Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 92–97 (2017)
DOI: 10.1501/Commua1_0000000778
ISSN 1303–5991

ON COFINITELY WEAK RAD-SUPPLEMENTED MODULES

FIGEN ERYILMAZ AND ŞENOL EREN

ABSTRACT. In this paper, necessary and sufficient conditions for a quotient module are found to be a cofinitely weak Rad-supplemented module under which circumstances. Nevertheless, some relations are investigated between cofinitely Rad-supplemented modules and cofinitely weak Rad-supplemented modules. Lastly, we show that an arbitrary ring R is a left Noetherian $V-$ring if and only if every weak Rad-supplemented $R-$module is injective.

1. INTRODUCTION

Throughout the paper, R will be an associative ring with identity, M will be an $R-$module and all modules will be unital left $R-$modules unless otherwise specified. By $N \leq M$, we mean that N is a submodule of M. Recall that a submodule L of M is *small* in M and denoted by $L \ll M$, if $M \neq L + K$ for every proper submodule K of M. A submodule S of M is said to be *essential* in M and denoted by $S \trianglelefteq M$, if $S \cap N \neq 0$ for every nonzero submodule $N \leq M$. We write $Rad(M)$ for the Jacobson radical of a module M. An $R-$module M is called *supplemented*, if every submodule N of M has a *supplement* in M, i.e. a submodule K is minimal with respect to $M = N + K$. K is supplement of N in M if and only if $M = N + K$ and $N \cap K \ll K$ [16].

If $M = N + K$ and $N \cap K \ll M$, then K and N are called *weak supplements* of each other. Also M is called a *weakly supplemented* module if every submodule of M has a weak supplement in M [13, 18]. By using this definition, Büyükaşık and Lomp showed in [6] that a ring R is left perfect if and only if every left $R-$module is weakly supplemented if and only if R is semilocal and the radical of the countably infinite free left $R-$module has a weak supplement. Furthermore Alizade and Büyükaşık showed that a ring R is semilocal if and only if every direct product of simple modules is weakly supplemented [3].

In [17], Xue introduced Rad-supplemented modules. Let M be an $R-$ module, N and K be any submodules of M with $M = N + K$. If $N \cap K \leq Rad(K)$

Received by the editors: Received: Jan. 28, 2016 Accepted: Aug. 11, 2016.
2010 *Mathematics Subject Classification.* Primary 16D10, 16L30; Secondary 16D99.
Key words and phrases. Cofinite Submodule, Cofinitely Weak Rad-Supplemented Module, Totally Cofinitely Weak Rad-Supplemented Module, Noetherian $V-$ring.

($N \cap K \leq Rad(M)$), then K is called a *(weak) Rad-supplement* of N in M. Besides M is called (*weakly*) *Rad-supplemented* module provided that each submodule has a (weak) Rad-supplement in M. For characterizations of Rad-supplemented and weak Rad-supplemented modules, we refer to [15] and [17]. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is a Rad-supplement.

Certain modules whose maximal submodules have supplements are studied in [1]. Also in the same paper, cofinitely supplemented modules are introduced. A submodule N of M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. M is called *cofinitely (weak) supplemented* if every cofinite submodule has a (weak) supplement in M [1, 2]. Nevertheless, it is known by [1, Theorem 2.8] and [2, Theorem 2.11] that an $R-$module M is cofinitely (weak) supplemented if and only if every maximal submodule of M has a (weak) supplement in M. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented ones.

M is called *cofinitely Rad-supplemented* if every cofinite submodule of M has a Rad-supplement [5]. Since every submodule of a finitely generated module is cofinite, a finitely generated module is Rad-supplemented if and only if it is cofinitely Rad-supplemented. According to [12], if every cofinite submodule of M has a Rad-supplement that is a direct summand of M, then M is called a $\oplus-$*cofinitely Rad-supplemented module.*

In a present paper [10], a module is called *cofinitely weak Rad-supplemented* if every cofinite submodule has a weak Rad-supplement and *totally cofinitely weak Rad-supplemented* if every submodule is *cofinitely weak Rad-supplemented.* Also it is proved in [10] that any arbitrary sum of cofinitely weak Rad-supplemented modules is a cofinitely weak Rad-supplemented module. Clearly this implies that any finite direct sum of cofinitely weak Rad-supplemented modules is also cofinitely weak Rad-supplemented. We will show that an infinite direct sum of totally cofinitely weak Rad-supplemented modules is totally cofinitely weak Rad-supplemented under certain conditions. Also we will prove that every torsion module over a Dedekind domain is a cofinitely weak Rad-supplemented module and find some conditions to show when any module over a Dedekind domain is cofinitely weak Rad-supplemented.

2. Main Results

Following [5], a module M is called $w-$*local* if it has a unique maximal submodule.

Theorem 1. *Every $w-$local module is cofinitely weak Rad-supplemented.*

Proof. Let M be a module and U be a cofinite submodule of M. Since $\frac{M}{U}$ is finitely generated, it has a maximal submodule such as $\frac{P}{U}$. Therefore P is a maximal submodule of M. Then we have $U + M = M$ and $U \cap M = U \subseteq P = Rad\,(M)$. Hence M is cofinitely weak Rad-supplemented. □

Recall that a module M is called *refinable* (or *suitable*), if for any submodules U,$V \leq M$ with $U + V = M$, there exists a direct summand U_1 of M with $U_1 \leq U$ and $U_1 + V = M$.

Theorem 2. *Let M be a refinable $R-$module. Then the following are equivalent:*
(i) M is $\oplus-$cofinitely Rad-supplemented,
(ii) M is cofinitely Rad-supplemented,
(iii) M is cofinitely weak Rad-supplemented.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$ Let M be a cofinitely weak Rad-supplemented module and N be a cofinite submodule of M. Then, we have $M = N + K$ and $N \cap K \leq Rad\,(M)$ where K is a submodule of M. Since M is a refinable module, it has a direct summand L such that $L \leq K$ and $M = L+N$. Following this, $N \cap L \leq N \cap K \leq Rad\,(M)$ implies that L is weak Rad-supplement of N. By using [14, Proposition 4], we get that L is Rad-supplement of N. Therefore, M is $\oplus-$cofinitely Rad-supplemented. □

A ring R is called a *left $V-$ring* if every simple left $R-$module is injective.

Theorem 3. *For an arbitrary ring R, the following are equivalent:*
(i) Every weakly Rad-supplemented $R-$module is injective,
(ii) R is a left Noetherian $V-$ring.

Proof. $(i) \Rightarrow (ii)$ Assume that M is a $\oplus-$supplemented $R-$module. Since M is weak Rad-supplemented, it is an injective module. By Proposition 5.3 in [11] we get that R is a left Noetherian $V-$ring.
$(ii) \Rightarrow (i)$ Let M be a weakly Rad-supplemented module. Since R is a left Noetherian $V-$ring, we get $Rad\,(M) = 0$ by Villamayor theorem in [7]. Then, M is semisimple and so $\oplus-$supplemented. Again using Proposition 5.3 in [11], we obtain M is an injective module. □

Corollary 1. *Let R be a commutative ring. Then, every weakly Rad-supplemented $R-$module is injective if and only if R is semisimple.*

Proof. Suppose that every weakly Rad-supplemented module is injective. By using the preceding theorem, we can say that R is a left Noetherian $V-$ring. Thus, R is semisimple by Proposition 1 and first corollary of [7]. The other side of the proof is obvious by [16, 20.3]. □

Theorem 4. *Let M be a module and N be a submodule of M. If every cofinite submodule containing N of M has a weak Rad-supplement in M, then $\frac{M}{N}$ is cofinitely weak Rad-supplemented.*

Proof. Let $\frac{U}{N}$ be a cofinite submodule of $\frac{M}{N}$. Since $\frac{\left(\frac{M}{N}\right)}{\left(\frac{U}{N}\right)} \cong \frac{M}{U}$, we get that U is a cofinite submodule of M containing N. Hence, we can find a submodule V of M such that $M = U + V$ and $U \cap V \leq Rad\,(M)$. By using Proposition 3.2 of [15], we can deduce that $\frac{(V+N)}{N}$ is a weak Rad-supplement of $\frac{U}{N}$ in $\frac{M}{N}$. Therefore, $\frac{M}{N}$ is a cofinitely weak Rad-supplemented module. □

Remark. While a quotient module of a module is a cofinitely weak Rad-supplemented module, it may not be a cofinitely weak Rad–supplemented module. For example, ${}_{\mathbb{Z}}\mathbb{Z}$ isn't cofinitely weak Rad-supplemented but $\mathbb{Z}_p$ is cofinitely weak Rad-supplemented for any prime number p.

Proposition 1. *Let M be a cofinitely weak Rad-supplemented $R-$module. Then every Rad-supplement in M is cofinitely weak Rad-supplemented.*

Proof. Let V be a $Rad-$supplement of U in M. That means $M = U + V$ and $U \cap V \leq Rad\,(V)$. Since $\frac{M}{U} = \frac{(U+V)}{U} \cong \frac{V}{U \cap V}$, we get that $\frac{V}{U \cap V}$ is a cofinitely weak Rad-supplemented module by [10, Proposition 6]. Theorem 4 in the same paper implies that V is cofinitely weak Rad-supplemented. □

Theorem 5. *Let R be a Dedekind domain and M be a torsion $R-$module. Then M is cofinitely weak Rad-supplemented.*

Proof. By [3, Corollary 2.7], we have $\frac{M}{Rad(M)}$ is semisimple and so cofinitely weak Rad-supplemented. □

Theorem 6. *Let R be a Dedekind domain, $\frac{M}{Rad(M)}$ be finitely generated and $Rad\,(M) \trianglelefteq M$. If $Rad\,(M)$ is cofinitely weak Rad-supplemented, then M is cofinitely weak Rad-supplemented.*

Proof. Suppose that $\frac{M}{Rad(M)}$ is generated by $m_1 + Rad\,(M)$, $m_2 + Rad\,(M)$,...,$m_n + Rad\,(M)$. Then, for finitely generated submodule $K = Rm_1 + Rm_2 + ... + Rm_n$, we have $M = Rad\,(M) + K$ and $K \cap Rad\,(M)$ is finitely generated as K is finitely generated. So $K \cap Rad\,(M) \ll M$ by Lemma 2.3 in [3]. That is to say, K is a weak supplement of $Rad\,(M)$ of M. Since $Rad\,(M) \trianglelefteq M$, we get $\frac{M}{Rad(M)}$ is torsion. Besides this, Proposition 9.15 of [4] implies that $Rad\left(\frac{M}{Rad(M)}\right) = 0$. Hence $\frac{M}{Rad(M)}$ is semisimple by Corollary 2.7 in [3]. If we consider $0 \to Rad\,(M) \to M \to \frac{M}{Rad(M)} \to 0$, then M is cofinitely weak Rad-supplemented by Theorem 7 in [10]. □

Proposition 2. *Let R be a non-semilocal commutative domain. If M is totally cofinitely weak Rad-supplemented, then M is torsion.*

Proof. Suppose that $Ann\,(m) = 0_R$ for some $m \in M$. Then we have $Rm \cong {}_RR$. Since Rm is cofinitely weak Rad-supplemented, ${}_RR$ is also (cofinitely) weak Rad-supplemented. Then by 17.2 of [8], R is a semilocal ring which gives a contradiction. Thus, M is a torsion module. □

Theorem 7. *Let R be an arbitrary ring and $M = \underset{i\in I}{\oplus} M_i$ such that M_i is totally cofinitely weak Rad-supplemented for all $i \in I$. If $U = \underset{i\in I}{\oplus} (U \cap M_i)$ for every submodule U of M, then M is totally cofinitely weak Rad-supplemented.*

Proof. Assume that U is a submodule of M and V is a cofinite submodule of U where $U = \underset{i\in I}{\oplus} (U \cap M_i)$. Since $V = \underset{i\in I}{\oplus} (V \cap M_i)$ and $\frac{U}{V} \cong \underset{i\in I}{\oplus} \left[\frac{(U\cap M_i)}{(V\cap M_i)}\right]$, we get that $V \cap M_i$ is a cofinite submodule of $U \cap M_i$ for all $i \in I$. We know that $U \cap M_i$ is cofinitely weak Rad-supplemented. Therefore $V \cap M_i$ has a weak Rad-supplement K_i in $U \cap M_i$ for all $i \in I$. Let $K = \underset{i\in I}{\oplus} K_i$. Then we obtain $U = V + K$ and $V \cap K \leq Rad\,(U)$. As a result, U is cofinitely weak Rad-supplemented and so M is totally cofinitely weak Rad-supplemented. □

Let R be a Dedekind domain and M be an $R-$module. By Ω, we denote the set of all maximal ideals of R. The submodule $T_P\,(M) = \{m \in M\,|P^n m = 0 \text{ for some } n \geq 1\}$ is called the $P-$primary part of M.

Theorem 8. *Let R be a non-semilocal Dedekind domain. Then, M is a totally cofinitely weak Rad$-$supplemented module if and only if M is torsion and $T_P\,(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$.*

Proof. Assume that M is a totally cofinitely weak Rad-supplemented module. Then M is torsion by Proposition 2. On the other hand $T_P\,(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. Because every submodule of a totally cofinitely weak Rad-supplemented module is a totally cofinitely weak Rad-supplemented module.

Conversely, we can write $M = \underset{P\in\Omega}{\oplus} T_P\,(M)$ by Proposition 6.9 in [9]. Let N be a submodule of M. Since M is torsion, N is also a torsion module. By using the same proposition, we can write that $N = \underset{P\in\Omega}{\oplus} T_P\,(N)$. Therefore, $\underset{P\in\Omega}{\oplus} T_P\,(N) = \underset{P\in\Omega}{\oplus} (N \cap T_P\,(M))$ and $T_P\,(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. As a result, M is totally cofinitely weak Rad-supplemented by the preceding theorem. □

Theorem 9. *Any torsion module over a Dedekind domain is totally cofinitely weak Rad-supplemented.*

Proof. Let R be a Dedekind domain, M be a torsion $R-$module and N be a submodule of M. Due to Corollary 2.7 of [3], $\frac{N}{Rad(N)}$ is semisimple and so it is cofinitely weak $Rad-$supplemented. Therefore N is cofinitely weak $Rad-$supplemented by Theorem 4 of [10]. □

References

[1] Alizade, R.,Bilhan G. and Smith, P.F., Modules whose maximal submodules have supplements, *Comm. Algebra*, (2001), 29(6), 2389-2405.

[2] Alizade, R. and Büyükaşık, E., Cofinitely weak supplemented modules, *Comm. Algebra*, (2003), 31(11), 5377-5390.

[3] Alizade, R. and Büyükaşık, E., Extensions of weakly supplemented modules, *Math. Scand.*, (2008), 103(2), 161-168.

[4] Anderson, F.W. and Fuller, K.R., Rings and Categories of Modules, New York, Springer-Verlag, 1974.

[5] Büyükaşık, E. and Lomp, C., On a recent generalization of semiperfect rings, *Bull. Aust. Math. Soc.*, (2008), 78(2), 317-325.

[6] Büyükaşık, E. and Lomp, C., Rings whose modules are weakly supplemented are perfect. Applications to certain ring extensions., *Math. Scand.*, (2009), 105(1), 25-30.

[7] Byrd, K. A., Rings whose quasi-injective modules are injective, Proc. Amer. Math. Soc., (1972), 33, 235–240.

[8] Clark, J., Lomp, C., Vajana, N., Wisbauer, R., Lifting modules, 1st. ed., Birkhauser Verlag Basel, Boston-Berlin, 2006.

[9] Cohn, P.M., Algebra, Vol.2, Wiley&Sons, 1989.

[10] Eryılmaz, F.Y. and Eren, Ş.,Totally cofinitely weak Rad-supplemented modules, *Int. J. Pure Appl. Math.*, (2012), 80(5), 683-692.

[11] Idelhadj, A. and Tribak, R.,On some properties of $\oplus-$supplemented modules, *Int. J. Math. Math. Sci.*, (2003), 69, 4373–4387.

[12] Koşan, M.T., Generalized cofinitely semiperfect modules, *Int. Electron J. Algebra*, (2009), 5, 58-69.

[13] Lomp, C., On semilocal modules and rings, *Comm. Algebra*,(1999), 27(4), 1921-1935.

[14] Türkmen, E. and Pancar, A., Some properties of Rad-supplemented modules, *Int. J. of Phy. Sci.*, (2011), 6(35), 7904-7909.

[15] Wang, Y. and Ding, N., Generalized supplemented modules, *Taiwanese J. Math.*, (2006), 10(6), 1589-1601.

[16] Wisbauer, R., Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia, 1991.

[17] Xue, W., Characterizations of semiperfect and perfect rings, *Publ. Math.*,(1996), 40(1), 115-125.

[18] Zöschinger, H., Invarianten wesentlicher überdeckungen, *Math. Ann.*, (1978), 237(3), 193-202.

Current address: Figen ERYILMAZ: Ondokuz Mayıs University, Faculty of Education, Deparment of Mathematics Education, 55139 Kurupelit, Samsun-TURKEY.

E-mail address: fyuzbasi@omu.edu.tr

Current address: Şenol EREN: Ondokuz Mayıs University, Faculty of Sciences and Arts, Deparment of Mathematics, 55139 Kurupelit, Samsun-TURKEY.

E-mail address: seren@omu.edu.tr