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INITIAL COEFFICIENTS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY SALAGEAN DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, we investigate a new subclass $\Sigma^n(\tau,\gamma,\varphi)$ of analytic and bi-univalent functions in the open unit disk $\mathcal{U}$ defined by Salagean differential operator. For functions belonging to this class, we obtain estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Salagean [18] introduced the following differential operator for $f(z) \in \mathcal{A}$ which is called the Salagean differential operator:

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \\ D^n f(z) &= D(D^{n-1}f(z)) \quad (n \in \mathbb{N} = 1, 2, 3, ...). \end{aligned}$$

We note that,

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.2}$$

It is well known that every $f \in \mathcal{S}$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

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and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). Lewin [13] introduced the bi-univalent function class and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [15], otherwise, showed that $\max\limits_{f\in\Sigma}|a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor Maclaurin coefficients: $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2\};\ \mathbb{N} = \{1,2,3,...\})$ is still an open problem. Recently, several researchers such as ([1]-[7], [9]-[16], [17], [19]-[24]) obtained the coefficients $|a_2|$, $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ . Motivating with their work, we introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass of the function class Σ employing the techniques used earlier by Srivastava et al. [19] and Frasin and Aouf [9].

Let φ be an analytic and univalent function with positive real part in $\mathcal{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor's series expansion of such function is

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ..., \tag{1.3}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function φ satisfies the above conditions unless otherwise stated.

Definition 1.1. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $\Sigma^n(\tau, \gamma, \varphi)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\tau}\left[(D^nf(z))' + \gamma z(D^nf(z))'' - 1\right] \prec \varphi(z)$$

$(0 \leq \gamma \leq 1,\ \tau \in \mathbb{C}/\{0\},\ n \in \mathbb{N},\ z \in \mathcal{U})$ *and*

$$1 + \frac{1}{\tau}\left[(D^ng(w))' + \gamma w(D^ng(w))'' - 1\right] \prec \varphi(w)$$

$(0 \leq \gamma \leq 1,\ \tau \in \mathbb{C}/\{0\},\ n \in \mathbb{N},\ w \in \mathcal{U})$, *where the function g is given by $g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$ and D^n is the Salagean differential operator.*

In this paper, we obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for $\Sigma^n(\tau, \gamma, \varphi)$ as well as its special classes.

Firstly, in order to derive our main results, we need the following lemma.

Lemma 1.1. [8] *Let* $p(z) = 1 + c_1 z + c_2 z^2 + ... \in P$, *where* P *is the family of all functions* p, *analytic in* $\mathcal{U}$, *for which* $\operatorname{Re} p(z) > 0$ $(z \in \mathcal{U})$. *Then*

$$|c_n| \leq 2; \ n = 1, 2, 3, ...$$

2. INITIAL COEFFICIENTS FOR THE CLASS $\Sigma^n(\tau, \gamma, \varphi)$

Theorem 2.1. *Let* $f(z) \in \Sigma^n(\tau, \gamma, \varphi)$ *be of the form (1.1). Then*

$$|a_2| \leq \frac{|\tau| \sqrt{B_1^3}}{\sqrt{\left|3^{n+1}\tau B_1^2 (1+2\gamma) + 4^{n+1}(1+\gamma)^2 (B_1 - B_2)\right|}} \tag{2.1}$$

and

$$|a_3| \leq B_1 |\tau| \left(\frac{B_1 |\tau|}{4^{n+1}(1+\gamma)^2} + \frac{1}{3^{n+1}(1+2\gamma)} \right). \tag{2.2}$$

Proof. Since $f \in \Sigma^n(\tau, \gamma, \varphi)$, there exist two analytic functions $u, v : \mathcal{U} \to \mathcal{U}$, with $u(0) = v(0) = 0$, such that

$$1 + \frac{1}{\tau}\left[(D^n f(z))' + \gamma z (D^n f(z))'' - 1\right] = \varphi(u(z)) \quad (z \in \mathcal{U}) \tag{2.3}$$

and

$$1 + \frac{1}{\tau}\left[(D^n g(w))' + \gamma w (D^n g(w))'' - 1\right] = \varphi(v(w)) \quad (w \in \mathcal{U}). \tag{2.4}$$

Define the function p and q as following:

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + b_1 w + b_2 w^2 + b_3 w^3 + ...$$

or equivalently,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{1}{2}\left(c_3 + \frac{c_1}{2}\left(\frac{c_1^2}{2} - c_2\right) - \frac{c_1 c_2}{2}\right) z^3 ... \tag{2.5}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{b_1}{2} w + \frac{1}{2}\left(b_2 - \frac{b_1^2}{2}\right) w^2 + \frac{1}{2}\left(b_3 + \frac{b_1}{2}\left(\frac{b_1^2}{2} - b_2\right) - \frac{b_1 b_2}{2}\right) w^3 .... \tag{2.6}$$

If we use (2.5) and (2.6) in (2.3) and (2.4) along with (1.3), we have

$$1 + \frac{1}{\tau}\left[(D^n f(z))' + \gamma z (D^n f(z))'' - 1\right]$$
$$= 1 + \frac{1}{2} B_1 c_1 z + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} B_2 c_1^2\right] z^2 + ... \tag{2.7}$$

and

$$1+\frac{1}{\tau}\left[\left(D^{n}g\left(w\right)\right)'+\gamma w\left(D^{n}g\left(w\right)\right)''-1\right]$$
$$=1+\frac{1}{2}B_1b_1w+\left[\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2\right]w^2+.... \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\frac{\left(1+\gamma\right)2^{n+1}a_2}{\tau}=\frac{1}{2}B_1c_1 \tag{2.9}$$

$$\frac{3^{n+1}\left(1+2\gamma\right)a_3}{\tau}=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2 \tag{2.10}$$

and

$$-\frac{\left(1+\gamma\right)2^{n+1}a_2}{\tau}=\frac{1}{2}B_1b_1 \tag{2.11}$$

$$\frac{3^{n+1}\left(1+2\gamma\right)\left(2a_2^2-a_3\right)}{\tau}=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{2.12}$$

From (2.9) and (2.11) we obtain

$$c_1=-b_1 \tag{2.13}$$

By adding (2.10) to (2.12) and combining this with (2.9) and (2.11), we get

$$a_2^2=\frac{\tau^2B_1^3\left(b_2+c_2\right)}{4\left[3^{n+1}\tau B_1^2\left(1+2\gamma\right)+4^{n+1}\left(1+\gamma\right)^2\left(B_1-B_2\right)\right]}. \tag{2.14}$$

Subtracting (2.10) from (2.12), if we use (2.9) and applying (2.13), we have

$$a_3=\frac{\tau^2B_1^2b_1^2}{2^{2n+4}\left(1+\gamma\right)^2}+\frac{\tau B_1\left(c_2-b_2\right)}{4.3^{n+1}\left(1+2\gamma\right)}. \tag{2.15}$$

Finally, in view of Lemma 1.1, we get results (2.1) to (2.2) asserted by the Theorem 2.1. □

3. COROLLARIES AND CONSEQUENCES

i) If we set

$$\tau=e^{i\beta}\cos\beta\quad\left(-\frac{\pi}{2}<\beta<\frac{\pi}{2}\right)$$

and

$$\varphi(z)=\frac{1+(1-2\kappa)z}{1-z}=1+2(1-\kappa)z+2(1-\kappa)z^2+...\quad(0\leq\kappa<1)$$

which gives $B_1=B_2=2(1-\kappa)$, in Theorem 2.1, we can have the following corollary.

Corollary 3.1. *Let* $f(z) \in \Sigma^n \left(e^{i\beta}\cos\beta\ , \gamma, \frac{1+(1-2\kappa)z}{1-z}\right)$ *be of the form (1.1). Then*

$$|a_2| \leq \sqrt{\frac{2\,(1-\kappa)}{3^{n+1}(1+2\gamma)}\cos\beta} \tag{3.1}$$

and

$$|a_3| \leq 2\,(1-\kappa)\left(\frac{(1-\kappa)\cos\beta}{2^{2n+1}\,(1+\gamma)^2} + \frac{1}{3^{n+1}\,(1+2\gamma)}\right)\cos\beta. \tag{3.2}$$

Remark 3.2. *For* $\gamma = 0$, *Corollary 3.1 simplifies to the following form.*

Corollary 3.3. *Let* $f(z) \in \Sigma^n \left(e^{i\beta}\cos\beta\ , 0, \frac{1+(1-2\kappa)z}{1-z}\right)$ *be of the form (1.1). Then*

$$|a_2| \leq \sqrt{\frac{2\,(1-\kappa)}{3^{n+1}}\cos\beta} \tag{3.3}$$

and

$$|a_3| \leq 2\,(1-\kappa)\left(\frac{(1-\kappa)\cos\beta}{2^{2n+1}} + \frac{1}{3^{n+1}}\right)\cos\beta. \tag{3.4}$$

ii) If we set $\tau = 1$ and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + ... \quad (0 < \alpha \leq 1)$$

which gives $B_1 = 2\alpha$, $B_2 = 2\alpha^2$, in Theorem 2.1, we can obtain the following corollary.

Corollary 3.4. *Let* $f(z) \in \Sigma^n \left(1, \gamma, \left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ *be of the form (1.1). Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{3^{n+1}(1+2\gamma)\,\alpha + 2^{2n+1}\,(1+\gamma)^2\,(1-\alpha)}} \tag{3.5}$$

and

$$|a_3| \leq \left(\frac{\alpha^2}{4^n\,(1+\gamma)^2} + \frac{2\alpha}{3^{n+1}\,(1+2\gamma)}\right). \tag{3.6}$$

Remark 3.5. *In its special case when* $\gamma = 0$ *in Corollary 3.4, we can get the following corollary.*

Corollary 3.6. *Let* $f(z) \in \Sigma^n \left(1, 0, \left(\frac{1+z}{1-z}\right)^{\alpha}\right)$ *be of the form (1.1). Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha 3^{n+1} + 2^{2n+1}\,(1-\alpha)}} \tag{3.7}$$

and

$$|a_3| \leq \left(\frac{\alpha^2}{4^n} + \frac{2\alpha}{3^{n+1}}\right). \tag{3.8}$$

Remark 3.7. **i:** *Putting $n = 0$ in Theorem 2.1, we obtain the corresponding result given earlier by Deniz* [7] *(also Srivastava and Bansal* [21]*).*

ii: *Putting $\tau = 1$, $\gamma = 0$, $n = 0$ in Theorem 2.1, we obtain the corresponding result given earlier by Ali et al* [1].

iii: *Putting $\beta = 0$, $n = 0$ in Corollary 3.3 and $\gamma = 0$, $n = 0$ in Corollary 3.4, we obtain the corresponding result given earlier by Srivastava et al* [19].

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