



ON THE EXISTENCE OF ε -OPTIMAL TRAJECTORIES OF THE CONTROL SYSTEMS WITH CONSTRAINED CONTROL RESOURCES

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ABSTRACT. The control system described by a Urysohn type integral equation is considered. It is assumed that the admissible control functions are chosen from the closed ball of the space L_p , $p > 1$, with radius r and centered at the origin. Precompactness of the set of trajectories of the control system in the space of continuous functions is shown. This allows to prove that optimal control problem with lower semicontinuous payoff functional has an ε -optimal trajectory for every $\varepsilon > 0$.

1. INTRODUCTION

Integral equations arise in many problems of contemporary physics and mechanics (see, e.g. [1], [3], [4], [11], [13], [15], [18] and references therein). Pointing out the importance of the integral equations, W. Heisenberg in his well known "Physics and Philosophy" writes: "The final equation of motion for matter will probably be some quantized nonlinear wave equation... This wave equation will probably be equivalent to rather complicated sets of integral equations..." (see, [6], page 68). Often the processes which are described by the integral equations have exterior influences called control efforts or uncertainties of the systems, depending on the characters of these influences. In this paper it will be assumed that exterior influences are control efforts and control functions characterizing the control efforts have an integral constraint. Integral constraint on the control functions is inevitable if the control resource is exhausted by consumption, such as energy, fuel, food and finance (see, e.g. [5], [14], [16], [17]).

In papers [8], [9] various topological properties of the sets of trajectories of the control systems described by the nonlinear Volterra type integral equations with integral constraint on the control functions are studied. In [7] the approximation of

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the sets of trajectories of the aforementioned systems is discussed. A similar problem for the systems described by the ordinary differential equations is considered in [5]. Existence of optimal controls and controllability of the systems described by the Urysohn type integral equations are discussed in [2], [12] where it is assumed that control functions have a geometric constraint.

In the presented paper existence of ε -optimal trajectories of the control systems described by the Urysohn type integral equations is investigated. The closed ball of the space L_p , $p > 1$, with radius r and centered at the origin is chosen as the set of admissible control functions which means that admissible control functions have an integral constraint. Precompactness of the set of trajectories generated by all admissible control functions is established. Using this result it is proved that optimal minimization control problem with lower semicontinuous payoff functional has an ε -optimal trajectory for every $\varepsilon > 0$.

The paper is organized as follows: In Section 2 the conditions are formulated which satisfy the system equation (Conditions A, B and C). In Section 3 it is shown that under accepted conditions, every admissible control function generates unique trajectory of the system (Theorem 3.1). In Section 4 it is proved that the set of trajectories of the system is bounded (Theorem 4.1). In Section 5 it is shown that the sections of the set of trajectories is continuous with respect to the Hausdorff metric (Proposition 5.2) and the set of trajectories is a precompact set in the space of continuous functions (Theorem 5.1). Existence of ε -optimal trajectories for optimal minimization control problem is proved (Theorem 5.2).

2. PRELIMINARIES

The control system described by a Urysohn type integral equation

$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_{\Omega} K(\xi, s, x(s), u(s)) ds \quad (2.1)$$

is considered, where $x \in \mathbb{R}^n$ is the state vector of the system, $u \in \mathbb{R}^m$ is the control vector, $\xi \in \Omega$, $\Omega \subset \mathbb{R}^k$ is a compact set, $\lambda > 0$ is a real number.

For given $p > 1$ and $r > 0$ we set

$$U_{p,r} = \left\{ u(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|u(\cdot)\|_p \leq r \right\}, \quad (2.2)$$

where $L_p(\Omega; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $u(\cdot) : \Omega \rightarrow \mathbb{R}^m$ such that $\|u(\cdot)\|_p < +\infty$, $\|u(\cdot)\|_p = \left(\int_{\Omega} \|u(s)\|^p ds \right)^{\frac{1}{p}}$, $\|\cdot\|$ denotes the Euclidean norm.

The set $U_{p,r} \subset L_p(\Omega, \mathbb{R}^m)$ is called the set of admissible control functions and every function $u(\cdot) \in U_{p,r}$ is called admissible control function.

We assume that the functions $f(\cdot) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $K(\cdot) : \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and number $\lambda \in (0, \infty)$ given in equation (2.1) satisfy the following conditions:

A. the functions $f(\cdot) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $K(\cdot) : \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ are continuous;

B. there exist $M_0 \in [0, 1)$, $M_1 \geq 0$, $H_1 \geq 0$, $M_2 \geq 0$, $H_2 \geq 0$, $M_3 \geq 0$ and $H_3 \geq 0$ such that

$$\|f(\xi, x_1) - f(\xi, x_2)\| \leq M_0 \|x_1 - x_2\|,$$

$$\begin{aligned} \|K(\xi_1, s, x_1, u_1) - K(\xi_2, s, x_2, u_2)\| &\leq [M_1 + H_1 (\|u_1\| + \|u_2\|)] \|\xi_1 - \xi_2\| \\ &\quad + [M_2 + H_2 (\|u_1\| + \|u_2\|)] \|x_1 - x_2\| \\ &\quad + [M_3 + H_3 (\|x_1\| + \|x_2\|)] \|u_1 - u_2\| \end{aligned}$$

for every $(\xi_1, s, x_1, u_1) \in \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m$, $(\xi_2, s, x_2, u_2) \in \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m$;

C. the inequality $0 \leq \lambda \left(M_2 \mu(\Omega) + 2H_* \mu(\Omega)^{\frac{p-1}{p}} r \right) < 1 - M_0$ is satisfied, where $\mu(\Omega)$ is the Lebesgue measure of the set Ω ,

$$H_* = \max \{H_1, H_2, H_3\}. \quad (2.3)$$

If the function $K(\cdot) : \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is Lipschitz continuous, then it satisfies the conditions A and B.

Now let us define the trajectory of the system (2.1) generated by a given admissible control function.

Let $u(\cdot) \in U_{p,r}$. A continuous function $x(\cdot) : \Omega \rightarrow \mathbb{R}^n$ satisfying the equation (2.1) for every $\xi \in \Omega$ is said to be a trajectory of the system (2.1) generated by the admissible control function $u(\cdot) \in U_{p,r}$.

We denote by $\mathbf{X}_{p,r}$ the set of all trajectories of the system (2.1) generated by all admissible control functions $u(\cdot) \in U_{p,r}$. The set $\mathbf{X}_{p,r}$ is called the set of trajectories of the system (2.1).

For each fixed $\xi \in \Omega$ we set

$$\mathbf{X}_{p,r}(\xi) = \{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,r}\}. \quad (2.4)$$

Now let us give a proposition which will be used in following arguments.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^k$ be a compact set, $\nu(\cdot) : \Omega \rightarrow \mathbb{R}$ and $r(\cdot) : \Omega \rightarrow \mathbb{R}$ be continuous functions, $\psi(\cdot) : \Omega \rightarrow [0, +\infty)$ be a Lebesgue integrable function,*

$\int_{\Omega} \psi(s) ds < 1$ and

$$\nu(\xi) \leq r(\xi) + \int_{\Omega} \psi(s) \nu(s) ds$$

for every $\xi \in \Omega$. Then the inequality

$$\nu(\xi) \leq r(\xi) + \frac{\int_{\Omega} r(s) \psi(s) ds}{1 - \int_{\Omega} \psi(s) ds} \quad (2.5)$$

holds for every $\xi \in \Omega$.

Moreover, if $r(\xi) = r_0$ for every $\xi \in \Omega$, then it follows from (2.5) that

$$\nu(\xi) \leq \frac{r_0}{1 - \int_{\Omega} \psi(s) ds}$$

for every $\xi \in \Omega$.

The proof of the Proposition 2.1 is similar to the proof of the Proposition 1 from [10].

3. EXISTENCE AND UNIQUENESS OF THE TRAJECTORIES

Denote

$$M(\lambda) = M_0 + \lambda \left[M_2 \mu(\Omega) + 2H_* \mu(\Omega)^{\frac{p-1}{p}} r \right]. \quad (3.1)$$

The following theorem shows that every admissible control function generates the unique trajectory of the system (2.1).

Theorem 3.1. *Let the functions $f(\cdot) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $K(\cdot) : \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and the number $\lambda \in (0, \infty)$ satisfy the conditions A - C. Then each $u_*(\cdot) \in U_{p,r}$ generates the unique trajectory $x_*(\cdot)$ of the system (2.1).*

Proof. Define a map $x(\cdot) \rightarrow F(x(\cdot))$, $x(\cdot) \in C(\Omega; \mathbb{R}^n)$ setting

$$F(x(\cdot))(\xi) = f(\xi, x(\xi)) + \lambda \int_{\Omega} K(\xi, s, x(s), u_*(s)) ds, \quad \xi \in \Omega, \quad (3.2)$$

where $C(\Omega; \mathbb{R}^n)$ is the space of continuous functions $x(\cdot) : \Omega \rightarrow \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(\xi)\| : \xi \in \Omega\}$.

Let us show that $F(x(\cdot)) \in C(\Omega; \mathbb{R}^n)$. Choose arbitrary $\xi_* \in \Omega$ and $\varepsilon > 0$. Since $x(\cdot) \in C(\Omega; \mathbb{R}^n)$, then from condition A it follows that there exists $\delta_1 = \delta_1(\varepsilon, x(\cdot)) > 0$ such that for every $\xi \in B_k(\xi_*, \delta_1) \cap \Omega$ the inequality

$$\|f(\xi, x(\xi)) - f(\xi_*, x(\xi_*))\| \leq \frac{\varepsilon}{2} \quad (3.3)$$

is verified where $B_k(\xi_*, \delta_1) = \{\xi \in \mathbb{R}^k : \|\xi - \xi_*\| \leq \delta_1\}$.

Denote

$$\delta_2 = \frac{\varepsilon}{2\lambda \left[M_1 \mu(\Omega) + 2H_1 \mu(\Omega)^{\frac{p-1}{p}} r \right]}.$$

Condition B and Hölder's inequality imply that for every $\xi \in B_k(\xi_*, \delta_2) \cap \Omega$ the inequality

$$\begin{aligned} & \left\| \int_{\Omega} K(\xi, s, x(s), u_*(s)) ds - \int_{\Omega} K(\xi_*, s, x(s), u_*(s)) ds \right\| \\ & \leq \int_{\Omega} [M_1 + 2H_1 \|u_*(s)\|] \|\xi - \xi_*\| ds \\ & \leq \left[M_1 \mu(\Omega) + 2H_1 \mu(\Omega)^{\frac{p-1}{p}} r \right] \delta_2 \leq \frac{\varepsilon}{2\lambda} \end{aligned} \quad (3.4)$$

is satisfied. Let $\delta_* = \min \{\delta_1, \delta_2\}$. (3.3) and (??) yield that for every $\xi \in B_k(\xi_*, \delta_*) \cap \Omega$ the inequality

$$\|F(x(\cdot))|(\xi) - F(x(\cdot))|(\xi_*)\| \leq \varepsilon$$

holds. This means that the function $\xi \rightarrow F(x(\cdot))|(\xi)$, $\xi \in \Omega$, is continuous at ξ_* . Since $\xi_* \in \Omega$ is arbitrarily chosen, we obtain that $F(x(\cdot)) \in C(\Omega; \mathbb{R}^n)$.

Let $x_1(\cdot) \in C(\Omega; \mathbb{R}^n)$ and $x_2(\cdot) \in C(\Omega; \mathbb{R}^n)$ be arbitrarily chosen functions. From condition B, (2.3), (3.1), (3.2) and Hölder's inequality it follows that

$$\begin{aligned} \|F(x_2(\cdot))|(\xi) - F(x_1(\cdot))|(\xi)\| & \leq M_0 \|x_2(\xi) - x_1(\xi)\| \\ & + \lambda \int_{\Omega} [M_2 + 2H_2 \|u_*(s)\|] \|x_2(s) - x_1(s)\| ds \\ & \leq \left[M_0 + \lambda M_2 \mu(\Omega) + 2\lambda H_2 r \mu(\Omega)^{\frac{p-1}{p}} \right] \|x_2(\cdot) - x_1(\cdot)\|_C \\ & = M(\lambda) \|x_2(\cdot) - x_1(\cdot)\|_C \end{aligned}$$

for every $\xi \in E$, and consequently

$$\|F(x_2(\cdot))|(\cdot) - F(x_1(\cdot))|(\cdot)\|_C \leq M(\lambda) \|x_2(\cdot) - x_1(\cdot)\|_C. \quad (3.5)$$

According to the condition C we have $M(\lambda) < 1$. (3.5) implies that the map $F(\cdot) : C(\Omega; \mathbb{R}^n) \rightarrow C(\Omega; \mathbb{R}^n)$ defined by (3.2) is contractive, and hence it has a unique fixed point $x_*(\cdot) \in C(\Omega; \mathbb{R}^n)$ which is unique continuous function satisfying the equation

$$x_*(\xi) = f(\xi, x_*(\xi)) + \lambda \int_{\Omega} K(\xi, s, x_*(s), u_*(s)) ds, \quad \xi \in \Omega.$$

□

4. BOUNDEDNESS OF THE SET OF TRAJECTORIES

In this section we will show that Conditions A - C guarantee boundedness of the set of trajectories $\mathbf{X}_{p,r}$. We set

$$\kappa_0 = \max\{\|f(\xi, 0)\| : \xi \in \Omega\},$$

$$\kappa_1 = \max\{\|K(\xi, s, 0, 0)\| : \xi \in \Omega, s \in \Omega\}$$

From conditions A and B it follows the validity of the following proposition.

Proposition 4.1. *Let the functions $f(\cdot) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $K(\cdot) : \Omega \times \Omega \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfy the conditions A and B. Then*

$$\|f(\xi, x)\| \leq \kappa_0 + M_0 \|x\|$$

$$\|K(\xi, s, x, u)\| \leq \kappa_1 + M_3 \|u\| + [M_2 + 2H_* \|u\|] \|x\|$$

for every $(\xi, s, x) \in \Omega \times \Omega \times \mathbb{R}^n$, where the constants M_0 , M_2 and M_3 are given in condition B, H_* is defined by (2.3).

Denote

$$\rho_* = \frac{\kappa_0 + \lambda \kappa_1 \mu(E) + \lambda M_3 \mu(E)^{\frac{p-1}{p}} r}{1 - M(\lambda)}, \quad (4.1)$$

where $M(\lambda)$ is defined by (3.1).

Theorem 4.1. *Let the conditions A - C be satisfied. Then for every $x(\cdot) \in \mathbf{X}_{p,r}$ the inequality*

$$\|x(\cdot)\|_C \leq \rho_*$$

holds.

Proof. Let $x(\cdot) \in \mathbf{X}_{p,r}$ be an arbitrary trajectory, generated by the admissible control function $u(\cdot) \in U_{p,r}$. From Proposition 4.1, Hölder's inequality and (2.2) we obtain

$$\begin{aligned} \|x(\xi)\| &\leq \kappa_0 + M_0 \|x(\xi)\| \\ &+ \lambda \int_{\Omega} [\kappa_1 + M_3 \|u(s)\| + (M_2 + 2H_* \|u(s)\|) \|x(s)\|] ds \\ &\leq \kappa_0 + M_0 \|x(\xi)\| + \lambda \kappa_1 \mu(\Omega) + \lambda M_3 \mu(\Omega)^{\frac{p-1}{p}} r \\ &+ \lambda \int_{\Omega} (M_2 + 2H_* \|u(s)\|) \|x(s)\| ds \end{aligned}$$

for every $\xi \in \Omega$. Since $M_0 \in [0, 1)$, then we have from the last inequality

$$\begin{aligned} \|x(\xi)\| &\leq \frac{\kappa_0 + \lambda \kappa_1 \mu(\Omega) + \lambda M_3 \mu(\Omega)^{\frac{p-1}{p}} r}{1 - M_0} \\ &+ \frac{\lambda}{1 - M_0} \int_{\Omega} [M_2 + 2H_* \|u(s)\|] \|x(s)\| ds \end{aligned} \quad (4.2)$$

for every $\xi \in \Omega$. Since $u(\cdot) \in U_{p,r}$, then (3.1), (4.1), (4.2), Condition C and Proposition 2.1 yield

$$\begin{aligned} \|x(\xi)\| &\leq \frac{\kappa_0 + \lambda\kappa_1\mu(\Omega) + \lambda M_3\mu(\Omega)^{\frac{p-1}{p}} r}{1 - M_0} \\ &\quad \times \frac{1}{1 - \frac{\lambda}{1 - M_0} \int_{\Omega} [M_2 + 2H_* \|u(s)\|] ds} \\ &\leq \frac{\kappa_0 + \lambda\kappa_1\mu(\Omega) + \lambda M_3\mu(\Omega)^{\frac{p-1}{p}} r}{1 - M_0} \\ &\quad \times \frac{1}{1 - \frac{\lambda}{1 - M_0} \left[M_2\mu(\Omega) + 2H_*\mu(\Omega)^{\frac{p-1}{p}} r \right]} = \rho_* \end{aligned}$$

for every $\xi \in \Omega$, and hence $\|x(\cdot)\|_C \leq \rho_*$. \square

From Theorem 4.1 it follows the validity of the following corollary.

Corollary 4.1. *The inclusion $\mathbf{X}_{p,r}(\xi) \subset B_n(\rho_*)$ holds for every $\xi \in \Omega$, where the set $\mathbf{X}_p(\xi)$ is defined by (2.4), the number $\rho_* > 0$ is defined by (4.1), $B_n(\rho_*) = \{x \in \mathbb{R}^n : \|x\| \leq \rho_*\}$.*

5. PRECOMPACTNESS OF THE SET OF TRAJECTORIES AND EXISTENCE OF ε -OPTIMAL TRAJECTORIES

In this section precompactness of the set of trajectories and existence of ε -optimal trajectories are studied. Denote

$$D_1 = \Omega \times B_n(\rho_*),$$

$$\begin{aligned} \omega_0(\Delta) &= \max \{ \|f(\xi_2, x) - f(\xi_1, x)\| : \|\xi_2 - \xi_1\| \\ &\leq \Delta, (\xi_1, x) \in D_1, (\xi_2, x) \in D_1 \}, \end{aligned} \quad (5.1)$$

$$\varphi(\Delta) = \frac{1}{1 - M_0} \left\{ \omega_0(\Delta) + \lambda \left[M_1\mu(\Omega) + 2H_1\mu(\Omega)^{\frac{p-1}{p}} r \right] \Delta \right\}. \quad (5.2)$$

By virtue of condition A, we have $\omega_0(\Delta) \rightarrow 0$, $\varphi(\Delta) \rightarrow 0$ as $\Delta \rightarrow 0^+$.

The Hausdorff distance between the sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ is denoted by $h(U, V)$ and defined as

$$h(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(v, U) \right\},$$

where $d(u, V) = \inf \{ \|u - v\| : v \in V \}$.

Proposition 5.1. *Let the conditions A - C be satisfied. Then for every $x(\cdot) \in \mathbf{X}_{p,r}$, $\xi_1 \in \Omega$, $\xi_2 \in \Omega$ the inequality*

$$\|x(\xi_2) - x(\xi_1)\| \leq \varphi(\|\xi_2 - \xi_1\|)$$

holds and hence

$$h(\mathbf{X}_{p,r}(\xi_2), \mathbf{X}_{p,r}(\xi_1)) \leq \varphi(\|\xi_2 - \xi_1\|)$$

where $\mathbf{X}_{p,r}(\xi_1)$ and $\mathbf{X}_{p,r}(\xi_2)$ are defined by (2.4).

Proof. Let $x(\cdot) \in \mathbf{X}_{p,r}$ be an arbitrarily chosen trajectory of the system (2.1). Then there exists $u(\cdot) \in U_{p,r}$ such that

$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_{\Omega} K(\xi, s, x(s), u(s)) ds, \quad \xi \in \Omega.$$

Now let $\xi_1 \in \Omega$ and $\xi_2 \in \Omega$. Since $x(\cdot) \in \mathbf{X}_{p,r}$, $u(\cdot) \in U_{p,r}$, then from (5.1), Condition B, Theorem 4.1 and Hölders inequality we have

$$\begin{aligned} \|x(\xi_2) - x(\xi_1)\| &\leq \|f(\xi_2, x(\xi_2)) - f(\xi_1, x(\xi_2))\| + \|f(\xi_1, x(\xi_2)) - f(\xi_1, x(\xi_1))\| \\ &+ \lambda \int_{\Omega} \|K(\xi_2, s, x(s), u(s)) - K(\xi_1, s, x(s), u(s))\| ds \\ &\leq \omega_0(\|\xi_2 - \xi_1\|) + M_0 \|x(\xi_2) - x(\xi_1)\| + \lambda \int_{\Omega} [M_1 + 2H_1 \|u(s)\|] \|\xi_2 - \xi_1\| ds \\ &\leq \omega_0(\|\xi_2 - \xi_1\|) + M_0 \|x(\xi_2) - x(\xi_1)\| + \lambda \left[M_1 \mu(\Omega) + 2H_1 \mu(\Omega)^{\frac{p-1}{p}} r \right] \|\xi_2 - \xi_1\|. \end{aligned}$$

Since $M_0 \in [0, 1)$, then the last inequality and (5.1) complete the proof. \square

Since $\varphi(\Delta) \rightarrow 0$ as $\Delta \rightarrow 0^+$, then Proposition 5.1 yields the validity of the following propositions.

Proposition 5.2. *Let the conditions A - C be satisfied. Then the set valued map $\xi \rightarrow \mathbf{X}_{p,r}(\xi)$, $\xi \in \Omega$, is continuous, where $\mathbf{X}_{p,r}(\xi)$ is defined by (2.4).*

Proposition 5.3. *Let the conditions A - C be satisfied. Then the set of trajectories $\mathbf{X}_{p,r}$ is a family of equicontinuous functions.*

Now, from Theorem 4.1 and Proposition 5.3 it follows precompactness of the set of trajectories.

Theorem 5.1. *Let the conditions A - C be satisfied. Then the set of trajectories $\mathbf{X}_{p,r}$ is a precompact subset of the space $C(\Omega; \mathbb{R}^m)$.*

Now, consider minimization of the lower semicontinuous functional $\gamma(x(\cdot)) : C(\Omega, \mathbb{R}^n) \rightarrow \mathbb{R}$ on the set of trajectories $\mathbf{X}_{p,r}$. Denote

$$I_* = \inf_{x(\cdot) \in \mathbf{X}_{p,r}} \gamma(x(\cdot)).$$

Since $\mathbf{X}_{p,r} \subset C(\Omega, \mathbb{R}^n)$ is nonempty and precompact set and $\gamma(x(\cdot))$ is a lower semicontinuous functional, we have that $|I_*| < +\infty$.

Let $\varepsilon > 0$ be a given number. A trajectory $x_\varepsilon(\cdot) \in \mathbf{X}_{p,r}$ satisfying the inequality $\gamma(x_\varepsilon(\cdot)) < I_* + \varepsilon$ is said to be an ε -optimal trajectory.

Theorem 5.2. *Let the conditions A - C be satisfied and $\gamma(x(\cdot)) : C(\Omega, \mathbb{R}^n) \rightarrow \mathbb{R}$ be a lower semicontinuous functional. Then for every $\varepsilon > 0$ there exists an ε -optimal trajectory.*

The proof of the theorem follows from precompactness of the set of trajectories $\mathbf{X}_{p,r}$, i.e. from Theorem 5.1 and lower semicontinuity of the functional $\gamma(x(\cdot))$.

6. CONCLUSION

Nonlinear control systems arise in different problems of theory and applications. Integral constraint on control functions appears if the control resource is exhausted by consumption. The precompactness property of the set of trajectories is a useful tool to study the existence of approximately optimal trajectories in the optimal control problems with semicontinuous payoff functionals. Note that control system described by an integral equation with geometric constraints on the control functions can be studied in the framework of integral inclusions. For control systems with integral constraint on the controls, the situation is different. The matter is that integral boundedness of the function does not guarantee geometric boundedness. Note that extending the system dimension, it is possible to write the control system described by integral equation with integral constraint on the controls in the form of integral inclusion with unbounded right hand side and with phase state constraint. But in this case, the new system turns out more complex than the original one. Therefore studying the considered system in its original form is more preferable, than the reduced one and it is one of the actual problems of control systems theory.

REFERENCES

- [1] Appell, J.M., A.S. Kalitvin, A.S. and Zabrejko, P.P., Partial integral operators and integro-differential equations, M. Dekker Inc., New York, 2000.
- [2] Balder, E.J., On existence problems for the optimal control of certain nonlinear integral equations of Urysohn type, *J. Optim. Theory Appl.* 42 (1984), 447-465.
- [3] Browder, F.E., Nonlinear functional analysis and nonlinear integral equations of Hammerstein and Urysohn type. Contributions to nonlinear functional analysis, Academic Press, New York, 1971, 425-500.
- [4] Gohberg, I.G. and Krein, M.G., Theory and applications of Volterra operators in Hilbert space, Amer. Math. Soc., Providence, R. I., 1970.
- [5] Guseinov, Kh.G., Approximation of the attainable sets of the nonlinear control systems with integral constraint on controls, *Nonlinear Anal. TMA*, 71 (2009), 622-645.
- [6] Heisenberg, W., Physics and philosophy: The revolution in modern science, George Allen & Unwin, London, 1958.
- [7] Huseyin, A., On the approximation of the set of trajectories of control system described by a Volterra integral equation, *Nonlin. Anal. Model. Contr.* 19 (2014), 199-208.
- [8] Huseyin, A. and Huseyin, N., Precompactness of the set of trajectories of the controllable system described by a nonlinear Volterra integral equation, *Math. Model. Anal.* 17 (2012), 686-695.

- [9] Huseyin, A. and Huseyin, N., Dependence on the parameters of the set of trajectories of the control system described by a nonlinear Volterra integral equation, *Appl. Math. Praha*, 59 (2014), 303-317.
- [10] Guseiin, N., Gusein, A. and Guseinov, Kh.G., Approximation of the set of trajectories of a control system described by the Urysohn integral equation. *Tr. Inst. Mat. Mekh.* 21 (2) (2015), 59-72.
- [11] Infante, G. and Webb, J.R.L., Nonlinear non-local boundary-value problems and perturbed Hammerstein integral equations, *Proc. Edinb. Math. Soc.* 49 (2) (2006), 637-656.
- [12] Joshi, M.C. and George, R.K. Controllability of nonlinear systems, *Numer. Funct. Anal. Optim.* 10 (1989), 139-166.
- [13] Krasnoselskii, M.A. and Krein, S.G., On the principle of averaging in nonlinear mechanics, *Uspekhi Mat. Nauk*, 10 (1955), 147-153. (In Russian)
- [14] Krasovskii, N.N., Theory of control of motion: Linear systems, Nauka, Moscow, 1968. (In Russian)
- [15] Polyanin, A.D. and Manzhirov, A.V., Handbook of integral equation, CRC Press, Boca Raton, FL, 1998.
- [16] Subbotin, A.I. and Ushakov, V.N., Alternative for an encounter-evasion differential game with integral constraints on the players controls, *J. Appl. Math. Mech.* 39 (1975), 367-375.
- [17] Ukhobotov, V.I., One dimensional projection method in linear differential games with integral constraints, Chelyabinsk State University press, Chelyabinsk, 2005. (In Russian)
- [18] Urysohn, P.S., On a type of nonlinear integral equation, *Mat. Sb.* 31 (1924), 236-255. (In Russian)

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