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A TRACE FORMULA FOR THE STURM-LIOUVILLE TYPE EQUATION WITH RETARDED ARGUMENT

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Abstract. In this paper, we deal with a discontinuous Sturm-Liouville problem with retarded argument and eigenparameter-dependent boundary conditions. We obtain the asymptotic formulas for the eigenvalues and the regularized trace formula for the problem.

1. Introduction

In this paper, we consider a discontinuous Sturm-Liouville problem which contains an eigenparameter not only differential equation, but also boundary conditions, with retarded argument. Namely, the problem consists of the Sturm-Liouville equation:

$$y''(x) + q(x)\, y(x - \Delta(x)) + \lambda^2 y(x) = 0,\ x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right], \tag{1.1}$$

with boundary conditions:

$$(\lambda\alpha_1' + \alpha_1)\, y(0) - (\lambda\alpha_2' + \alpha_2)\, y'(0) = 0, \tag{1.2}$$

$$\left(\lambda\beta_1' + \beta_1\right) y(\pi) - \left(\lambda\beta_2' + \beta_2\right) y'(\pi) = 0, \tag{1.3}$$

and transmission conditions:

$$y\left(\frac{\pi}{2} - 0\right) - \delta y\left(\frac{\pi}{2} + 0\right) = 0, \tag{1.4}$$

$$y'\left(\frac{\pi}{2} - 0\right) - \delta y'\left(\frac{\pi}{2} + 0\right) = 0, \tag{1.5}$$

where the real-valued function $q(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $q\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} q(x)$; the real-valued function $\Delta(x) \geq 0$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $\Delta\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x)$; $x - \Delta(x) \geq 0$, if $x \in \left[0, \frac{\pi}{2}\right)$, $x - \Delta(x) \geq \frac{\pi}{2}$, if $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is an eigenparameter; $\delta \neq 0$, $\alpha_i, \alpha_i',\ \beta_i,\ \beta_i'$ $(i = 1, 2)$ are arbitrary real numbers.

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Gelfand and Levitan [1] firstly obtained a trace formula for a self adjoint Sturm-Liouville differential equation. This investigation was continued in many directions, such as Dirac systems [2-4], the case of continuous [5-11], discontinuous [12,13] or matrix Sturm-Liouville operator [14-16] and also Sturm-Liouville problems with retarded argument [17-19]. In the survey paper [20], the history and the current state of the theory of the regularized trace of the linear operators were presented. There are lots of works about computation of eigenvalues and eigenfunctions of continuous and discontinuous boundary value problems with retarded argument [21-25]. A discontinuous boundary value problem with retarded argument and with transmission conditions at the points of discontinuity was investigated in [25]. For the same problem, regularized sums from the eigenvalues, oscillations of eigenfunctions and the solutions of inverse nodal problem was given in [19]. That problem is a special case when the boundary conditions do not contain an eigenparameter in our problem.

Firstly, we obtain the asymptotic formula of the characteristic function $\omega(\lambda)$ whose zeros are eigenvalues of the problem. Then, we calculate the asymptotic formulas for the eigenvalues. Finally, we get a formula of the regularized trace for the problem (1.1)-(1.5). To derive this trace formula we will use similar contour integration method in [12,19] with some modifications.

2. The Asymptotic Formulas for the Eigenvalues and Trace Formula

We define a solution of (1.1) by

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in\left[0,\frac{\pi}{2}\right),\\ \phi_2(x,\lambda), & x\in\left(\frac{\pi}{2},\pi\right],\end{cases}$$

as follows: Let $\phi_1(x,\lambda)$ be a solution of (1.1) on $\left[0,\frac{\pi}{2}\right]$, satisfying the initial conditions

$$\phi_1(0,\lambda)=\lambda\alpha_2'+\alpha_2,\ \phi_1'(0,\lambda)=\lambda\alpha_1'+\alpha_1. \tag{2.1}$$

After defining this solution, we shall define the solution $\phi_2(x,\lambda)$ of (1.1) on $\left[\frac{\pi}{2},\pi\right]$ by means of the solution $\phi_1(x,\lambda)$ by the initial conditions

$$\phi_2\left(\frac{\pi}{2},\lambda\right)=\delta^{-1}\phi_1\left(\frac{\pi}{2},\lambda\right),\qquad \phi_2'\left(\frac{\pi}{2},\lambda\right)=\delta^{-1}\phi_1'\left(\frac{\pi}{2},\lambda\right). \tag{2.2}$$

Consequently, the function $\phi(x,\lambda)$ defined on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ is a solution of (1.1), which satisfies the boundary condition (1.2) and the transmission conditions (1.4) and (1.5).

Then the following integral equations hold:

$$\begin{aligned}\phi_1(x,\lambda)=&(\lambda\alpha_2'+\alpha_2)\cos(\lambda x)+\frac{1}{\lambda}(\lambda\alpha_1'+\alpha_1)\sin(\lambda x)\\ &-\frac{1}{\lambda}\int_0^x q(\tau)\sin(\lambda(x-\tau))\,\phi_1(\tau-\Delta(\tau),\lambda)\,d\tau,\end{aligned} \tag{2.3}$$

and

$$\phi_2(x,\lambda)=\delta^{-1}\phi_1\left(\tfrac{\pi}{2},\lambda\right)\cos\left(\lambda\left(x-\tfrac{\pi}{2}\right)\right)+\frac{1}{\lambda}\delta^{-1}\phi_1'\left(\tfrac{\pi}{2},\lambda\right)\sin\left(\lambda\left(x-\tfrac{\pi}{2}\right)\right)$$
$$-\frac{1}{\lambda}\int_{\frac{\pi}{2}}^{x}q(\tau)\sin(\lambda(x-\tau))\,\phi_2(\tau-\Delta(\tau),\lambda)\,d\tau. \tag{2.4}$$

Solving the equations (2.3) and (2.4) by the method of successive approximation, we obtain the following asymptotic representation for $|\lambda|\to\infty$:

$$\begin{aligned}\phi_1(x,\lambda) &= \lambda\alpha_2'\cos(\lambda x)+\alpha_2\cos(\lambda x)+\alpha_1'\sin(\lambda x)\\ &-\frac{\alpha_2'}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\\ &-\frac{\alpha_2'}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau\\ &+\frac{1}{\lambda}\left\{\alpha_1\sin(\lambda x)-\frac{\alpha_2}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\right.\\ &\qquad-\frac{\alpha_2}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau\\ &\qquad+\frac{\alpha_1'}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\\ &\qquad\left.-\frac{\alpha_1'}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau\right\}+O\left(\frac{e^{|\operatorname{Im}\lambda|x}}{\lambda^2}\right),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\phi_1'(x,\lambda) &= -\lambda^2\alpha_2'\sin(\lambda x)\\ &+\lambda\left\{-\alpha_2\sin(\lambda x)+\alpha_1'\cos(\lambda x)-\tfrac{\alpha_2'}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\right.\\ &\qquad\left.-\tfrac{\alpha_2'}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau\right\}\\ &+\alpha_1\cos(\lambda x)-\tfrac{\alpha_2}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\\ &-\tfrac{\alpha_2}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau-\tfrac{\alpha_1'}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\\ &+\tfrac{\alpha_1'}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau+O\left(\tfrac{e^{|\operatorname{Im}\lambda|x}}{\lambda}\right),\end{aligned} \tag{2.6}$$

$$\begin{aligned}\phi_2(x,\lambda) &= \lambda\alpha_2'\delta^{-1}\cos(\lambda x)+\alpha_2\delta^{-1}\cos(\lambda x)+\alpha_1'\delta^{-1}\sin(\lambda x)\\ &-\tfrac{\alpha_2'\delta^{-1}}{2}\int_0^x q(\tau)\sin(\lambda(x-\Delta(\tau)))\,d\tau\\ &-\tfrac{\alpha_2'\delta^{-1}}{2}\int_0^x q(\tau)\sin(\lambda(x-2\tau+\Delta(\tau)))\,d\tau+O\left(\tfrac{e^{|\operatorname{Im}\lambda|x}}{\lambda}\right),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\phi_2'(x,\lambda) &= -\lambda^2\alpha_2'\delta^{-1}\sin(\lambda x)\\ &+\lambda\left\{-\alpha_2\delta^{-1}\sin(\lambda x)+\alpha_1'\delta^{-1}\cos(\lambda x)-\tfrac{\alpha_2'\delta^{-1}}{2}\int_0^x q(\tau)\cos(\lambda(x-\Delta(\tau)))\,d\tau\right.\\ &\left.-\tfrac{\alpha_2'\delta^{-1}}{2}\int_0^x q(\tau)\cos(\lambda(x-2\tau+\Delta(\tau)))\,d\tau\right\}+O\left(e^{|\mathrm{Im}\,\lambda|x}\right).\end{aligned} \tag{2.8}$$

The solution $\phi(x,\lambda)$ defined above is a nontrivial solution of (1.1) satisfying the boundary condition (1.2) and the transmission conditions (1.4) and (1.5). Putting $\phi(x,\lambda)$ into the boundary condition (1.3), we get the following characteristic equation

$$\omega(\lambda) \equiv \left(\lambda\beta_1'+\beta_1\right)\phi(\pi,\lambda)-\left(\lambda\beta_2'+\beta_2\right)\phi'(\pi,\lambda)=0. \tag{2.9}$$

The eigenvalues of the problem (1.1)-(1.5) are the zeros of (2.9), and the eigenvalues are discrete and simple (see [22, 23, 26]).

Putting the expressions (2.7) and (2.8) into (2.9), we obtain

$$\begin{aligned}\omega(\lambda) &= \lambda^3\alpha_2'\beta_2'\delta^{-1}\sin(\lambda\pi)+\lambda^2\delta^{-1}\left\{\left[\alpha_2'\beta_1'-\alpha_1'\beta_2'+\tfrac{\alpha_2'\beta_2'}{2}\int_0^\pi q(\tau)\cos(\lambda\Delta(\tau))\,d\tau\right.\right.\\ &\left.+\tfrac{\alpha_2'\beta_2'}{2}\int_0^\pi q(\tau)\cos(\lambda(2\tau-\Delta(\tau)))\,d\tau\right]\cos(\lambda\pi)+\left[\alpha_2\beta_2'+\alpha_2'\beta_2\right.\\ &\left.\left.+\tfrac{\alpha_2'\beta_2'}{2}\int_0^\pi q(\tau)\sin(\lambda\Delta(\tau))\,d\tau+\tfrac{\alpha_2'\beta_2'}{2}\int_0^\pi q(\tau)\sin(\lambda(2\tau-\Delta(\tau)))\,d\tau\right]\sin(\lambda\pi)\right\}\\ &+O\left(\lambda e^{|\mathrm{Im}\,\lambda|\pi}\right).\end{aligned} \tag{2.10}$$

Let we define

$$\begin{aligned}A(\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int_0^\pi q(\tau)\cos(\lambda\Delta(\tau))\,d\tau,\\ B(\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int_0^\pi q(\tau)\cos(\lambda(2\tau-\Delta(\tau)))\,d\tau,\\ C(\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int_0^\pi q(\tau)\sin(\lambda\Delta(\tau))\,d\tau,\\ D(\lambda,\Delta(\tau)) &= \tfrac{1}{2}\int_0^\pi q(\tau)\sin(\lambda(2\tau-\Delta(\tau)))\,d\tau.\end{aligned} \tag{2.11}$$

From (2.11), (2.10) can be written as

$$\begin{aligned}\omega(\lambda) &= \lambda^3\alpha_2'\beta_2'\delta^{-1}\sin(\lambda\pi)+\lambda^2\delta^{-1}\left\{\left[\alpha_2'\beta_1'-\alpha_1'\beta_2'+\alpha_2'\beta_2'(A(\lambda,\Delta(\tau))\right.\right.\\ &\left.+B(\lambda,\Delta(\tau)))\right]\cos(\lambda\pi)+\left[\alpha_2\beta_2'+\alpha_2'\beta_2+\alpha_2'\beta_2'(C(\lambda,\Delta(\tau))\right.\\ &\left.\left.+D(\lambda,\Delta(\tau)))\right]\sin(\lambda\pi)\right\}+O\left(\lambda e^{|\mathrm{Im}\,\lambda|\pi}\right).\end{aligned} \tag{2.12}$$

Theorem 1. *The eigenvalues of the problem (1.1)-(1.5) have the following asymptotic representation for* $n \to \infty$:

$$\lambda_n = \lambda_n^0 - \frac{1}{\lambda_n^0 \pi} \left\{ \frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(n, \Delta\left(\tau\right)\right) + B\left(n, \Delta\left(\tau\right)\right) \right\} + O\left(\frac{1}{n^2}\right), \tag{2.13}$$

where $A\left(\lambda, \Delta\left(\tau\right)\right)$ *and* $B\left(\lambda, \Delta\left(\tau\right)\right)$ *are given by (2.11) and* λ_n^0 *will be defined below.*

Proof. Let we define

$$\omega_0\left(\lambda\right) = \lambda^3 \alpha_2' \beta_2' \delta^{-1} \sin\left(\lambda \pi\right), \tag{2.14}$$

and denote by $\lambda_{\pm n}^0$, $n \in \mathbb{N} \cup \{0\}$, the zeros of (2.14) except that zero is multiplicity 4, then $\lambda_{\pm 0}^0 = \lambda_{\pm 1}^0 = 0$ and

$$\lambda_n^0 = \begin{cases} n-1, & n \geq 1, \\ n+1, & n \leq -1, \end{cases} \tag{2.15}$$

[similar to ref 27]. Denote by C_n, the circle of radius ε, $0 < \varepsilon < \frac{1}{2}$ with the centers at the points λ_n^0. Thus, on the contour C_n, from (2.12) and (2.14), we have

$$\begin{aligned} \frac{\omega\left(\lambda\right)}{\omega_0\left(\lambda\right)} = 1 + \frac{1}{\lambda} \Bigg\{ & \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right) \right) \cot\left(\lambda\pi\right) \\ & + \frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right) \Bigg\} + O\left(\frac{1}{\lambda^2}\right). \end{aligned} \tag{2.16}$$

Expanding $\ln \frac{\omega(\lambda)}{\omega_0(\lambda)}$ by the Maclaurin formula, we obtain

$$\begin{aligned} \ln \frac{\omega\left(\lambda\right)}{\omega_0\left(\lambda\right)} = & \frac{1}{\lambda} \Bigg\{ \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right) \right) \cot\left(\lambda\pi\right) \\ & + \frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right) \Bigg\} \\ & - \frac{1}{2\lambda^2} \Bigg\{ \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right) \right)^2 \cot^2\left(\lambda\pi\right) \\ & + \left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right) \right)^2 \\ & + 2 \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right) \right) \\ & \times \left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right) \right) \cot\left(\lambda\pi\right) \Bigg\} + O\left(\frac{1}{\lambda^3}\right). \end{aligned} \tag{2.17}$$

Using the Rouche theorem in (2.12), it follows that $\omega\left(\lambda\right)$ has the same number of zeros inside the contour as $\omega_0\left(\lambda\right)$. Then, we have

$$\lambda_n = \lambda_n^0 + \epsilon_n, \tag{2.18}$$

for sufficiently large n, where $\left|\epsilon_n\right| < \frac{\pi}{2}$. Substituting from (2.18) into (2.12), we get $\epsilon_n = O\left(\frac{1}{n}\right)$.We continue making λ_n more precise. Using the residue theorem, we have

$$\begin{aligned}\lambda_n-\lambda_n^0&=-\frac{1}{2\pi i}\oint\limits_{C_n}\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}d\lambda\\&=-\frac{1}{2\pi i}\oint\limits_{C_n}\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A(\lambda,\Delta(\tau))+B(\lambda,\Delta(\tau))\right)\frac{\cot(\lambda\pi)}{\lambda}d\lambda\\&\quad-\frac{1}{2\pi i}\oint\limits_{C_n}\left(\frac{\alpha_2}{\alpha_2'}+\frac{\beta_2}{\beta_2'}+C(\lambda,\Delta(\tau))+D(\lambda,\Delta(\tau))\right)\frac{1}{\lambda}d\lambda+O\left(\frac{1}{n^2}\right)\\&=-\frac{1}{\lambda_n^0\pi}\left\{\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A(n,\Delta(\tau))+B(n,\Delta(\tau))\right\}+O\left(\frac{1}{n^2}\right),\end{aligned}\tag{2.19}$$

thus we have the asymptotic formula (2.13). □

Theorem 2. *The following formula of the regularized trace for the problem (1.1)-(1.5) holds:*

$$\begin{aligned}\lambda_{-0}^2+\lambda_0^2+\sum_{n=1}^{\infty}\Big(\lambda_n^2+\lambda_{-n}^2-2(n-1)^2\\+\frac{4}{\pi}\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A(n,\Delta(\tau))+B(n,\Delta(\tau))\right)\Big)\\=-\frac{2}{\pi}\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A(0,\Delta(\tau))+B(0,\Delta(\tau))\right)\\-\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A(0,\Delta(\tau))+B(0,\Delta(\tau))\right)^2\\+\left(\frac{\alpha_2}{\alpha_2'}+\frac{\beta_2}{\beta_2'}+C(0,\Delta(\tau))+D(0,\Delta(\tau))\right)^2,\end{aligned}\tag{2.20}$$

where $A(\lambda,\Delta(\tau))$, $B(\lambda,\Delta(\tau))$, $C(\lambda,\Delta(\tau))$ and $D(\lambda,\Delta(\tau))$ are given by (2.11).

Proof. Denote by Γ_{N_0}, the counterclockwise square contours EFGH with $E=(N_0-1+\varepsilon)(1-i)$, $F=(N_0-1+\varepsilon)(1+i)$, $G=(N_0-1+\varepsilon)(-1+i)$, $H=(N_0-1+\varepsilon)(-1-i)$ and N_0 is an integer. Asymptotic formula of λ_n implies that for all sufficiently large N_0, the numbers λ_n, with $|n|\le N_0$, are inside Γ_{N_0}, and the numbers λ_n, with $|n|>N_0$, are outside Γ_{N_0}. Obviously, λ_n^0 do not lie on the contour Γ_{N_0}. It follows that

$$\begin{aligned}
\sum_{\Gamma_{N_0}} \left(\lambda_n^2 - \left(\lambda_n^0\right)^2\right) &= \lambda_{-0}^2 + \lambda_0^2 + \sum_{n=1}^{N_0} \left(\lambda_n^2 + \lambda_{-n}^2 - 2\left(n-1\right)^2\right) \\
&= -\frac{1}{2\pi i} \oint_{\Gamma_{N_0}} 2\lambda \ln \frac{\omega\left(\lambda\right)}{\omega_0\left(\lambda\right)} d\lambda \\
&= -\frac{1}{2\pi i} \oint_{\Gamma_{N_0}} 2\left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right)\right) \cot\left(\lambda\pi\right) d\lambda \\
&\quad - \frac{1}{2\pi i} \oint_{\Gamma_{N_0}} 2\left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right)\right) d\lambda \\
&\quad + \frac{1}{2\pi i} \oint_{\Gamma_{N_0}} \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right)\right)^2 \frac{\cot^2\left(\lambda\pi\right)}{\lambda} d\lambda \\
&\quad + \frac{1}{2\pi i} \oint_{\Gamma_{N_0}} \left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right)\right)^2 \frac{1}{\lambda} d\lambda \\
&\quad + \frac{1}{2\pi i} \oint_{\Gamma_{N_0}} 2\left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(\lambda, \Delta\left(\tau\right)\right) + B\left(\lambda, \Delta\left(\tau\right)\right)\right) \\
&\quad \times \left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(\lambda, \Delta\left(\tau\right)\right) + D\left(\lambda, \Delta\left(\tau\right)\right)\right) \frac{\cot\left(\lambda\pi\right)}{\lambda} d\lambda \\
&\quad + O\left(\frac{1}{N_0}\right).
\end{aligned} \tag{2.21}$$

By residue calculations, we get

$$\begin{aligned}
&\lambda_{-0}^2 + \lambda_0^2 + \sum_{n=1}^{N_0} \left(\lambda_n^2 + \lambda_{-n}^2 - 2\left(n-1\right)^2\right) \\
&\quad = -2\left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(n, \Delta\left(\tau\right)\right) + B\left(n, \Delta\left(\tau\right)\right)\right) \frac{\left(2\left(N_0 - 1\right) + 1\right)}{\pi} \\
&\qquad - \frac{2}{\pi}\left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(0, \Delta\left(\tau\right)\right) + B\left(0, \Delta\left(\tau\right)\right)\right) \\
&\qquad - \left(\frac{\beta_1'}{\beta_2'} - \frac{\alpha_1'}{\alpha_2'} + A\left(0, \Delta\left(\tau\right)\right) + B\left(0, \Delta\left(\tau\right)\right)\right)^2 \\
&\qquad + \left(\frac{\alpha_2}{\alpha_2'} + \frac{\beta_2}{\beta_2'} + C\left(0, \Delta\left(\tau\right)\right) + D\left(0, \Delta\left(\tau\right)\right)\right)^2 + O\left(\frac{1}{N_0}\right),
\end{aligned} \tag{2.22}$$

which implies that

$$\begin{aligned}
&\lambda_{-0}^2+\lambda_0^2+\sum_{n=1}^{N_0}\Big(\lambda_n^2+\lambda_{-n}^2-2\left(n-1\right)^2\\
&\qquad\qquad+\frac{4}{\pi}\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A\left(n,\Delta\left(\tau\right)\right)+B\left(n,\Delta\left(\tau\right)\right)\right)\Big)\\
&\quad=-\frac{2}{\pi}\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A\left(0,\Delta\left(\tau\right)\right)+B\left(0,\Delta\left(\tau\right)\right)\right)\\
&\qquad-\left(\frac{\beta_1'}{\beta_2'}-\frac{\alpha_1'}{\alpha_2'}+A\left(0,\Delta\left(\tau\right)\right)+B\left(0,\Delta\left(\tau\right)\right)\right)^2\\
&\qquad+\left(\frac{\alpha_2}{\alpha_2'}+\frac{\beta_2}{\beta_2'}+C\left(0,\Delta\left(\tau\right)\right)+D\left(0,\Delta\left(\tau\right)\right)\right)^2+O\left(\frac{1}{N_0}\right).
\end{aligned}\tag{2.23}$$

Passing to the limit as $N_0\to\infty$ in (2.23), we get the regularized trace formula (2.20). □

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