



## UNIVALENCE OF CERTAIN INTEGRAL OPERATORS INVOLVING NORMALIZED WRIGHT FUNCTIONS

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ABSTRACT. In this paper our main aim is to give some sufficient conditions for functions represented with normalized Wright functions to be univalent in the open unit disk. The key tools in our proofs are the Becker's and the generalized version of the well-known Ahlfors's and Becker's univalence criteria.

### 1. INTRODUCTION

Let  $A$  be the class of analytic functions  $f(z)$  in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ , normalized by  $f(0) = 0 = f'(0) - 1$  of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

It is well-known that a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is said to be univalent if the following condition is satisfied:  $z_1 = z_2$  if  $f(z_1) = f(z_2)$ . We denote by  $S$  the subclass of  $A$  consisting of functions which are also univalent in  $U$ .

For some recent investigations of various subclasses of the univalent functions class  $S$ , see the works by Altıntaş *et al.* [1], Gao *et al.* [7], and Owa *et al.* [8]. In recent years there have been many studies (see for example [2-6, 9, 10]) on the univalence of the following integral operators:

$$G_p(z) = \left\{ p \int_0^z t^{p-1} f'(t) dt \right\}^{1/p}, \quad (1.2)$$

$$G_{p,q}(z) = \left\{ p \int_0^z t^{p-1} \left( \frac{f(t)}{t} \right)^q dt \right\}^{1/p} \quad (1.3)$$

and

$$G_q(z) = \left\{ p \int_0^z t^{p-1} \left( e^{f(t)} \right)^q dt \right\}^{1/p} \quad (1.4)$$

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where the function  $f(z)$  belong to the class  $A$  and the parameters  $p, q$  are complex numbers such that the integrals in (1.2)-(1.4) exist. Furthermore, Breaz *et al.* [5] have obtained various sufficient conditions for the univalence of the following integral operator:

$$G_{n,\alpha}(z) = \left\{ [n(\alpha - 1) + 1] \int_0^z \left( \prod_{k=1}^n f_k(t) \right)^{\alpha-1} dt \right\}^{1/[n(\alpha-1)+1]} \quad (1.5)$$

where  $n$  is a natural number,  $\alpha$  is a real number and functions  $f_k \in A$ ,  $k = 1, \dots, n$ . By Baricz and Frasin [2] was obtained some sufficient conditions for the univalence of the integral operators of the type (1.3)-(1.5) when the function  $f(z)$  is the normalized Bessel function with various parameters.

The Wright function is defined by the following infinite series:

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!} \quad (1.6)$$

where  $\Gamma$  is Euler gamma function,  $\lambda > -1$ ,  $\mu, z \in \mathbb{C}$ . This series is absolutely convergent in  $\mathbb{C}$ , when  $\lambda > -1$  and absolutely convergent in open unit disk for  $\lambda = -1$ . Furthermore, for  $\lambda > -1$  the Wright function  $W_{\lambda,\mu}(z)$  is an entire function. The Wright function was introduced by Wright in [12] and has appeared for the first time in the case  $\lambda > 0$  in connection with his investigation in the asymptotic theory of partitions. Later on, it has found many other applications, first of all, in the Mikusinski operational calculus and in the theory of integral transforms of Hankel type. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright functions and of the integral operators involving Wright functions.

Note that Wright function  $W_{\lambda,\mu}(z)$ , defined by (1.6) does not belong to the class  $A$ . Thus, it is natural to consider the following two kinds of normalization of the Wright function:

$$W_{\lambda,\mu}^{(1)}(z) := \Gamma(\mu)zW_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda n + \mu)} \frac{z^{n+1}}{n!}, \quad \lambda > -1, \mu > 0, z \in U$$

and

$$W_{\lambda,\mu}^{(2)}(z) := \Gamma(\lambda + \mu) \left[ W_{\lambda,\mu}(z) - \frac{1}{\Gamma(\mu)} \right] = \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda n + \lambda + \mu)} \frac{z^{n+1}}{(n+1)!},$$

$$\lambda > -1, \lambda + \mu > 0, z \in U.$$

Easily, we write

$$W_{\lambda,\mu}^{(1)}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1) + \mu)} \frac{z^n}{(n-1)!}, \quad \lambda > -1, \mu > 0, z \in U \quad (1.7)$$

and

$$W_{\lambda,\mu}^{(2)}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda n + \mu)} \frac{z^n}{n!}, \quad \lambda > -1, \lambda + \mu > 0, z \in U. \quad (1.8)$$

Note that

$$W_{1,p+1}^{(1)}(-z) = -J_p^{(1)}(z) = \Gamma(p+1)z^{1-p/2}J_p(2\sqrt{z})$$

where  $J_p(z)$  is the Bessel function and  $J_p^{(1)}(z)$  the normalized Bessel function.

In this paper, we give various sufficient conditions for integral operators of type (1.2)-(1.4) when the function  $f(z)$  is the normalized Wright functions to be univalent in the open unit disk  $U$ . We would like to show that the univalence of integral operators which involve normalized Wright functions can be derived easily via some well-known univalence criteria.

## 2. PRELIMINARIES

In this section, we give the necessary information and lemmas, which shall need in our investigation.

In our investigation, we shall need the following lemmas.

**Lemma 1** ([3]). *If  $f \in A$  and the following condition is satisfied:*

$$(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

for all  $z \in U$  then the function  $f(z)$  is univalent in  $U$ .

**Lemma 2** ([10]). *Let  $q \in \mathbb{C}$  and  $a \in \mathbb{R}$  such that  $\operatorname{Re}(q) \geq 1$ ,  $a > 1$  and  $2a|q| \leq 3\sqrt{3}$ . If  $f \in A$  satisfies the inequality  $|zf'(z)| \leq a$  for all  $z \in U$  then the function  $G_q : U \rightarrow \mathbb{C}$  defined by (1.4) univalent in  $U$ .*

**Lemma 3** ([9]). *Let  $p$  and  $c$  be complex numbers such that  $\operatorname{Re}(p) > 0$  and  $|c| \leq 1$ ,  $c \neq -1$ . If the function  $f \in A$  satisfies the inequality*

$$\left| c|z|^{2p} + (1 - |z|^{2p}) \frac{zf''(z)}{pf'(z)} \right| \leq 1$$

for all  $z \in U$  then the function  $G_p : U \rightarrow \mathbb{C}$  defined by (1.2) is univalent in  $U$ .

We shall need, also the following results.

**Lemma 4.** *Let  $\lambda \geq 1$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation*

$$2\mu - (\mu + 1)e^{\frac{1}{\mu+1}} + 1 = 0. \quad (2.1)$$

Then, the following inequalities hold for all  $z \in U$

$$\left| \frac{z \left( W_{\lambda,\mu}^{(1)}(z) \right)'}{W_{\lambda,\mu}^{(1)}(z)} - 1 \right| \leq \frac{e^{1/(\mu+1)}}{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}, \quad (2.2)$$

$$\left| z \left( W_{\lambda,\mu}^{(1)}(z) \right)' \right| \leq 1 + \frac{1}{\mu} \left\{ (\mu + 2)e^{\frac{1}{\mu+1}} - (\mu + 1) \right\}. \quad (2.3)$$

*Proof.* By using the definition of the normalized Wright function  $W_{\lambda,\mu}^{(1)}(z)$ , we obtain for all  $z \in U$

$$\left| \frac{z \left( W_{\lambda, \mu}^{(1)}(z) \right)' }{W_{\lambda, \mu}^{(1)}(z)} - 1 \right| = \left| \frac{z \left( W_{\lambda, \mu}^{(1)}(z) \right)' - W_{\lambda, \mu}^{(1)}(z)}{W_{\lambda, \mu}^{(1)}(z)} \right| \leq \frac{\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-2)!}}{1 - \sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!}}.$$

Under hypothesis  $\lambda \geq 1$ , the inequality  $\Gamma(n-1+\mu) \leq \Gamma(\lambda(n-1)+\mu)$ ,  $n \in \mathbb{N}$  holds, which is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \leq \frac{1}{(\mu)_{n-1}}, \quad n \in \mathbb{N} \quad (2.4)$$

where  $(\mu)_n = \Gamma(n+\mu)/\Gamma(\mu) = \mu(\mu+1) \cdots (\mu+n-1)$ ,  $(\mu)_0 = 1$  is Pochhammer (or Appell) symbol, defined in terms of Euler gamma function.

Using (2.4), we obtain

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-2)!} \leq \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \frac{1}{(\mu)_{n-1}}.$$

Further, the inequality

$$(\mu)_{n-1} = \mu(\mu+1) \cdots (\mu+n-2) \geq \mu(\mu+1)^{n-2}, \quad n \in \mathbb{N} \quad (2.5)$$

is true, which is equivalent to  $1/(\mu)_{n-1} \leq 1/\mu(\mu+1)^{n-2}$ ,  $n \in \mathbb{N}$ . Using (2.5), we get

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-2)!} \leq \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \frac{1}{\mu(\mu+1)^{n-2}} = \frac{e^{1/(\mu+1)}}{\mu}. \quad (2.6)$$

Similarly, we have

$$\sum_{n=2}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} \frac{1}{(n-1)!} \leq \frac{\mu+1}{\mu} \left( e^{1/(\mu+1)} - 1 \right). \quad (2.7)$$

Combining inequalities (2.6) and (2.7), we immediately get that first assertion (2.2) of Lemma 4 holds.

Let's prove second assertion of lemma. From the definition of the normalized Wright function  $W_{\lambda, \mu}^{(1)}(z)$ , we have

$$\left| z \left( W_{\lambda, \mu}^{(1)}(z) \right)' \right| \leq 1 + \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)} + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)}.$$

Using (2.6) and (2.7), we get

$$\begin{aligned} \left| z \left( W_{\lambda, \mu}^{(1)}(z) \right)' \right| &\leq 1 + \frac{e^{1/(\mu+1)}}{\mu} + \frac{\mu+1}{\mu} \left( e^{1/(\mu+1)} - 1 \right) \\ &= 1 + \frac{1}{\mu} \left[ (\mu+2)e^{1/(\mu+1)} - (\mu+1) \right]. \end{aligned}$$

Thus, the proof of Lemma 4 is complete.

For the normalized Wright function  $W_{\lambda, \mu}^{(2)}(z)$ , we can give the following lemma.

**Lemma 5.** *Let  $\lambda \geq 1$  and  $\lambda + \mu > x_0$  where  $x_0 \cong 1.2581$  is the root of the equation*

$$2x - (x+1)e^{\frac{1}{x+1}} + 1 = 0. \quad (2.8)$$

*Then, the following inequalities hold for all  $z \in U$*

$$\left| \frac{z \left( W_{\lambda, \mu}^{(2)}(z) \right)'}{W_{\lambda, \mu}^{(1)}(z)} - 1 \right| \leq \frac{(\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)}{(\lambda + \mu) - (\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)}, \quad (2.9)$$

$$\left| z \left( W_{\lambda, \mu}^{(2)}(z) \right)' \right| \leq 1 + \frac{\lambda + \mu + 1}{\lambda + \mu} (e^{1/(\mu+1)} - 1). \quad (2.10)$$

*Proof.* The proof of this lemma is very similar to the proof of Lemma 4, so the details of the proof may be omitted.

### 3. UNIVALENCE OF INTEGRAL OPERATORS INVOLVING WRIGHT FUNCTIONS

In this section our main aim is to give sufficient conditions for the integral operators of the type (1.2)–(1.4) when the function  $f(z)$  is the normalized Wright functions to be univalent in the open unit disk  $U$ . To this end, firstly we consider the following integral operator:

$$G_{\lambda, \mu}^q(z) = \int_0^z \left( \frac{W_{\lambda, \mu}^{(1)}(t)}{t} \right)^q dt, \quad \lambda > -1, \quad \mu > 0, \quad z \in U. \quad (3.1)$$

For this integral operator, we can give the following theorem.

**Theorem 1.** *Let  $\lambda \geq 1$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation (2.1). Moreover, suppose that  $q$  is a complex number such that*

$$|q| \leq \frac{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}{e^{1/(\mu+1)}}.$$

*Then, the function  $G_{\lambda, \mu}^q : U \rightarrow \mathbb{C}$  defined by (3.1) is univalent in  $U$ .*

*Proof.* Since  $W_{\lambda, \mu}^{(1)} \in A$ , clearly  $G_{\lambda, \mu}^q \in A$ , i. e.  $G_{\lambda, \mu}^q(0) = \left( G_{\lambda, \mu}^q(0) \right)' - 1 = 0$ . On the other hand, it is easy to see that

$$\left( G_{\lambda, \mu}^q(z) \right)' = \left( \frac{W_{\lambda, \mu}^{(1)}(z)}{z} \right)^q$$

and

$$\frac{z \left( G_{\lambda, \mu}^q(z) \right)''}{\left( G_{\lambda, \mu}^q(z) \right)'} = q \left[ \frac{z \left( W_{\lambda, \mu}^{(1)}(z) \right)'}{W_{\lambda, \mu}^{(1)}(z)} - 1 \right]. \quad (3.2)$$

By using first assertion (2.2) of Lemma 4, we obtain

$$\left| \frac{z \left( G_{\lambda, \mu}^q(z) \right)''}{\left( G_{\lambda, \mu}^q(z) \right)'} \right| \leq |q| \left| \frac{z \left( W_{\lambda, \mu}^{(1)}(z) \right)'}{W_{\lambda, \mu}^{(1)}(z)} - 1 \right| \leq \frac{|q| e^{1/(\mu+1)}}{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}$$

for all  $z \in U$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation

$$(2\mu + 1) - (\mu + 1)e^{\frac{1}{\mu+1}} = 0.$$

Hence, for all  $z \in U$  and  $\mu > \mu_0$ , we write the following inequality:

$$(1 - |z|^2) \left| \frac{z \left( G_{\lambda, \mu}^q(z) \right)''}{\left( G_{\lambda, \mu}^q(z) \right)'} \right| \leq (1 - |z|^2) \frac{|q| e^{1/(\mu+1)}}{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}.$$

This last expression is bounded by 1 if

$$|q| \leq \frac{(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}}{e^{1/(\mu+1)}}.$$

But, this is true by hypothesis of theorem. Thus, according to the Lemma 1, function  $G_{\lambda, \mu}^q(z)$  is univalent in  $U$ . With this the proof of Theorem 1 is complete. By setting  $q = 1$  in Theorem 1, we have the following result.

**Corollary 1.** *Let  $\lambda \geq 1$  and  $\mu > \mu_1$  where  $\mu_1 \cong 2.4898$  is the root of the equation*

$$(2\mu + 1) - (\mu + 2)e^{\frac{1}{\mu+1}} = 0. \quad (3.3)$$

*Then, the uncton  $G_{\lambda, \mu} : U \rightarrow \mathbb{C}$  defined by*

$$G_{\lambda, \mu}(z) = \int_0^z \frac{W_{\lambda, \mu}^{(1)}(t)}{t} dt$$

*is univalent in  $U$ .*

If we take  $\lambda = 1$ ,  $\mu = p + 1$  in Theorem 1, we arrive at the following corollary.

**Corollary 2.** *The function  $G_p^q : U \rightarrow \mathbb{C}$  defined by*

$$G_p^q(z) = \int_0^z \left( \frac{J_p^{(1)}(-t)}{t} \right)^q dt$$

*is univalent in  $U$  if  $p > \mu_0 - 1$  where  $\mu_0 \cong 1.2581$  is the root of the equation (2.1) and  $q$  is a complex number such that*

$$|q| \leq \frac{(2p + 3) - (p + 2)e^{1/(p+2)}}{e^{1/(p+2)}}$$

*Here, function  $J_p^{(1)}(z)$  is normalized Bessel function.*

By taking  $q = 1$  in Corollary 2, we obtain the following result.

**Corollary 3.** *Let  $p > \mu_1 - 1$  where  $\mu_1 \cong 2.4898$  is the root of the equation (3.3).*

*Then, the function  $G_p : U \rightarrow \mathbb{C}$  defined by*

$$G_p(z) = \int_0^z \frac{J_p^{(1)}(-t)}{t} dt$$

*is univalent in  $U$ . Here,  $J_p^{(1)}(z)$  is normalized Bessel function.*

For the integral operator

$$F_{\lambda, \mu}^q(z) = \int_0^z \left( \frac{W_{\lambda, \mu}^{(2)}(t)}{t} \right)^q dt, \quad \lambda > -1, \lambda + \mu > 0, z \in U \quad (3.4)$$

we can give the following theorem.

**Theorem 2.** *Let  $\lambda \geq 1$  and  $\lambda + \mu > x_0$  where  $x_0 \cong 1.2581$  is the root of the equation (2.8). Moreover, suppose that  $q$  is a complex number such that*

$$|q| \leq \frac{(\lambda + \mu) - (\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}{(\lambda + \mu + 1)(e^{1/(\mu+1)} - 1)}.$$

Then, the function  $F_{\lambda,\mu}^q : U \rightarrow \mathbb{C}$  defined by (3.4) is univalent in  $U$ .

*Proof.* The proof of this theorem is very similar to the proof of Theorem 1, so the details of the proof may be omitted.

By setting  $q = 1$  in Theorem 2, we have the following corollary.

**Corollary 4.** *Let  $\lambda \geq 1$  and  $\lambda + \mu > x_1$  where  $x_1 \cong 2.3325$  is the root of the equation*

$$3x - 2(x + 1)e^{\frac{1}{x+1}} + 2 = 0. \quad (3.5)$$

Then, the function  $F_{\lambda,\mu} : U \rightarrow \mathbb{C}$  defined by

$$F_{\lambda,\mu}(z) = \int_0^z \frac{W_{\lambda,\mu}^{(2)}(t)}{t} dt$$

is univalent in  $U$ .

Now, we consider the following integral operator:

$$G_{\lambda,\mu}^{p,q}(z) = \left\{ p \int_0^z t^{p-1} \left( \frac{W_{\lambda,\mu}^{(1)}(t)}{t} \right)^q dt \right\}^{1/p}, \quad \lambda > -1, \mu > 0, z \in U. \quad (3.6)$$

On the univalence of the function  $G_{\lambda,\mu}^{p,q}(z)$ , we give the following theorem.

**Theorem 3.** *Let  $\lambda \geq 1$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation (2.1). Moreover, suppose that  $p, q$  and  $c$  be complex numbers such that  $\operatorname{Re}(p) > 0$ ,  $|c| < 1$  and the following condition is satisfied:*

$$|c| \leq 1 - \frac{|q| e^{1/(\mu+1)}}{|p| [(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}]}.$$

Then, the integral operator  $G_{\lambda,\mu}^{p,q} : U \rightarrow \mathbb{C}$  defined by (3.6) is univalent in  $U$ .

*Proof.* We can rewrite the integral operator (3.6) as

$$G_{\lambda,\mu}^{p,q}(z) = \left\{ p \int_0^z t^{p-1} \left( G_{\lambda,\mu}^q(t) \right)' dt \right\}^{1/p} \quad (3.7)$$

where function  $G_{\lambda,\mu}^q : U \rightarrow \mathbb{C}$  is defined in (3.1).

Under hypothesis of theorem, using (3.2) and (2.2), we obtain

$$\left| c |z|^{2p} + (1 - |z|^{2p}) \frac{z \left( G_{\lambda,\mu}^q(z) \right)''}{p \left( G_{\lambda,\mu}^q(z) \right)'} \right| \leq |c| + \frac{|q| e^{1/(\mu+1)}}{|p| [(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}]}.$$

This last expression is bounded by 1 if

$$|c| \leq 1 - \frac{|q| e^{1/(\mu+1)}}{|p| [(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}]}.$$

But this is true by hypothesis of theorem. Thus, according to the Lemma 3, function  $G_{\lambda,\mu}^{p,q}(z)$  defined by (3.7) is univalent in  $U$ . With this, the proof of Theorem 3 is complete.

By setting  $q = 1$  in Theorem 3, we arrive at the following result.

**Corollary 5.** Let  $\lambda \geq 1$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation (2.1). Moreover, suppose that  $p$  and  $c$  be complex numbers such that  $\text{Re}(p) > 0$ ,  $|c| < 1$  and the following condition is satisfied:

$$|c| \leq 1 - \frac{e^{1/(\mu+1)}}{|p| [(2\mu + 1) - (\mu + 1)e^{1/(\mu+1)}]}.$$

Then, the integral operator  $G_{\lambda,\mu}^p : U \rightarrow \mathbb{C}$  defined by

$$G_{\lambda,\mu}^p(z) = \left\{ p \int_0^z t^{p-2} W_{\lambda,\mu}^{(1)}(t) dt \right\}^{1/p}. \quad (3.8)$$

is univalent in  $U$ .

**Remark 1.** Note that, recently the function  $G_{\lambda,\mu}^p : U \rightarrow \mathbb{C}$  defined by (3.8) was investigated by Prajapat [11] and he obtained some sufficient conditions for the univalence this function.

Now, on the univalence of the integral operator

$$F_{\lambda,\mu}^{p,q}(z) = \left\{ p \int_0^z t^{p-1} \left( \frac{W_{\lambda,\mu}^{(2)}(t)}{t} \right)^q dt \right\}^{1/p}, \quad \lambda > -1, \lambda + \mu > 0, z \in U \quad (3.9)$$

we can give the following theorem.

**Theorem 4.** Let  $\lambda \geq 1$  and  $\lambda + \mu > x_0$  where  $x_0 \cong 1.2581$  is the root of the equation (2.8). Moreover, suppose that  $p, q$  and  $c$  be complex numbers such that  $\text{Re}(p) > 0$ ,  $|c| < 1$  and the following condition is satisfied:

$$|c| \leq 1 - \frac{|q| (\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)}{|p| [(\lambda + \mu) - (\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)]}.$$

Then, the integral operator  $F_{\lambda,\mu}^{p,q} : U \rightarrow \mathbb{C}$  defined by (3.9) is univalent in  $U$ .

*Proof.* The proof of Theorem 4 is similar to the proof of Theorem 3. Hence, the details of the proof of Theorem 4 may be omitted.

By setting  $q = 1$  in Theorem 4, we obtain the following corollary.

**Corollary 6.** Let  $\lambda \geq 1$  and  $\lambda + \mu > x_0$  where  $x_0 \cong 1.2581$  is the root of the equation (2.8). Moreover, suppose that  $p$  and  $c$  be complex numbers such that  $\text{Re}(p) > 0$ ,  $|c| < 1$  and the following condition is satisfied:

$$|c| \leq 1 - \frac{(\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)}{|p| [(\lambda + \mu) - (\lambda + \mu + 1) (e^{1/(\mu+1)} - 1)]}.$$

Then, the function  $F_{\lambda,\mu}^p : U \rightarrow \mathbb{C}$  defined by

$$F_{\lambda,\mu}^p(z) = \left\{ p \int_0^z t^{p-2} W_{\lambda,\mu}^{(2)}(t) dt \right\}^{1/p}, \quad z \in U \quad (3.10)$$

is univalent in  $U$ .

Now, we consider integral operator of the type (1.4) when the function  $f(z)$  is the normalized Wright function.

Let

$$H_{\lambda,\mu}^q(z) = \left\{ q \int_0^z t^{q-1} \left( e^{W_{\lambda,\mu}^{(1)}(t)} \right)^q \right\}^{1/q}, \quad \lambda > -1, \lambda + \mu > 0, z \in U. \quad (3.11)$$



On univalence of the function (3.11), we give the following theorem.

**Theorem 5.** *Let  $q \in \mathbb{C}$ ,  $\lambda \geq 1$  and  $\mu > \mu_0$  where  $\mu_0 \cong 1.2581$  is the root of the equation (2.1). If  $\operatorname{Re}(q) \geq 1$  and the following condition is satisfied:*

$$|q| \leq \frac{3\sqrt{3}\mu}{2[(\mu+2)e^{1/(\mu+1)} - 1]} \quad (3.12)$$

then the function  $H_{\lambda,\mu}^q : U \rightarrow \mathbb{C}$  defined by (3.11) is univalent in  $U$ .

*Proof.* From (2.3), we write

$$\left| z \left( W_{\lambda,\mu}^{(1)}(z) \right)' \right| \leq 1 + \frac{1}{\mu} \left\{ (\mu+2)e^{\frac{1}{\mu+1}} - (\mu+1) \right\}$$

for all  $z \in U$ . Taking

$$a = 1 + \frac{1}{\mu} \left\{ (\mu+2)e^{\frac{1}{\mu+1}} - (\mu+1) \right\},$$

we easily see that  $2a|q| \leq 3\sqrt{3}$  if provided (3.12). Thus, under hypothesis of theorem, all hypothesis of the Lemma 2 is provided. Hence, the proof of Theorem 5 is complete.

By setting  $q = 1$  in Theorem 5, we have the following result.

**Corollary 7.** *Let  $\lambda \geq 1$  and  $\mu > \mu_1$  where  $\mu_1 \cong 1.6692$  is the root of the equation*

$$3\sqrt{3}\mu - 2(\mu+2)e^{1/(\mu+1)} + 2 = 0. \quad (3.13)$$

Then, the function  $H_{\lambda,\mu} : U \rightarrow \mathbb{C}$  defined by

$$H_{\lambda,\mu}(z) = \int_0^z e^{W_{\lambda,\mu}^{(1)}(t)} dt$$

is univalent in  $U$ .

Now, let

$$Q_{\lambda,\mu}^q(z) = \left\{ q \int_0^z t^{q-1} \left( e^{W_{\lambda,\mu}^{(2)}(t)} \right)^q \right\}^{1/q}, \quad \lambda > -1, \lambda + \mu > 0, z \in U. \quad (3.14)$$

For the function (3.14), we can give the following theorem which will be proved similarly to the Theorem 5.

**Theorem 6.** *Let  $q \in \mathbb{C}$ ,  $\lambda \geq 1$  and  $\lambda + \mu > x_0$  where  $x_0 \cong 1.2581$  is the root of the equation (2.1). If  $\operatorname{Re}(q) \geq 1$  and the following condition is satisfied:*

$$|q| \leq \frac{3\sqrt{3}(\lambda + \mu)}{2[(\lambda + \mu + 1)e^{1/(\lambda + \mu + 1)} - 1]}$$

then the function  $Q_{\lambda,\mu}^q : U \rightarrow \mathbb{C}$  defined by (3.14) is univalent in  $U$ .

By setting  $q = 1$  in Theorem 6, we obtain the following corollary.

**Corollary 8.** *Let  $\lambda \geq 1$  and  $\lambda + \mu > x_1$  where  $x_1 \cong 0.83232$  is the root of the equation*

$$3\sqrt{3}x - 2(x+1)e^{1/(x+1)} + 2 = 0.$$

Then, the function  $Q_{\lambda,\mu} : U \rightarrow \mathbb{C}$  defined by

$$Q_{\lambda,\mu}(z) = \int_0^z e^{W_{\lambda,\mu}^{(2)}(t)} dt$$

is univalent in  $U$ .

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