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QUANTITATIVE ESTIMATES FOR JAIN-KANTOROVICH OPERATORS

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Abstract. By using given arbitrary sequences, $\beta_n > 0$, $n \in \mathbb{N}$ with the property that $lim_{n\to\infty} n\beta_n = 0$ and $lim_{n\to\infty}\beta_n = 0$, we give a Kantorovich type generalization of Jain operator based on the a Poisson disrtibition. Fristly we give the quantitative Voronovskaya type theorem. Then we also obtain the Grüss Voronovskaya type theorem in quantitative form .We show that they have an arbitrary good order of weighted approximation.

1. Introduction

With the help of a Poisson type distribution,

$$\omega_\beta(k,\alpha) = \frac{\alpha(\alpha+k\beta)^{k-1}}{k!} e^{-(\alpha+k\beta)}; \quad k = 0,1,2,.. \tag{1.1}$$

for $0 < \alpha < \infty$, $|\beta| < 1$, in 1970, G. C. Jain [14] introduced a positive linear operator defined for $f \in C(\mathbb{R}^+)$ as

$$P_n^{[\beta]}(f)(x) = \sum_{k=0}^{\infty} \omega_\beta(k,nx) f\left(\frac{k}{n}\right), \tag{1.2}$$

where $\beta \in [0,1)$ and

$$\sum_{k=0}^{\infty} \omega_\beta(k,\alpha) = 1.$$

As a particular case $\beta = 0$, we obtain the well-known Szasz-Mirakyan operators studied in [6], [11] and [15];

$$P_n^{[0]}(f)(x) \equiv S_n(f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.3}$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$.

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Recently, Agratini [3] studied class of integral type positive linear operators of $P_n^{[\beta]}$ and obtained some approximation properties of them in weighted spaces. Also, some authors studied generalizations of Jain's operators ([10], [13],[16] and [17]).

Now, we define and investigate Kantorovich variant of $P_n^{[\beta]}$ operator , in order to obtain an approximation process for spaces of locally integrable functions on unbounded interval, replacing the sample values $f(k/n)$ by the mean values of f in the intervals $\left[\frac{k}{n},\frac{k+1}{n}\right]$ as follows:

Definition 1. *For* $\beta\in[0,1)$

$$S_n^{[\beta]}\left(f\right)\left(x\right)=n\sum_{k=0}^{\infty}\omega_{\beta}\left(k,nx\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f\left(t\right)dt \tag{1.4}$$

where $\omega_{\beta}\left(k,nx\right)=\frac{nx}{k!}\left(nx+k\beta\right)^{k-1}e^{-(nx+k\beta)}$.

The Kantorovich method was applied to many generalizations of the Bernstein polynomials like for example Szász-Mirakyan, Baskakov and other operators. A recent contribution in this direction was given in [4]. We note that, P. L. Butzer [5] introduced and studied Szasz-Mirakyan-Kantorovich operators defined by

$$S_n^{[0]}\left(f\right)\left(x\right)\equiv K_n\left(f\right)\left(x\right)=n\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f\left(t\right)dt$$

for $f\in L_1(0,\infty)$, the space of integrable functions on unbounded interval $[0,\infty)$.

In this paper we study some approximation properties of the sequence of linear positive operators given by (1.4) in a weighted space.

The structure of the paper is as follows. In the second section, we calculate some moment of our operator in Definition 1. In the third section, a Voronovskaya type theorem in quantitative form is obtained as well. In the fourth section, we also give a Grüss Voronovskaya type theorem in quantitative form. In the last section, some weighted approximation theorems are presented.

2. Moments of the Operators $S_n^{[\beta]}$

We begin with the following lemma which is necessary to prove the main result. Taking in view Lemma 1 in [2] has been established the following moments:

Lemma 1. *Let* e_j, $j\in\mathbb{N}\cup\{0\}$, *be the* j*-th monomial,* $e_j\left(t\right)=t^j$. *For the operators defined by (1.2) (see also* [14, Eq.(2.11)]*) the moments are as follows:*

$$P_n^{[\beta]}\left(e_0\right)\left(x\right)=1,\ P_n^{[\beta]}\left(e_1\right)\left(x\right)=\frac{x}{1-\beta},\ P_n^{[\beta]}\left(e_2\right)\left(x\right)=\frac{x^2}{\left(1-\beta\right)^2}+\frac{x}{n\left(1-\beta\right)^3},$$

$$P_n^{[\beta]}(e_3)(x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} + \frac{x(2\beta+1)}{n^2(1-\beta)^5},$$
$$P_n^{[\beta]}(e_4)(x) = \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} + \frac{x^2(8\beta+7)}{n^2(1-\beta)^6} + \frac{x\left(6\beta^2+8\beta+1\right)}{n^3(1-\beta)^7},$$

$$\begin{aligned} P_n^{[\beta]}(e_5)(x) &= \frac{x^5}{(1-\beta)^5} + x^4\frac{10}{n(1-\beta)^6} + x^3\frac{5(5+4\beta)}{n^2(1-\beta)^7} \\ &\quad + x^2\frac{15\left[1+4\beta+2\beta^2\right]}{n^3(1-\beta)^8} + x\frac{1+22\beta+58\beta^2+24\beta^3}{n^4(1-\beta)^9}, \end{aligned}$$

$$\begin{aligned} P_n^{[\beta]}(e_6)(x) &= \frac{x^6}{(1-\beta)^6} + \frac{15x^5}{n(1-\beta)^7} + x^4\frac{5(13+8\beta)}{n^2(1-\beta)^8} + x^3\frac{30\left(3+8\beta+3\beta^2\right)}{n^3(1-\beta)^9} \\ &\quad + x^2\frac{31+292\beta+478\beta^2+144\beta^3}{n^4(1-\beta)^{10}} \\ &\quad + x\frac{1+4\beta\left(13+82\beta+111\beta^2+30\beta^3\right)}{n^5(1-\beta)^{11}}. \end{aligned}$$

Lemma 2. *The operators $S_n^{[\beta]}$, defined by (1.4) the moments are as follows:*

$$S_n^{[\beta]}(e_0)(x) = 1, \quad S_n^{[\beta]}(e_1)(x) = \frac{x}{1-\beta} + \frac{1}{2n},$$
$$S_n^{[\beta]}(e_2)(x) = \frac{x^2}{(1-\beta)^2} + \frac{\left(2-2\beta+\beta^2\right)x}{n(1-\beta)^3} + \frac{1}{3n^2},$$
$$\begin{aligned} S_n^{[\beta]}(e_3)(x) &= \frac{x^3}{(1-\beta)^3} + \frac{\left(9-6\beta+3\beta^2\right)x^2}{2n(1-\beta)^4} + \frac{\left(7-10\beta+15\beta^2-8\beta^3+2\beta^4\right)x}{2n^2(1-\beta)^5} \\ &\quad + \frac{1}{4n^3}, \end{aligned}$$

$$\begin{aligned} S_n^{[\beta]}(e_4)(x) &= \frac{x^4}{(1-\beta)^4} + \frac{2\left(4-2\beta+\beta^2\right)x^3}{n(1-\beta)^5} + \frac{\left(15-12\beta+18\beta^2-8\beta^3+2\beta^4\right)x^2}{n^2(1-\beta)^6} \\ &\quad + \frac{\left(6-6\beta+27\beta^2-24\beta^3+17\beta^4-6\beta^5+\beta^6\right)x}{n^3(1-\beta)^7} + \frac{1}{5n^4}, \end{aligned}$$

$$\begin{aligned} S_n^{[\beta]}(e_5)(x) &= \frac{x^5}{(1-\beta)^5} + \frac{5\left(5-2\beta+\beta^2\right)x^4}{2n(1-\beta)^6} + \frac{5\left(26-14\beta+21\beta^2-8\beta^3+2\beta^4\right)x^3}{3n^2(1-\beta)^7} \\ &\quad + \frac{5\left(18-4\beta+42\beta^2-28\beta^3+19\beta^4-6\beta^5+\beta^6\right)x^2}{2n^3(1-\beta)^8} \\ &\quad + \left(\frac{62+44\beta+566\beta^2-392\beta^3+595\beta^4-386\beta^5}{6n^4(1-\beta)^9}\right. \\ &\qquad \left. + \frac{183\beta^6-48\beta^7+6\beta^8}{6n^4(1-\beta)^9}\right)x + \frac{1}{6n^5}, \end{aligned}$$

$$\begin{aligned} & S_n^{[\beta]}(e_6)(x) \\ = & \frac{x^6}{(1-\beta)^6} + \frac{3\left(6-2\beta+\beta^2\right)x^5}{n(1-\beta)^7} + \frac{5\left(2+\beta^2\right)\left(10-4\beta+\beta^2\right)x^4}{n^2(1-\beta)^8} \\ & + \frac{5\left(40+60\beta^2-32\beta^3+21\beta^4-6\beta^5+\beta^6\right)x^3}{n^3(1-\beta)^9} \\ & + \frac{\left(129+168\beta+612\beta^2-224\beta^3+400\beta^4-218\beta^5+99\beta^6-24\beta^7+3\beta^8\right)x^2}{n^4(1-\beta)^{10}} \\ & + \left(\frac{18+78\beta+417\beta^2+96\beta^3+470\beta^4-308\beta^5}{n^5(1-\beta)^{11}}\right. \\ & \qquad \left. + \frac{269\beta^6-134\beta^7+48\beta^8-10\beta^9+\beta^{10}}{n^5(1-\beta)^{11}}\right)x + \frac{1}{7n^6}. \end{aligned}$$

Proof. Obviously by (1.4), we obtain $S_n^{[\beta]}(e_0)(x) = 1$. With a simple calculation, we obtain that

$$S_n^{[\beta]}(e_1) = P_n^{[\beta]}(e_1)(x) + \frac{1}{2n}P_n^{[\beta]}(e_0)(x) = \frac{x}{1-\beta} + \frac{1}{2n},$$

$$\begin{aligned} S_n^{[\beta]}(e_2)(x) &= P_n^{[\beta]}(e_2)(x) + \frac{1}{n}P_n^{[\beta]}(e_1)(x) + \frac{1}{3n^2}P_n^{[\beta]}(e_0)(x) \\ &= \frac{x^2}{(1-\beta)^2} + \frac{x\left(2-2\beta+\beta^2\right)}{n(1-\beta)^3} + \frac{1}{3n^2}, \end{aligned}$$

$$\begin{aligned} S_n^{[\beta]}(e_3)(x) &= P_n^{[\beta]}(e_3)(x) + \frac{3}{2n}P_n^{[\beta]}(e_2)(x) + \frac{1}{n^2}P_n^{[\beta]}(e_1)(x) + \frac{1}{4n^3}P_n^{[\beta]}(e_0)(x) \\ &= \frac{x^3}{(1-\beta)^3} + \frac{x^2 3\left(3-2\beta+\beta^2\right)}{2n(1-\beta)^4} + \frac{x\left(7-10\beta+15\beta^2-8\beta^3+2\beta^4\right)}{2n^2(1-\beta)^5} \\ &\quad + \frac{1}{4n^3} \end{aligned}$$

Similarly, for $j \geq 4$, the proof can be done. □

Lemma 3. *The j-th order central moment of the operators $S_n^{[\beta]}$ are as following*

$$S_n^{[\beta]}\left(\varphi_x^0(t)\right)(x) = 1, \quad \left(S_n^{[\beta]}\varphi_x^1(t)\right)(x) = x\frac{\beta}{1-\beta} + \frac{1}{2n},$$

$$S_n^{[\beta]}\left(\varphi_x^2(t)\right)(x) = x^2\frac{\beta^2}{(1-\beta)^2} + x\frac{\left(1+\beta-2\beta^2+\beta^3\right)}{n(1-\beta)^3} + \frac{1}{3n^2},$$

$$\begin{aligned} S_n^{[\beta]}\left(\varphi_x^3(t)\right)(x) = x^3\frac{\beta^3}{(1-\beta)^3} &+x^2\frac{3\beta\left(2+\beta-2\beta^2+\beta^3\right)}{2n(1-\beta)^4} \\ &+x\frac{5-5\beta^2+12\beta^3-8\beta^4+2\beta^5}{2n^2(1-\beta)^5} + \frac{1}{4n^3}, \end{aligned}$$

$$\begin{aligned} S_n^{[\beta]}\left(\varphi_x^4(t)\right)(x) &= x^4\frac{\beta^4}{(1-\beta)^4} + x^3\frac{2\beta^2\left(3+\beta-2\beta^2+\beta^3\right)}{n(1-\beta)^5} \\ &+x^2\frac{\left(3+10\beta-2\beta^2-2\beta^3+12\beta^4-8\beta^5+2\beta^6\right)}{n^2(1-\beta)^6} \\ &+x\frac{\left(5+\beta+6\beta^2+11\beta^3-18\beta^4+15\beta^5-6\beta^6+\beta^7\right)}{n^3(1-\beta)^7} + \frac{1}{5n^4}, \end{aligned}$$

$$\begin{aligned} &S_n^{[\beta]}\left(\varphi_x^6(t)\right)(x) \\ = \; &x^6\frac{\beta^6}{(1-\beta)^6} + x^5\frac{3\beta^4\left(5+\beta-2\beta^2+\beta^3\right)}{n(1-\beta)^7} \\ &+x^4\frac{5\beta^2\left(9+10\beta-3\beta^2+2\beta^3+6\beta^4-4\beta^5+\beta^6\right)}{n^2(1-\beta)^8} \\ &+x^3\frac{5\left(3+21\beta+21\beta^2+10\beta^3+30\beta^4+3\beta^5-14\beta^6+15\beta^7-6\beta^8+\beta^9\right)}{n^3(1-\beta)^9} \\ &+x^2\left(\frac{70+156\beta+225\beta^2+374\beta^3+43\beta^4}{n^4(1-\beta)^{10}}\right. \\ &\qquad\left.+\frac{7\beta^5+160\beta^6-153\beta^7+84\beta^8-24\beta^9+3\beta^{10}}{n^4(1-\beta)^{10}}\right) \\ &+x\left(\frac{17+89\beta+362\beta^2+261\beta^3+140\beta^4+154\beta^5-193\beta^6}{n^5(1-\beta)^{11}}\right. \\ &\qquad\left.+\frac{196\beta^7-117\beta^8+45\beta^9-10\beta^{10}+\beta^{11}}{n^5(1-\beta)^{11}}\right) + \frac{1}{7n^6} \end{aligned}$$

where $\varphi_x^j(t) = (t-x)^j$, $j = 0, 1, 2, \dots$.

3. Voronovskaya Theorems

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f(1+x^2)$ with some constant M_f, depending only on f, but independent of x. $B_{x^2}[0,\infty)$ is called weighted space and it is a Banach space endowed with the norm

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{f(x)}{1+x^2}.$$

Let $C_{x^2}[0,\infty) = C[0,\infty) \cap B_{x^2}[0,\infty)$ and by $C_{x^2}^k[0,\infty)$, we denote subspace of all continuous functions $f \in B_{x^2}[0,\infty)$ for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite.

We know that usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta \to 0$, on infinite interval. Thus we use weighted modulus of continuity $\Omega(f,\delta)$ defined on infinite interval $[0,\infty)$ (see [12]). Let

$$\mathbf{\Omega}(f,\delta) = \sup_{|h|<\delta,\ x\in[0,\infty)} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)} \text{ for each } f \in C_{x^2}[0,\infty).$$

Now some elementary properties of $\Omega(f,\delta)$ are collected in the following Lemma.

Lemma 4. *Let* $f \in C_{x^2}^k[0,\infty)$. *Then,*

i) $\Omega(f,\delta)$ is a monotonically increasing function of δ, $\delta \ge 0$.
ii) For every $f \in C_{x^2}^k[0,\infty)$, $\lim_{\delta\to 0} \Omega(f,\delta) = 0$.
iii) For each $\lambda > 0$,

$$\Omega(f,\lambda\delta) \le 2(1+\lambda)\left(1+\delta^2\right)\Omega(f,\delta). \tag{3.1}$$

From the inequality (3.1) and definition of $\Omega(f,\delta)$ we get

$$|f(t) - f(x)| \le 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f,\delta) \tag{3.2}$$

for every $f \in C_{x^2}[0,\infty)$ and $x, t \in [0,\infty)$.

Next, we give the quantitative Voronovskaya type theorem in weighted spaces, which states the following:

Theorem 1. *Let* $f'' \in C_{x^2}^k[0,\infty)$ *and* $0 \le \beta_n < 1$. *Then, we have*

$$\sup_{x\ge 0} \frac{\left| n\left[S_n^{[\beta_n]}(f)(x) - f(x)\right] - \frac{1}{2}\left[f'(x) + xf''(x)\right]\right|}{(1+x^2)^4} \le \left\|f'\right\|_{x^2} \frac{n\beta_n}{1-\beta_n} + \frac{\left\|f''\right\|_{x^2}}{2}\alpha_n + C\alpha_n \Omega\left(f''; \frac{1}{\sqrt{n}}\right),$$

where $\alpha_n \to 0$, *depending on* β_n, *as* $n \to \infty$ *and* C *is a positive constant.*

Proof. By the local Taylor's formula there exist η lying between x and y such that

$$f(y) = f(x) + f^{'}(x)(y-x) + \frac{f^{''}(x)}{2}(y-x)^2 + h(y,x)(y-x)^2,$$

where

$$h(\eta, x) := \frac{f''(\eta) - f''(x)}{2}$$

and h is a continuous function which vanishes at 0. Applying the operator $S_n^{[\beta_n]}$ to above equality, we obtain the equality

$$\begin{aligned} S_n^{[\beta_n]}(f)(x) - f(x) &= f^{'}(x) S_n^{[\beta_n]}\left(\varphi_x^1(t)\right)(x) + \frac{f^{''}(x)}{2} S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x) \\ &\quad + S_n^{[\beta_n]}\left(h(\eta,x)(y-x)^2\right)(x), \end{aligned}$$

also we can write that

$$\begin{aligned} &\left| S_n^{[\beta_n]}(f)(x) - f(x) - \frac{f^{'}(x)}{2n} - \frac{f^{''}(x)}{2n}x \right| \\ \leq\ & f^{'}(x)\left[S_n^{[\beta_n]}\left(\varphi_x^1(t)\right)(x) - \frac{1}{2n}\right] + \frac{f^{''}(x)}{2}\left[S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x) - \frac{x}{n}\right] \\ & + S_n^{[\beta_n]}\left(h(\eta,x)(y-x)^2\right)(x). \end{aligned}$$

To estimate last inequality using the inequality (3.2) and the inequality $|\eta - x| \leq |y-x|$, we can write that

$$|h(\eta,x)| \leq \left(1 + (y-x)^2\right)\left(1+x^2\right)\left(1 + \frac{|y-x|}{\delta}\right)\left(1+\delta^2\right)\Omega\left(f'';\delta\right).$$

Since

$$|h(\eta,x)| \leq \begin{cases} 2\left(1+x^2\right)\left(1+\delta^2\right)^2\Omega\left(f'';\delta\right), & |y-x| < \delta \\ 8\left(1+x^2\right)\frac{(y-x)^4}{\delta^4}\left(1+\delta^2\right)^2\Omega\left(f'';\delta\right), & |y-x| \geq \delta \end{cases}$$

choosing $\delta < 1$, we have

$$\begin{aligned} |h(\eta,x)| &\leq 2\left(1+x^2\right)\left(1 + \frac{(y-x)^4}{\delta^4}\right)\left(1+\delta^2\right)^2\Omega\left(f'';\delta\right) \\ &\leq 8\left(1+x^2\right)\left(1 + \frac{(y-x)^4}{\delta^4}\right)\Omega\left(f'';\delta\right). \end{aligned}$$

We deduce that

$$S_n^{[\beta_n]}\left(|h(\eta,x)|(y-x)^2\right)(x) = 8\left(1+x^2\right)\Omega\left(f'';\delta\right)\left\{S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x) + \frac{1}{\delta^4}S_n^{[\beta_n]}\left(\varphi_x^6(t)\right)(x)\right\}$$
$$= 8\left(1+x^2\right)\Omega\left(f'';\delta\right)S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x) \times\left\{1+\frac{1}{\delta^4}\frac{S_n^{[\beta_n]}\left(\varphi_x^6(t)\right)(x)}{S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x)}\right\}. \quad (3.3)$$

Using Lemma 1 and calculating with simple, we have

$$\frac{S_n^{[\beta_n]}\left(\varphi_x^6(t)\right)(x)}{S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x)} \le x^4\frac{\beta_n^4}{(1-\beta_n)^4} + x^3\frac{21\beta_n^2}{n(1-\beta_n)^5} + x^2\frac{660}{n^2(1-\beta_n)^6} + x\frac{1122}{n^3(1-\beta_n)^7} + \frac{1268}{n^4(1-\beta_n)^8}.$$

In (3.3), choosing

$$\delta = \frac{1}{\sqrt{n}}$$

then, we have

$$S_n^{[\beta]}\left(|h(\eta,x)|(y-x)^2\right)(x) \le C\left(1+x^2\right)\left(x^4+x^3+x^2+x+1\right) \times S_n^{[\beta_n]}\left(\varphi_x^2(t)\right)(x)\,\Omega\left(f'';\frac{1}{\sqrt{n}}\right),$$

where C is a positive constant. Thus we have desired result. □

Remark 1. *It is seen that $S_n^{[\beta]}$ does not form an approximation process. In order to transform it into an approximation process, the constant β will be replaced by a number $\beta_n \in [0,1)$ and also*

$$\lim_{n\to\infty} n\beta_n = 0.$$

The following estimate is Voronovskaya type asymptotic formula.

Corollary 1. *Let $f'' \in C_{x^2}^k[0,\infty)$, $x>0$ be fixed and $0 \le \beta_n < 1$. Then, we have*

$$\lim_{n\to\infty} n\left[S_n^{[\beta_n]}(f)(x) - f(x)\right] = \frac{1}{2}\left[f'(x) + xf''(x)\right].$$

4. Grüss Type Approximation

Let us to prove the following result called by us Grüss-Voronovskaya type theorem in quantitative form (see [9]).

Theorem 2. *Suppose that the first and second derivative f', g', f'', g'' and $(fg)''$ exist at a point $x \in [0,\infty)$, we have*

$$
\begin{aligned}
&\sup_{x\geq 0} \frac{n\left|S_n^{[\beta_n]}(fg;x) - S_n^{[\beta_n]}(f;x)\,S_n^{[\beta_n]}(g;x) - xf'(x)\,g'(x)\right|}{(1+x^2)^6} \\
\leq\; & \left\|(fg)^{'}\right\|_{x^2} \frac{n\beta_n}{1-\beta_n} + \frac{\left\|(fg)^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left((fg)'';\frac{1}{\sqrt{n}}\right) \\
& + \|f\|_{x^2}\left\{\left\|g^{'}\right\|_{x^2}\frac{n\beta_n}{1-\beta_n} + \frac{\left\|g^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left(g'';\frac{1}{\sqrt{n}}\right)\right\} \\
& + \|g\|_{x^2}\left\{\left\|f^{'}\right\|_{x^2}\frac{n\beta_n}{1-\beta_n} + \frac{\left\|f^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left(f'';\frac{1}{\sqrt{n}}\right)\right\} \\
& + nA_n(f)\,A_n(g),
\end{aligned}
$$

where

$$
A_n(f) = \|f'\|_{x^2}\left(\frac{\beta_n}{1-\beta_n} + \frac{1}{2n}\right) + \frac{\|f''\|_{x^2}}{2}\gamma_n
$$

and $\gamma_n \to 0$, depending on β_n, $n \to \infty$.

Proof. For $x \in [0,\infty)$, we have

$$
\begin{aligned}
& S_n^{[\beta_n]}(fg;x) - S_n^{[\beta_n]}(f;x)\,S_n^{[\beta_n]}(g;x) - \frac{x}{n}f'(x)\,g'(x) \\
=\; & S_n^{[\beta_n]}(fg;x) - f(x)\,g(x) - \frac{1}{2n}\left(f(x)\,g(x)\right)' - \frac{x}{2n}\left(f(x)\,g(x)\right)'' \\
& - f(x)\left[S_n^{[\beta_n]}(g;x) - g(x) - \frac{1}{2n}g'(x) - \frac{x}{2n}g''(x)\right] \\
& - g(x)\left[S_n^{[\beta_n]}(f;x) - f(x) - \frac{1}{2n}f'(x) - \frac{x}{2n}f''(x)\right] \\
& + \left(g(x) - S_n^{[\beta_n]}(g;x)\right)\left(S_n^{[\beta_n]}(f;x) - f(x)\right) \\
=\; & A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

Thus we can write

$$
\begin{aligned}
& \frac{\left|nS_n^{[\beta_n]}\left((fg;x) - S_n^{[\beta_n]}(f;x)\,S_n^{[\beta_n]}(g;x)\right) - xf'(x)\,g'(x)\right|}{(1+x^2)^4} \\
\leq\; & \frac{n}{(1+x^2)^6}\left\{|A_1| + |A_2| + |A_3| + |A_4|\right\},
\end{aligned}
$$

where

$$\frac{n}{(1+x^2)^6}|A_1| \le \frac{\left| nS_n^{[\beta_n]}((fg;x)-f(x)g(x)) - \frac{1}{2}(f(x)g(x))' - \frac{x}{2}(f(x)g(x))'' \right|}{(1+x^2)^4},$$

$$\frac{n}{(1+x^2)^6}|A_2| \le \frac{|f(x)|}{(1+x^2)} \frac{\left| n\left(S_n^{[\beta_n]}(g;x)-g(x)\right) - \frac{1}{2}g'(x) - \frac{x}{2}g''(x) \right|}{(1+x^2)^4},$$

$$\frac{n}{(1+x^2)^6}|A_3| \le \frac{|g(x)|}{(1+x^2)} \frac{\left| n\left(S_n^{[\beta_n]}(f;x)-f(x)\right) - \frac{1}{2}f'(x) - \frac{x}{2}f''(x) \right|}{(1+x^2)^4},$$

$$\frac{n}{(1+x^2)^6}|A_4| \le n\frac{\left|S_n^{[\beta_n]}(g;x)-g(x)\right|}{(1+x^2)^3} \frac{\left|S_n^{[\beta_n]}(f;x)-f(x)\right|}{(1+x^2)^3}.$$

From Theorem 1, we have

$$\sup_{x\ge 0} \frac{n|A_1|}{(1+x^2)^6} \le \left\|(fg)^{'}\right\|_{x^2} \frac{n\beta_n}{1-\beta_n} + \frac{\left\|(fg)^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left((fg)''; \frac{1}{\sqrt{n}}\right),$$

$$\sup_{x\ge 0} \frac{n|A_2|}{(1+x^2)^6} \le \|f\|_{x^2} \left\{ \left\|g^{'}\right\|_{x^2} \frac{n\beta_n}{1-\beta_n} + \frac{\left\|g^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left(g''; \frac{1}{\sqrt{n}}\right) \right\},$$

$$\sup_{x\ge 0} \frac{n|A_3|}{(1+x^2)^6} \le \|g\|_{x^2} \left\{ \left\|f^{'}\right\|_{x^2} \frac{n\beta_n}{1-\beta_n} + \frac{\left\|f^{''}\right\|_{x^2}}{2}\alpha_n + C\alpha_n\Omega\left(f''; \frac{1}{\sqrt{n}}\right) \right\}.$$

On the other hand, we can write

$$S_n^{[\beta_n]}(f;x) - f(x) = f'(x)\, S_n^{[\beta]}\left(\varphi_x^1(t)\right)(x) + \frac{1}{2} S_n^{[\beta_n]}\left(f''(\xi)(t-x)^2; x\right).$$

Therefore, we have

$$\begin{aligned} \frac{\left|S_n^{[\beta_n]}(f;x)-f(x)\right|}{(1+x^2)^3} &\le \frac{|f'(x)|\, S_n^{[\beta]}\left(\varphi_x^1(t)\right)(x)}{(1+x^2)^3} + \frac{S_n^{[\beta_n]}\left(|f''(\xi)|(t-x)^2; x\right)}{2(1+x^2)^3} \\ &\le \|f'\|_{x^2} \frac{S_n^{[\beta]}\left(\varphi_x^1(t)\right)(x)}{(1+x^2)^2} \\ &\quad + \frac{1}{2}\|f''\|_{x^2} \frac{S_n^{[\beta_n]}\left(\left(1+\zeta^2\right)(t-x)^2; x\right)}{(1+x^2)^2} \end{aligned}$$

where ζ is a number between t and x.

Case 1: $t < \zeta < x$;

$$\frac{\left|S_n^{[\beta_n]}(f;x)-f(x)\right|}{(1+x^2)^3} \le \|f'\|_{x^2} \frac{S_n^{[\beta]}\left(\varphi_x^1(t)\right)(x)}{(1+x^2)^2} + \frac{1}{2}\|f''\|_{x^2} \frac{S_n^{[\beta]}\left(\varphi_x^2(t)\right)(x)}{(1+x^2)}.$$

Case 2: $x < \zeta < t$;

$$\begin{aligned}
&\frac{\left|S_n^{[\beta_n]}(f;x) - f(x)\right|}{\left(1+x^2\right)^3} \\
\leq\ & \|f'\|_{x^2} \frac{S_n^{[\beta]}\left(\varphi_x^1(t)\right)(x)}{\left(1+x^2\right)^2} + \frac{1}{2}\|f''\|_{x^2} \frac{S_n^{[\beta_n]}\left(\left(1+t^2\right)(t-x)^2;x\right)}{\left(1+x^2\right)^2} \\
=\ & \|f'\|_{x^2} \frac{\left(S_n^{[\beta]}\varphi_x^1(t)\right)(x)}{\left(1+x^2\right)^2} + \frac{1}{2}\|f''\|_{x^2} \left\{\frac{S_n^{[\beta]}\left(\varphi_x^2(t)\right)(x)}{1+x^2}\right. \\
& \left. +2x\frac{S_n^{[\beta]}\left(\varphi_x^3(t)\right)(x)}{\left(1+x^2\right)^2} + \frac{S_n^{[\beta]}\left(\varphi_x^4(t)\right)(x)}{\left(1+x^2\right)^2}\right\}.
\end{aligned}$$

Thus, we obtain for two cases of ζ that

$$\begin{aligned}
&\sup_{x\geq 0}\frac{\left|S_n^{[\beta_n]}(f;x) - f(x)\right|}{\left(1+x^2\right)^3} \\
\leq\ & \|f'\|_{x^2} \sup_{x\geq 0}\frac{\left(S_n^{[\beta]}\varphi_x^1(t)\right)(x)}{\left(1+x^2\right)^2} + \frac{1}{2}\|f''\|_{x^2}\left\{\sup_{x\geq 0}\frac{\left(S_n^{[\beta]}\varphi_x^2(t)\right)(x)}{1+x^2}+\right. \\
& \left. 2\sup_{x\geq 0} x\frac{\left(S_n^{[\beta]}\varphi_x^3(t)\right)(x)}{\left(1+x^2\right)^2} + \sup_{x\geq 0}\frac{\left(S_n^{[\beta]}\varphi_x^4(t)\right)(x)}{\left(1+x^2\right)^2}\right\} \\
=\ & \|f'\|_{x^2}\left(\frac{\beta_n}{1-\beta_n}+\frac{1}{2n}\right) + \frac{\|f''\|_{x^2}}{2}\gamma_n := A_n(f).
\end{aligned}$$

Thus the proof is completed. □

5. Weight Approximation

Now, in this section we give some weight approximation theorems for the functions which belong to weighted space $C_{x^2}^k[0,\infty)$ by $S_n^{[\beta]}$ operators. For details of proofs see [2] and [8].

Theorem 3. *If $f \in C_{x^2}^k[0,\infty)$. then the inequality*

$$\sup_{x\geq 0}\frac{\left|S_n^{[\beta_n]}(f)(x) - f(x)\right|}{\left(1+x^2\right)^{\frac{5}{2}}} \leq K\Omega\left(f;\sqrt{\frac{1-\beta_n}{n}}\right)$$

is satisfied for a sufficiently large n, where K is a constant.

Theorem 4. *For each $f \in C_{x^2}^k[0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|S_n^{[\beta_n]}(f) - f\right\|_{x^2} = 0.$$

Now, we give the following theorem to approximation all functions in $C_{x^2}[0,\infty)$. This type of results is given in Gadjiev et al. [7] for locally integrable functions.

Theorem 5. *For each* $f \in C_{x^2}[0,\infty)$ *and* $\alpha > 0$, *we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{\left| S_n^{[\beta_n]}(f)(x) - f(x) \right|}{\left(1+x^2\right)^{1+\alpha}} = 0.$$

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