the theorems which is mentioned above then we give following result:

The isometry group of $\mathbb{R}^3_{CL}$ is $O(3)$ orthogonal group where $O(3)$ is the symmetry group of Euclidean sphere.

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