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PARA-CONTACT PRODUCT SEMI-RIEMANNIAN SUBMERSIONS

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Abstract. We introduce the concept of para-contact product semi-Riemannian submersions from an almost para-contact metric manifold onto a semi-Riemannian product manifold. We provide an example and show that the vertical and horizontal distributions of such submersions are invariant with respect to the almost para-contact structure of the total manifold. Moreover, we investigate various properties of the O'Neill's tensors of such submersions, find the integrability of the horizontal distribution. The paper is also focused on the transference of structures defined on the total manifold.

Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [20] and [13], respectively. Later, Riemannian submersions were considered between almost complex manifolds by Watson in [23] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. For instances, Riemannian submersions between almost contact manifolds were studied by Chinea in [6] under the name of almost contact submersions. Riemannian submersions have been also considered for quaternionic Kähler manifolds [14] and para-quaternionic Kähler manifolds [5]. This kind of submersions have been studied with different names by many authors(see [1], [2], [3], [11], [12],[15], [18] and more). On the other hand, in[16] Kaneyuki and Williams defined the almost para-contact structure on pseudo-Riemannian manifold M of dimension (2m+1) and constructed the almost para-complex structure on $M^{2m+1} \times R$.

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Semi-Riemannian submersions were introduced by O'Neill in his book[21]. It is known that such submersions have their applications in Klauza-Klein theories, Yang-Mills equations, strings, supergravity. For applications of semi-Riemannian submersions, see:[9]. Since semi-Riemannian Product manifolds and para-contact manifolds are semi-Riemannian manifolds, one should consider semi-Riemannian submersions between such manifolds. In this paper, we define para-contact product semi-Riemannian submersions between almost para-contact metric manifold and semi-Riemannian product manifold, and study the geometry of such submersions. We observe that para-contact product semi-Riemannian submersion has also rich geometric properties.

The paper is organized as follows. In Section 2 we collect basic definitions, some formulas and results for later use. In section 3 we introduce the notion of para-contact product semi-Riemannian submersions and give an example of para-contact product semi-Riemannian submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for para-contact product semi-Riemannian submersions. We find the integrability of the horizontal distribution.

Preliminaries

In this section we are going to recall main definitions and properties of almost para-contact metric manifolds, semi-Riemannian product manifolds and semi-Riemannian submersions.

2.1. Almost para-contact metric manifolds. Let M be a $(2m+1)$- dimensional differentiable manifold. Let φ be a $(1,1)-$tensor field, ξ a vector field and η a 1-form on M. Then (φ, ξ, η) is called an almost para-contact structure on M if

(i) $\eta(\xi) = 1$, $\varphi^2 = Id - \eta \otimes \xi$,

(ii) the tensor field φ induces an almost para-complex structure on the distribution $\mathcal{D} = ker\eta$, that is, the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues 1, -1 of φ, respectively, have equal dimension m.
M is said to be almost para-contact manifold if it is endowed with an almost para-contact structure([7], [16],[19],[26]).

Let M be an almost para-contact manifold. M is called an almost para-contact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of signature $(m+1, m)$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \chi(M). \tag{1}$$

For such a manifold, we additionally have $\eta(X) = g(X,\xi)$, $\varphi\xi = 0$, $\eta\circ\varphi = 0$. Moreover, we can define a skew-symmetric 2-form Φ by $\Phi(X,Y) = g(X,\varphi Y)$, which is called the fundamental 2-form corresponding to the structure. Note that $\eta\wedge\Phi$ is, up to a constant factor, the Riemannian volume element of M.
On an almost para-contact manifold, one defines the $(1,2)-$tensor field $N^{(1)}$ by

$$N^{(1)}(X,Y) = [\varphi,\varphi](X,Y) - 2d\eta(X,Y)\xi, \tag{2}$$

where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi,\varphi](X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y]. \tag{3}$$

If $N^{(1)}$ vanishes identically, then the almost para-contact manifold (structure) is said to be normal ([26]). The normality condition says that the almost para-complex structure J defined on $M\times R$ by

$$J(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\frac{d}{dt}) \tag{4}$$

is integrable.

We note that an almost para-contact metric manifold (M,g,φ,ξ,η) is called

- (a) *normal*, if $N_\varphi - 2d\eta\otimes\xi = 0$, where N_φ is the Nijenhuis tensor of φ;
- (b) *para-contact*, if $\Phi = d\eta$;
- (c) *K-para-contact*, if M is para-contact and ξ Killing;
- (d) *para–cosymplectic,* if $\nabla\Phi = 0$ which implies $\nabla\eta = 0$, where ∇ is the Levi-Civita connection on M;
- (f) *almost para–cosymplectic,* if $d\eta = 0$ and $d\Phi = 0$;
- (g) *weakly para–cosymplectic,* if M is almost para–cosymplectic and $[R(X,Y),\varphi] = R(X,Y)\varphi - \varphi R(X,Y) = 0$;
- (h) *para–Sasakian,* if $\Phi = d\eta$ and M is normal;
- (j) *quasi-para–Sasakian,* if $d\Phi = 0$ and M is normal([8],[24],[26]).

It is known that an almost para-contact manifold is a para-Sasakian manifold if and only if $(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X$, for $X,Y\in\Gamma(TM)$.

Lemma 2.1 ([26]). Let (M,φ,ξ,η,g) be an almost para-contact metric manifold. Then we have

$$\begin{aligned} 2g((\nabla_X\varphi)Y,Z) &= -d\Phi(X,Y,Z) - d\Phi(X,\varphi Y,\varphi Z) - N^{(1)}(Y,Z,\varphi X) \\ &\quad + N^{(2)}(Y,Z)\eta(X) - 2d\eta(\varphi Z,X)\eta(Y) \\ &\quad + 2d\eta(\varphi Y,X)\eta(Z), \end{aligned} \tag{5}$$

where Φ is the fundamental 2-form and

$$N^{(2)}(X,Y) = (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \tag{6}$$

where L is the Lie derivative.
Moreover if M is para-contact, then we have

$$\begin{aligned} 2g((\nabla_X\varphi)Y,Z) &= -N^{(1)}(Y,Z,\varphi X) - 2d\eta(\varphi Z,X)\eta(Y) \\ &+ 2d\eta(\varphi Y,X)\eta(Z). \end{aligned} \tag{7}$$

For an almost para-contact metric manifold, the following identities are well known:

$$(\nabla_X\varphi)Y = \nabla_X\varphi Y - \varphi(\nabla_X Y), \tag{8}$$

$$(\nabla_X\Phi)(Y,Z) = g(Y,(\nabla_X\varphi)Z), \tag{9}$$

$$(\nabla_X\eta)Y = g(Y,\nabla_X\xi). \tag{10}$$

2.2. Semi-Riemannian product manifolds. Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indexes $q_1 > 0$, $q_2 > 0$, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ the projections which are given by $\pi(x,y) = x$ and $\sigma(x,y) = y$ for any $(x,y) \in M_1 \times M_2$, respectively. We denote the product manifold by $M = (M_1 \times M_2, g)$, where

$$g(X,Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(TM)$ and $*$ means tangent mapping. Then we have $\pi_*^2 = \pi_*$, $\sigma_*^2 = \sigma_*$, $\pi_*\sigma_* = \sigma_*\pi_* = 0$ and $\pi_* + \sigma_* = I$, where I is identity tranformation. Thus (M, g) is an $(m_1 + m_2)-$dimensional semi-Riemannian manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $M = M_1 \times M_2$ is characterized by M_1 and M_2 are totally geodesic submanifolds of M[25].

Now, if we put $F = \pi_* - \sigma_*$, then we can easily see that $F^2 = I$ and

$$g(FX,Y) = g(X,FY), \tag{11}$$

for any $X, Y \in \Gamma(TM)$.

Denote the Levi-Civita connection on M with respect to g by ∇. Then, M is called a locally semi-Riemannian product manifold if F is parallel with respect to ∇, i.e.,

$$\nabla_X F = 0, X \in \Gamma(TM)([17],[22]). \tag{12}$$

2.3. Semi-Riemannian submersions. Let (M, g) and (B, g') be two connected semi-Riemannian manifolds of index $s(0 \le s \le dimM)$ and $s'(0 \le s' \le dimB)$ respectively, with $s > s'$. Roughly speaking, a semi-Riemannian submersion is a smooth map $\pi : M \to B$ which is onto and satisfies the following conditions:
(i) $\pi_{*p} : T_pM \to T_{\pi(p)}B$ is onto for all $p \in M$;
(ii) The fibres $\pi^{-1}(p'), p' \in B$, are semi-Riemannian submanifolds of M;
(iii) π_* preserves scalar products of vectors normal to fibres.
The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is

basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A semi-Riemannian submersion $\pi : M \to B$ determines two $(1,2)$ tensor field T and A on M, by the formulas:

$$T(E,F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{13}$$

and

$$A(E,F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{14}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [4],[10]). From (13) and (14), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{15}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{16}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{17}$$

for any $X, Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if X is basic then $h(\nabla_U X) = h(\nabla_X U) = A_X U$.

We note that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2} v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.2 (see [10],[21]). *If $\pi : M \to B$ is a semi-Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on B, then we have the following properties*

(1) *$h[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X', Y'] \circ \pi$;*
(2) *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
(3) *$[E, U] \in \Gamma(\mathcal{V})$, for any $U \in \Gamma(\mathcal{V})$ and for any basic vector field E.*

Para-contact product semi-Riemannian submersions

In this section, we define the notion of a para-contact product semi-Riemannian submersion, give an example and study the geometry of such submersions. We now define a $(\varphi, F)-$ para-holomorphic map which is similar to the notion of a $(\varphi, F)-$ holomorphic map between almost contact metric manifold and almost Hermitian manifold.

Definition 3.1. Let $(M^{2m+1}, \varphi, \xi, \eta)$ be an almost para-contact manifold and (B^n, F) an almost product manifold, respectively. The map $\pi : M \to B$ is $(\varphi, F)-$para-holomorphic if $\pi_* \circ \varphi = F \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

Definition 3.2. Let $(M, \varphi, \xi, \eta, g)$ be an almost para-contact metric manifold and (B, F, g') be a semi-Riemannian product manifold. A semi-Riemannian submersion $\pi : M \to B$ is a called para-contact product semi-Riemannian submersion if it is (φ, F)-para-holomorphic, as well.

We give an example of a para-contact product semi-Riemannian submersion.

Example 3.1. Consider the following submersion defined by

$$\begin{aligned} \pi : R^5_2 &\to R^2_1 \\ (x_1, x_2, y_1, y_2, z) &\to (\frac{x_1 + x_2}{\sqrt{2}}, \frac{y_1 + y_2}{\sqrt{2}}). \end{aligned}$$

Then, the kernel of π_* is

$$\mathcal{V} = Ker\pi_* = Span\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi = \frac{\partial}{\partial z}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (Ker\pi_*)^\perp = Span\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\}.$$

Hence, we have

$$g(X, X) = g'(\pi_* X, \pi_* X) = -2, \quad g(Y, Y) = g'(\pi_* Y, \pi_* Y) = 2$$

Thus, π is a semi-Riemannnian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_* \varphi X = F \pi_* X, \quad \pi_* \varphi Y = F \pi_* Y.$$

Thus, π is a para-contact product semi-Riemannian submersion.

By using Definition 3.1, we have the following result.

Proposition 3.1. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para-contact metric manifold M onto a semi-Riemannian product manifold B, and let X be a basic vector field on M, $\pi-$related to X' on B. Then, φX is also a basic vector field $\pi-$related to FX'.*

The following result can be proved in a standard way.

Proposition 3.2. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para-contact metric manifold M onto a semi-Riemannian product manifold B. If X, Y are basic vector fields on M, $\pi-$related to X', Y' on B, then, we have*

(i) *$h(\nabla_X \varphi)Y$ is the basic vector field $\pi-$related to $(\nabla'_{X'} F)Y'$;*
(ii) *$h[X, Y]$ is the basic vector field $\pi-$related to $[X', Y']$.*

Next proposition shows that a para-contact product semi-Riemannian submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 3.3. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para-contact metric manifold M onto a semi-Riemannian product manifold B. Then, the horizontal and vertical distributions are $\varphi-$ invariant.*

Proof. Consider a vertical vector field U; it is known that $\pi_*(\varphi U) = F(\pi_* U)$. Since U is vertical and π is a semi-Riemannian submersion, we have $\pi_* U = 0$ from which $\pi_*(\varphi U) = 0$ follows and implies that φU is vertical, being in the kernel of π_*.
As concerns the horizontal distribution, let X be a horizontal vector field. We have $g(\varphi X, U) = -g(X, \varphi U) = 0$ because φU is vertical and X is horizontal. From $g(\varphi X, U) = 0$ we deduce that φX is orthogonal to U and then φX is horizontal.

Proposition 3.4. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para-contact metric manifold M onto a semi-Riemannian product manifold B. Then, we have*

(i) *$\pi^*\Phi' = \Phi$ holds on the horizontal distribution, only;*
(ii) *ξ is a vertical vector field;*
(iii) *$\eta(X) = 0$, for all horizontal vector fields X;*

Proof. We prove only statement (i), the other assertions can be obtained in a similar way. If X and Y are basic vector fields on M, $\pi-$related to X', Y' on B, then using the definition of a para-contact product semi-Riemannian submersion, we have

$$\begin{aligned} \pi^*\Phi'(X, Y) &= \Phi'(\pi_* X, \pi_* Y) = g'(\pi_* X, F\pi_* Y) = g'(\pi_* X, \pi_* \varphi Y) \\ &= \pi^* g'(X, \varphi Y) = g(X, \varphi Y) = \Phi(X, Y) \end{aligned}$$

which gives the proof of assertion(i).

Theorem 3.1. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion. If the total space M is an almost para-contact metric manifold with $(\nabla_X \varphi)Y = 0$, for $X, Y \in \Gamma(\mathcal{H})$, then the base space B is a locally semi-Riemannian* *product manifold.*

Proof. Let X, Y and Z be basic vector fields on M, $\pi-$related to X', Y' and Z' on B. Since $(\nabla_X \varphi)Y = 0$ for $X, Y \in \Gamma(\mathcal{H})$. we get

$$0 = g(Z, \nabla_X \varphi Y - \varphi \nabla_X Y)$$

for $Z \in \Gamma(\mathcal{H})$. Using (17) we obtain

$$0 = g(Z, h\nabla_X \varphi Y) - g(Z, h\varphi \nabla_X Y)$$

Then, by using $\pi_* \varphi = F\pi_*$, we get

$$0 = g'(Z', \nabla'_{X'} FY) - g'(Z', F\nabla'_{X'} Y').$$

Hence $0 = g'(Z', (\nabla'_{X'} F)Y')$ which shows that B is a locally semi-Riemannian product manifold.

Theorem 3.2. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion. If the total space M is a para-Sasakian manifold, then the base space B is a locally semi-Riemannian product manifold.*

Proof. Let X and Y be basic vector fields on M, $\pi-$related to X' and Y' on B. Since M is a para–Sasakian manifold, we have

$$\begin{aligned} (\nabla_X \varphi)Y &= -g(X, Y)\xi + \eta(Y)X \\ &= -g(X, Y)\xi. \end{aligned}$$

Since π is a semi-Riemannian submersion, we get

$$\pi_*((\nabla_X \varphi)Y) = -g(X, Y)\pi_* \xi = 0.$$

Then, by using $\pi_* \varphi = F\pi_*$, we obtain $\pi_*((\nabla_X \varphi)Y) = (\nabla'_{X'} F)Y' = 0$,
which proves the assertion.

Theorem 3.3. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion. If the total space M is a quasi para-Sasakian manifold, then the base space B is a locally semi-Riemannian product manifold.*

Proof. Let X and Z be basic vector fields on M, $\pi-$related to X' and Z' on B. Since (5), we have

$$\begin{aligned} 2g((\nabla_X \varphi)X, Z) &= -d\Phi(X, X, Z) - d\Phi(X, \varphi X, \varphi Z) - N^{(1)}(X, Z, \varphi X) \\ &\quad + N^{(2)}(X, Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(X) + 2d\eta(\varphi X, X)\eta(Z). \end{aligned}$$

Since M is a quasi-para-Sasakian manifold, we obtain

$$2g((\nabla_X \varphi)X, Z) = -2d\eta(\varphi Z, X)\eta(X) + 2d\eta(\varphi X, X)\eta(Z).$$

Since η vanishes on the horizontal distribution, we have

$$g((\nabla_X \varphi)X, Z) = 0.$$

Thus, we deduce that

$$\pi_*((\nabla_X \varphi)X) = 0 = (\nabla'_{X'} F)X',$$

which shows that the base space is a locally semi-Riemannian product manifold.

Proposition 3.5. *Let $\pi : (M^{2m+1}, \varphi, \xi, \eta, g) \to (B^{2n}, F, g')$ be a para-contact product semi-Riemannian submersion from an almost para-contact metric manifold M onto a semi-Riemannian product manifold B. Then, the fibres are almost para-contact metric manifolds.*

Proof. Denoting by F the fibres, it is clear that $dimF = 2(m-n) + s = 2r + 1$, where $r = m - n$. We define an almost para-contact structure $(\hat{g}, \hat{\varphi}, \hat{\eta}, \hat{\xi})$, by setting $\varphi = \hat{\varphi}$, $\eta = \hat{\eta}$and $\xi = \hat{\xi}$. Then, we get

$$\hat{\varphi}^2 U = \varphi^2 U = U - \eta(U)\xi,$$

for $U \in \Gamma(\mathcal{V})$.

On the other hand, for $U, V \in \Gamma(\mathcal{V})$ we obtain

$$\begin{aligned} \hat{g}(\hat{\varphi}V, \hat{\varphi}U) &= \hat{g}(\varphi V, \varphi U) = -\hat{g}(V, \varphi^2 U) = -\hat{g}(V, U - \eta(U)\xi) \\ &= -\hat{g}(V, U) + \hat{\eta}(U)\hat{\eta}(V), \end{aligned}$$

which gives the proof of assertion.

We now check the properties of the tensor fields T and A for a para-contact product semi-Riemannian submersion, we will see that such tensors have extra properties for such submersions.

Proposition 3.6. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from a para- cosymplectic manifold M onto a semi-Riemannian product manifold B, and let X and Y be horizontal vector fields. Then, we have*

(i) $A_X \varphi Y = \varphi A_X Y$,

(ii) $A_{\varphi X} Y = \varphi A_X Y$.

Proof. (i) Let X and Y be horizontal vector fields, and U vertical. Since M is a Para cosymplectic manifold, we have

$$\begin{aligned} (\nabla_X \Phi)(U, Y) &= g((\nabla_X \varphi)Y, U) \\ &= g(\nabla_X \varphi Y - \varphi \nabla_X Y, U) = 0 \end{aligned}$$

Thus, since the vertical and the horizontal distributions are invariant, from (17) we obtain

$$g(A_X\varphi Y - \varphi A_X Y, U) = 0.$$

Then, we have

$$A_X\varphi Y = \varphi A_X Y.$$

In a similar way, we obtain (ii).

For the tensor field T we have the following.

Proposition 3.7. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from a para-cosymplectic manifold M onto a semi-Riemannian product manifold B, and let U and V be vertical vector fields. Then, we have*

(i) $T_U\varphi V = \varphi T_U V$,

(ii) $T_{\varphi U}V = \varphi T_U V$.

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

Theorem 3.4. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para-cosymplectic manifold M onto a semi-Riemannian product manifold B. Then, the horizontal distribution is integrable.*

Proof. Let X and Y be basic vector fields. It suffices to prove that $v([X,Y]) = 0$, for basic vector fields on M. Since M is an almost para–cosymplectic manifold, it implies $d\Phi(X,Y,V) = 0$, for any vertical vector V. Then, one obtains

$$X(\Phi(Y,V)) - Y(\Phi(X,V)) + V(\Phi(X,Y))$$
$$-\Phi([X,Y],V) + \Phi([X,V],Y) - \Phi([Y,V],X) = 0.$$

Since $[X,V],[Y,V]$ are vertical and the two distributions are $\varphi-$invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X,Y],\varphi V) = V(g(X,\varphi Y)).$$

On the other hand, if X is basic then $h(\nabla_V X = h(\nabla_X V) = A_X V$, thus we have

$$\begin{aligned} V(g(X,\varphi Y)) &= g(\nabla_V X,\varphi Y) + g(\nabla_V\varphi Y, X) \\ &= g(A_X V,\varphi Y) + g(A_{\varphi Y}V, X). \end{aligned}$$

Since, A is skew-symmetric and alternating operator, we get $V(g(X,\varphi Y)) = 0$. This proves the assertion.

Since for a quasi-para-Sasakian manifold $d\Phi = 0$, applying Theorem 3.4, we have the following result.

Corollary 3.1 *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto a semi-Riemannian product manifold B. Then, the horizontal distribution is integrable.*

Corollary 3.2. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from a para–cosymplectic manifold M onto a semi-Riemannian product manifold B. Then, the horizontal distribution is completely integrable.*

Theorem 3.5. *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from an almost para–cosymplectic manifold M onto a semi-Riemannian product manifold B with $dim\mathcal{V}_p \geq 2$, $\forall p \in M$. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then $T_V X = 0$, for any $V \in \Gamma(\mathcal{V})$, if and only if $\eta(\nabla_X V) = \eta([X, V])$.*

Proof. Let W and V be vertical vector fields on M, X horizontal. Since M is an almost para–cosymplectic manifold, it implies $d\Phi = 0$. Then, we obtain

$$\begin{aligned} d\Phi(W, \varphi V, X) &= W(\Phi(\varphi V, X)) - \varphi V(\Phi(W, X)) + X(\Phi(W, \varphi V)) \\ &\quad -\Phi([W, \varphi V], X) + \Phi([W, X], \varphi V) - \Phi([\varphi V, X], W) = 0. \end{aligned}$$

Since $[W, \varphi V]$ is vertical and the two distributions are $\varphi-$invariant, the first two terms vanish. Thus, we get

$$X(\Phi(W, \varphi V)) + \Phi([W, X], \varphi V) - \Phi([\varphi V, X], W) = 0.$$

By direct computations, one obtains:

$$\begin{aligned} 0 &= X(g(W, V) - \eta(V)g(W, \xi)) + g([W, X], V - \eta(V)\xi) - g([\varphi V, X], \varphi W) \\ 0 &= g(W, \nabla_V X + [X, V]) + g(\nabla_W X, V) - g(\varphi[X, \varphi V], W) - \eta(V)(g(\nabla_X \xi, W) \\ &+ g(\nabla_W X, \xi)) - X(\eta(V))\eta(W). \end{aligned}$$

Using (15) we derive

$$\begin{aligned} 0 &= g(T_V X, W) + g(T_W X, V) - 2g(T_\xi X, W)\eta(V) + \eta(W)(\eta([X, V]) \\ &- X(\eta(V)) - \eta(V)\eta([X, \xi])). \end{aligned} \tag{18}$$

Moreover, we have

$$\begin{aligned} \eta([X, V]) - X(\eta(V)) - \eta(V)\eta([X, \xi]) &= -\eta(\nabla_V X) - g(\nabla_X \xi, V) + \eta(V)\eta(\nabla_\xi X) \\ &= g(T_V \xi, X) - g(\nabla_X \xi, V) - \eta(V)g(T_\xi \xi, X). \end{aligned}$$

Substituting in (18), we obtain

$$\begin{aligned} 0 &= 2g(T_V X, W) - 2g(T_\xi X, W)\eta(V) + \eta(W)(g(T_V \xi, X) \\ &+ \eta(V)g(T_\xi X, \xi) - g(\nabla_X \xi, V)). \end{aligned} \tag{19}$$

Now, assume that $T_V X = 0$, for any $X \in \Gamma(\mathcal{V})$. Then (19) implies $g(\nabla_X \xi, V) = 0$, for any V and we have

$$\begin{aligned} \eta([X,V]) &= g(\nabla_X V, \xi) - g(\nabla_V X, \xi) \\ &= X(\eta(V)) - g(\nabla_X \xi, V) - g(T_V X, \xi) \\ &= \eta(\nabla_X V). \end{aligned}$$

On the other hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, the hypothesis $\eta([X,V]) = \eta(\nabla_X V)$ implies $g(T_V X, \xi) = g(\nabla_V X, \xi) = g(\nabla_X V + [V,X], \xi) = 0$. So, (19) reduces to

$$0 = 2g(T_V X, W) - \eta(W) g(\nabla_X \xi, V),$$

for any $V, W \in \Gamma(\mathcal{V})$. Thus, for any vertical vector field W orthogonal to ξ, we get $g(T_V X, W) = 0$. Since $g(T_V X, \xi) = 0$, one has $T_V X = 0$, $V \in \Gamma(\mathcal{V})$ and the proof is completed.

From Theorem 3.5, we have the following result.

Corollary 3.3 *Let $\pi : M \to B$ be a para-contact product semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto a semi-Riemannian product manifold B with $dim\mathcal{V}_p \geq 2$, $\forall p \in M$. If X horizontal vector field is an infinitesimal automorphism of $\varphi-$tensor field, then $T_V X = 0$, for any $V \in \Gamma(\mathcal{V})$, if and only if $\eta(\nabla_X V) = \eta([X,V])$.*

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