



COMMON FIXED POINT RESULTS FOR A BANACH OPERATOR PAIR IN $CAT(0)$ SPACES WITH APPLICATIONS

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ABSTRACT. In this paper, sufficient conditions for the existence of a common fixed point for a Banach operator pair of mappings satisfying generalized contractive conditions in the frame work of $CAT(0)$ spaces are obtained. As an application, related results on best approximation are derived. Our results generalize various known results in contemporary literature.

1. INTRODUCTION AND PRELIMINARIES

Metric fixed point theory is a branch of fixed point theory which finds its primary applications in functional analysis. The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric conditions on mappings and/or underlying spaces play a crucial role in metric fixed point problems. Although it has a purely metric flavor, it is also a major branch of nonlinear functional analysis with close ties to Banach space geometry, see for example [11, 12] and references mentioned therein. Several results concerning the existence and approximation of a fixed point of a mapping rely on convexity hypotheses and geometric properties of the Banach spaces. Gromov [13] introduced the notion of $CAT(0)$ spaces. For application of these spaces in various branches of mathematics and for a vigorous discussion on these spaces, we refer to Bridson and Haefliger [4] and Burago-Burago-Ivanov [6]. The results obtained in this direction were the starting point for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. Applying fixed point theorems, useful results have been established in approximation theory. Meinardus [22] was the first to employ fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results appeared in the literature of approximation theory ([2] and [25]). Recently, Chen and Li [7] introduced the class of Banach operator pairs as a new class of noncommuting

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maps. For some more study, see for example [1, 27, 28]. In this paper, common fixed points for Banach operator pair of mappings which are more general than C_q -commuting mappings, are obtained in the setting of a CAT(0) spaces. As an application, invariant approximation results for these mappings are also derived.

2. PRELIMINARIES

First we recall some basics.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y . When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\Delta(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane \mathbb{E}^2 such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0) : Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$. If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \quad (\text{CN})$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact (cf. [4, p. 163]), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

A metric space X is called a CAT(0) space [13] if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in Euclidean plane. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space, see [11, 23].

Following are some elementary facts about CAT(0) spaces, see Dhompongsa and Panyanak [9].

Lemma 2.1. Let (X, d) be a CAT(0) space. Then

- (i): (X, d) is uniquely geodesic (see [4, pp.160]).

- (ii): Let p, x, y be points of X , let $\alpha \in [0, 1]$, and let m_1 and m_2 denote, respectively, the points of $[p, x]$ and $[p, y]$ satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then $d(m_1, m_2) \leq \alpha d(x, y)$ (see [19, Lemma 3]).
- (iii): Let $x, y \in X, x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$.
- (iv): Let $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \quad (1.1)$$

For convenience, from now on we will use the notation $(1 - t)x \oplus ty$ for the unique point z satisfying (1.1).

Lemma 2.2. ([9]) Let X be a CAT(0) space. Then

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Let us recall the definitions and related concepts about Δ -convergence. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X . For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see, e.g. [8], Proposition 7) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and call x the Δ -limit of $\{x_n\}$, see [20, 21].

A self mapping T on CAT(0) space X is said to be Δ -continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $\Delta - \lim_n x_n = x$, we have $\Delta - \lim_n Tx_n = Tx$.

A subset K of X is said to be Δ -closed if any sequence $\{x_n\}$ in K with $\Delta - \lim_n x_n = x$ implies that $x \in K$. A subset K of X is said to be Δ -compact if for any sequence $\{x_n\}$ in K , there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\Delta - \lim_m x_m = x \in K$.

The following lemma can be found, for example, in [9].

Lemma 2.3. Every bounded sequence in a CAT(0) space X has a Δ -convergent subsequence.

(ii) If C is a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in C , then the asymptotic center of $\{x_n\}$ is in C .

Let X be a CAT(0) space. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A set Y is said to be q -starshaped if there exists q in Y such that Y includes every geodesic segment joining any of its point with q . Obviously q -starshaped subsets of X contain all convex subsets of X as a proper subclass.

For the sake of convenience, we gather some basic definitions and set out the terminology needed in the sequel.

Definition 2.4. Let $T, S : X \rightarrow X$. A point $x \in X$ is called:

- (1) a *fixed point of T* if $Tx = x$;
- (2) a *coincidence point of the pair $\{T, S\}$* if $Tx = Sx$;
- (3) a *common fixed point of the pair $\{T, S\}$* if $x = Tx = Sx$.

$F(T)$, $C(T, S)$ and $F(T, S)$ denote the set of all fixed points of T , the set of all coincidence points of the pair $\{T, S\}$, and the set of all common fixed points of the pair $\{T, S\}$, respectively.

Let Y be a q -starshaped subset of a CAT(0) space X and $T, S : Y \rightarrow Y$. Put,

$$Y_q^{T(x)} = \{y_\lambda : y_\lambda = (1 - \lambda)q \oplus \lambda Tx \text{ and } \lambda \in [0, 1]\}.$$

Now, for each x in X , $d(S(x), Y_q^{T(x)}) = \inf_{\lambda \in [0, 1]} d(S(x), y_\lambda)$. Moreover if for $u \in X$, $x \in Y$, $Y_u^x \cap Y$ is nonempty then $x \in \partial Y$ (boundary of Y).

Definition 2.5. A self mapping T on a CAT(0) space X is said to satisfy a *property (I)*, if for $\lambda \in [0, 1]$, we have $T((1 - \lambda)x \oplus \lambda y) = (1 - \lambda)Tx \oplus \lambda Ty$.

Definition 2.6. Let X be a CAT(0) space and Y a q -starshaped subset of X , S and T be self mappings on X and $q \in F(S)$, then T is said to be:

- (1) an *S -contraction* if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$;
- (2) an *asymptotically S -nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n \rightarrow \infty} k_n = 1$ such that $d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy)$ for each x, y in Y and each $n \in \mathbb{N}$. If $k_n = 1$, for all $n \in \mathbb{N}$, then T is known as an S -nonexpansive mapping. If $S = I$ (identity map), then T is asymptotically nonexpansive mapping;
- (3) *R -weakly commuting* if there exists a real number $R > 0$ such that $d(TSx, STx) \leq Rd(Tx, Sx)$ for all x in Y ;
- (4) *R -subweakly commuting* if there exists a real number $R > 0$ such that $d(TSx, STx) \leq Rd(Sx, Y_q^{T(x)})$ for all $x \in Y$;
- (5) *C_q -commuting* if $STx = TSx$ for all $x \in C_q(S, T)$, where $C_q(S, T) = \cup\{C(S, T_k) : 0 \leq k \leq 1\}$ and $T_k x = (1 - k)q \oplus kTx$.

Definition 2.7. Let X be a metric space and K be any closed subset of X . If there exists a $y_0 \in K$ such that $d(x, y_0) = d(x, K) = \inf_{y \in K} d(x, y)$, then y_0 is called a *best approximation to x out of K* . We denote by $P_K(x)$, the set of all best approximations to x out of K .

A self mapping T on a CAT(0) space X is said to be *uniformly asymptotically regular* on E if for each $\varepsilon > 0$, there exists a positive integer N such that $d(T^n x, T^{n+1} x) < \varepsilon$ for all $n \geq N$ and for all x in E .

The ordered pair (T, I) of two self maps of a metric space (X, d) is called a *Banach operator pair* if the set $F(I)$ is T -invariant, namely $T(F(I)) \subseteq F(I)$. Obviously, any commuting pair (T, I) is a Banach operator pair but not conversely in general, see [7]. If (T, I) is a Banach operator pair then (I, T) need not be a Banach operator pair (cf. Example 1 [7]). If the self-maps T and I of X satisfy $d(fTx, Tx) \leq kd(fx, x)$ for all $x \in X$ and $k \geq 0$, then (T, f) is a Banach operator pair.

3. COMMON FIXED POINT RESULTS

In this section, the existence of common fixed points of Banach operator pair of mappings is established in a CAT(0) space. The following result is a consequence of ([16], Theorem 2.1).

Theorem 3.1. Let K be a subset of a metric space (X, d) , and f and T be weakly compatible selfmaps of K . Assume that $clT(K) \subset f(K)$, $clT(K)$ is complete, and T and f satisfy for all $x, y \in K$ and $0 \leq h < 1$,

$$d(Tx, Ty) \leq h \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}.$$

Then $K \cap F(f) \cap F(T)$ is singleton.

The following result extends and improves Lemma 3.1 of [7] and Theorem 1 in [18].

Lemma 3.2. Let K be a nonempty subset of a metric space (X, d) , and (T, f) be a Banach operator pair on K . Assume that $clT(K)$ is complete, and T and f satisfy for all $x, y \in K$ and $0 \leq h < 1$,

$$d(Tx, Ty) \leq h \max \{d(fx, fy), d(Tx, fx), d(Ty, fy), d(Tx, fy), d(Ty, fx)\}. \quad (3.1)$$

If f is continuous and $F(f)$ is nonempty, then there exists a unique common fixed point of T and f .

Proof. By our assumptions, $T(F(f)) \subseteq F(f)$ and $F(f)$ is nonempty and closed. Moreover, $cl(T(F(f)))$ being subset of $cl(T(K))$ is complete. Further, for all $x, y \in F(f)$, we have by inequality (3.1),

$$\begin{aligned} d(Tx, Ty) &\leq h \max \{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fy, Tx), d(fx, Ty)\} \\ &= h \max \{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}. \end{aligned}$$

Hence T is a generalized contraction on $F(f)$ and $cl(T(F(f))) \subseteq cl(F(f)) = F(f)$. By Theorem 3.1, T has a unique fixed point z in $F(f)$ and consequently $F(f) \cap F(T)$ is singleton.

The following result presents an analogue of Lemma 3.3 [3] for Banach operator pair without imposing the condition that f satisfies property I .

Lemma 3.3. Let f and T be self-maps on a nonempty q -starshaped subset K of a CAT(0) space X . Assume that f is continuous and $F(f)$ is q -starshaped with

$q \in F(f)$, (T, f) is a Banach operator pair on K and satisfy for each $n \geq 1$

$$d(T^n x, T^n y) \leq k_n \max \left\{ \begin{array}{l} d(fx, fy), \text{dist}(fx, Y_q^{T^n x}), \text{dist}(fy, Y_q^{T^n y}), \\ \text{dist}(fx, Y_q^{T^n y}), \text{dist}(fy, Y_q^{T^n x}) \end{array} \right\} \quad (3.2)$$

for all $x, y \in K$, where $\{k_n\}$ is a sequence of real numbers with $k_n \geq 1$ and $\lim_{n \rightarrow \infty} k_n = 1$. For each $n \geq 1$, define a mapping T_n on K by

$$T_n x = (1 - \mu_n)q \oplus \mu_n T^n x,$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \lambda_n = 1$.

Then for each $n \geq 1$, T_n and f have exactly one common fixed point x_n in K such that $fx_n = x_n = (1 - \mu_n)q \oplus \mu_n T^n x_n$ provided $cl(T_n(K))$ is complete for each n .

Proof. By definition,

$$T_n x = (1 - \mu_n)q \oplus \mu_n T^n x.$$

As (T, f) is a Banach operator pair, for each $n \geq 1$, $T^n(F(f)) \subseteq F(f)$ and $F(f)$ is nonempty and closed. Since $F(f)$ is q -starshaped and $T^n x \in F(f)$, for each $x \in F(f)$, $T_n x = (1 - \mu_n)q \oplus \mu_n T^n x \in F(f)$. Thus (T_n, f) is Banach operator pair for each n . Since $T_n x \in [q, T^n x]$ and $T_n y \in [q, T^n y]$ such that $d(T_n x, q) = \mu_n d(q, T^n x)$ and $d(T_n y, q) = \mu_n d(q, T^n y)$, therefore by (3.2),

$$\begin{aligned} d(T_n x, T_n y) &= d((1 - \mu_n)q \oplus \mu_n T^n x, (1 - \mu_n)q \oplus \mu_n T^n y) \\ &\leq \mu_n d(T^n x, T^n y) \\ &\leq \lambda_n \max \{d(fx, fy), \text{dist}(fx, Y_q^{T^n x}), \text{dist}(fy, Y_q^{T^n y}), \\ &\quad \text{dist}(fx, Y_q^{T^n y}), \text{dist}(fy, Y_q^{T^n x})\} \\ &\leq \lambda_n \max \{d(fx, fy), d(fx, T^n x), d(fy, T^n y), \\ &\quad d(fx, T^n y), d(fy, T^n x)\} \end{aligned}$$

for each $x, y \in K$. By Lemma 3.2, for each $n \geq 1$, there exists a unique $x_n \in K$ such that $x_n = fx_n = T_n x_n$. Thus for each $n \geq 1$, $K \cap F(T_n) \cap F(f) \neq \emptyset$. The following result extends the recent results due to Chen and Li ([7], Theorems 3.2-3.3) to asymptotically f -nonexpansive maps.

Theorem 3.4. Let f and T be self-maps on a q -starshaped subset K of a CAT(0) space X . Assume that (T, f) is Banach operator pair on K , $F(f)$ is q -starshaped with $q \in F(f)$, f is continuous, T is uniformly asymptotically regular and asymptotically f -nonexpansive. Then $F(T) \cap F(f) \neq \emptyset$, provided $cl(T(K))$ is compact and T is continuous or f is Δ -continuous and $T(K)$ is Δ -compact and complete.

Proof. Notice that compactness of $cl(T(K))$ implies that $cl T_n(K)$ is compact and thus complete. From Lemma 3.3, for each $n \geq 1$, there exists $x_n \in K$ such that $x_n = fx_n = (1 - \mu_n)q \oplus \mu_n T^n x_n$. As $T(K)$ is bounded, so $d(x_n, T^n x_n) = d((1 - \mu_n)q \oplus \mu_n T^n x_n, T^n x_n) \leq (1 - \mu_n)d(T^n x_n, q) \rightarrow 0$ as $n \rightarrow \infty$. Since (T, f) is

Banach operator pair and $fx_n = x_n$, so $fT^n x_n = T^n fx_n = T^n x_n$ and thus we get

$$\begin{aligned} d(x_n, Tx_n) &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(fT^n x_n, fx_n) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n). \end{aligned}$$

Further, T is uniformly asymptotically regular, therefore we have

$$d(x_n, Tx_n) \leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \rightarrow 0,$$

as $n \rightarrow \infty$. Now the compactness of $cl(T_n(K))$ further implies that there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that $x_k \rightarrow y \in K$ as $k \rightarrow \infty$. Now $d(y, Ty) \leq d(Ty, Tx_k) + d(Tx_k, x_k) + d(x_k, y)$ and continuity of T and the fact $d(x_k, Tx_k) \rightarrow 0$, gives that $y \in F(T)$. Also by the continuity of f , we have $y \in F(T) \cap F(f)$. Thus $F(T) \cap F(f) \neq \phi$.

The Δ -compactness and completeness of $T(K)$ implies that $T_n(K)$ is Δ -compact and complete. From Lemma 3.3, for each $n \geq 1$, there exists $x_n \in K$ such that $x_n = fx_n = (1 - \mu_n)q \oplus \mu_n T^n x_n$. The analysis in (i), implies that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. The Δ -compactness of $T(K)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ Δ -converging to $y \in K$ as $m \rightarrow \infty$. Δ -continuity of f implies that $fy = y$. Now we show that $y = Ty$. Suppose that $y \neq Ty$, then by uniqueness of asymptotic centers we have

$$\begin{aligned} \limsup_{m \rightarrow \infty} d(x_m, y) &< \limsup_{m \rightarrow \infty} d(x_m, Ty) \\ &\leq \limsup_{m \rightarrow \infty} d(x_m, Tx_m) + \limsup_{m \rightarrow \infty} d(Tx_m, Ty) \\ &= \limsup_{m \rightarrow \infty} d(Tx_m, Ty) \leq \limsup_{m \rightarrow \infty} (k_m d(fx_m, fy)) \\ &= \limsup_{m \rightarrow \infty} d(x_m, y), \end{aligned}$$

which is a contradiction. Thus $fy = Ty = y$ and hence $F(T) \cap F(I) \neq \phi$.

Corollary 3.5 Let f and T be self-maps on a q -starshaped subset K of a CAT(0) space X . Assume that (T, f) is Banach operator pair on K , $F(f)$ is q -starshaped with $q \in F(f)$, f is continuous and T is f -nonexpansive. Then $F(T) \cap F(f) \neq \phi$, provided $cl(T(K))$ is compact.

Corollary 3.6 Let f and T be self-maps on a q -starshaped subset K of a CAT(0) space X . Assume that (T, f) is commuting pair on K , $F(f)$ is q -starshaped with $q \in F(f)$, f is continuous and T is f -nonexpansive. Then $F(T) \cap F(f) \neq \phi$, provided $cl(T(K))$ is compact.

Definition 3.7. Let X be a metric space and K be a closed subset of X . If there exists a $y_0 \in K$ such that $d(x, y_0) = d(x, K) = \inf\{d(x, y) : y \in K\}$, then y_0 is called a *best approximation to x out of K* . We denote by $P_K(x)$, the set of all best approximations to x out of K .

Remark 3.8. Let K be a closed convex subset of a CAT(0) space. As $(1 - \lambda)u \oplus \lambda v \in K$ for $u, v \in K$, $\lambda \in [0, 1]$, note that $(1 - \lambda)u \oplus \lambda v \in P_K(x)$. Hence $P_K(x)$

is a convex subset of X . Also, $P_K(x)$ is a closed subset of X . Moreover, it can be shown that $P_K(x) \subset \partial K$, where ∂K stands for the boundary of K .

Now we obtain results on best approximation as a fixed point of Banach operator pair of mappings in a CAT(0) space.

Theorem 3.9. Let K be a subset of a CAT(0) space X and $f, T : X \rightarrow X$ be mappings such that $u \in F(f) \cap F(T)$ for some $u \in X$ and $T(\partial K \cap K) \subseteq K$. Suppose that $P_K(u)$ is nonempty and q -starshaped, f is continuous on $P_K(u)$, $d(Tx, Tu) \leq d(fx, fu)$ for each $x \in P_K(u)$ and $f(P_K(u)) \subseteq P_K(u)$. If (T, f) is a Banach operator pair on $P_K(u)$, $F(f)$ is nonempty and q -starshaped for $q \in F(f)$, T is uniformly asymptotically regular and asymptotically f -nonexpansive then $P_K(u) \cap F(f) \cap F(T) \neq \phi$, provided T is continuous and $cl(T(P_K(u)))$ is compact or f is Δ -continuous on $P_K(u)$ and $T(P_K(u))$ is Δ -compact and complete.
Proof. Let $x \in P_K(u)$. Then for any $h \in (0, 1)$, $d((1-h)u \oplus hx, x) \leq (1-h)d(x, u) < dist(u, K)$. It follows that $\{(1-h)u \oplus hx : 0 < h < 1\}$ and the set K are disjoint. Thus x is not in the interior of K and so $x \in \partial K \cap K$. Since $T(\partial K \cap K) \subseteq K$, Tx must be in K . Also $fx \in P_K(u)$, $u \in F(f) \cap F(T)$ and f and T satisfy $d(Tx, Tu) \leq d(fx, fu)$, thus we have

$$\begin{aligned} d(Tx, u) &= d(Tx, Tu) \leq d(fx, fu) \\ &= d(fx, u) = dist(u, K). \end{aligned}$$

It further implies that $Tx \in P_K(u)$. Therefore T is a self map of $P_K(u)$. The result now follows from Theorem 3.4.

The above result extends Theorem 3.2 of [2], Theorems 4.1-4.2 of [7], Theorem 7 of [15], Theorem 3 of [24], the corresponding results of [17], [18], [25], and [26].

Remarks 3.10.

- (1) Theorem 3.4 extends and improves Theorems 1 and 2 of Dotson [10], Theorem 2.2 of Al-Thagafi [2], Theorem 4 of Habiniak [14] and Theorem 1 of Khan and Khan [18].
- (2) Theorem 3.7 extends and improves Theorem 3.4 of Beg et al [3] to CAT(0) spaces.

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