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η-RICCI SOLITONS IN TRANS-SASAKIAN MANIFOLDS

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Abstract. The aim of this paper is to study the η-Ricci solitons in 3-dimensional trans-Sasakian manifolds.

1. Introduction

A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by Oubino[11] in 1985. In [4], Gray-Hervella classification of almost Hermite manifolds appears as a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures. The local structures of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [7]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or Sasakian or Kenmotsu manifold.

In 1982, Hamilton [5] made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of a manifold. It is a process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. It is given by

$$\frac{\partial g}{\partial t} = -2Ric\ g. \tag{1.1}$$

Ricci soliton in a Riemannian manifold (M, g) is a special solution to the Ricci flow and is a natural generalization of an Einstein metric which is defined as a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1.2}$$

where S is the Ricci tensor of M and L_V denote the Lie derivative operator along the vector field V.

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The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. In [13], Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later Tripathi [14], Nagaraja et al.[10] and others extensively studied Ricci solitons in contact metric manifolds.

It is well known that, if the potential vector field is zero or Killing then the Ricci soliton is an Einstein metric. In [6], [2] and [9], the authors proved that there are no Einstein real hypersurfaces of non-flat complex space forms. Motivated by this the authors Cho and Kimura [3] introduced the notion of η- Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η- Ricci solitons. Later Blaga [1] studied η-Ricci solitons in para-Kenmotsu manifolds. Recently, Prakasha and Hadimani [12] studied η-Ricci solitons on para-Sasakian manifolds. It is quite interesting to study η- Ricci solitons in trans-Sasakian manifolds not as real hypersurfaces of complex space forms but as special contact structures. In this paper, we derive the condition for a 3 dimensional trans-Sasakian manifold as an η- Ricci soliton and derive expression for the scalar curvature.

2. Preliminaries

A differentiable manifold M is said to be an almost contact metric manifold if it admits a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and Riemannian metric g, which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \ \ \eta \circ \phi = 0, \ \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \ \ g(X,\xi) = \eta(X), \tag{2.2}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be trans-Sasakian manifold, if $(M \times \mathbb{R}, J, G)$ belongs to class W_4 of the Hermitian manifold, where J is the almost complex structure of $M \times \mathbb{R}$ defined by

$$J(Z, f d/dt) = (\phi Z - f\xi, \eta(Z) d/dt), \tag{2.3}$$

for all vector fields Z on M and smooth function f on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This is expressed by the following condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

where α and β are some scalar functions and such a structure is said to be the trans-Sasakian structure of type (α, β). We note that trans-Sasakian manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are the cosymplectic, α-Sasakian and β-Kenmotsu manifold respectively. In particular, if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds respectively. From (2.4), it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

and

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

The trans-Sasakian manifold with structure tensor (ϕ, ξ, η, g) on M satisfies the following relations:

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] + (Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y,$$

$$2\alpha\beta+\xi\alpha=0, \tag{2.7}$$

$$S(X,\xi) = \left((n-1)(\alpha^2-\beta^2)-\xi\beta)\right)\eta(X)-(n-2)(X\beta)-(\phi X)\alpha,$$
$$Q\xi = \left((n-1)(\alpha^2-\beta^2)-\xi\beta)\right)\xi-(n-1)grad\beta+\phi(grad\alpha), \tag{2.8}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.
Further in a trans-Sasakian manifold of type (α, β), we have

$$\phi(grad\alpha) = (n-1)grad\beta. \tag{2.9}$$

Using (2.7) and (2.9), for constants α and β, we have

$$R(\xi,X)Y = (\alpha^2-\beta^2)[g(X,Y)\xi-\eta(Y)X] \tag{2.10}$$
$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y] \tag{2.11}$$
$$\eta(R(X,Y)Z) = (\alpha^2-\beta^2)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)] \tag{2.12}$$
$$S(X,\xi) = (n-1)(\alpha^2-\beta^2)\eta(X) \tag{2.13}$$
$$Q\xi = (n-1)(\alpha^2-\beta^2)\xi. \tag{2.14}$$

An important consequence of (2.5) is that ξ is a geodesic vector field.

$$\text{i.e.,}\quad \nabla_\xi\xi = 0. \tag{2.15}$$

For arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0.$$

The ξ-sectional curvature K_ξ of (M, g) is the sectional curvature of a plane spanned by ξ and a unitary vector field X. From (2.11), we have

$$K_\xi = g(R(X,\xi)\xi, X) = (\alpha^2-\beta^2). \tag{2.16}$$

It follows from (2.16) that ξ sectional curvature does not depend on X.

3. η-Ricci solitons on (M, ϕ, ξ, η, g)

Consider now a symmetric tensor field h of $(0, 2)$ - type which is parallel with respect to Levi-Civita connection ($\nabla h = 0$). Applying the Ricci commutation identity $\nabla^2h(X, Y; Z, W) - \nabla^2h(X, Y; W, Z) = 0$, we obtain

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0. \tag{3.1}$$

Replacing $Z = W = \xi$ in (3.1) and by the symmetry of h, we have

$$h(R(X,Y)\xi,\xi) = 0. \tag{3.2}$$

Taking $X = \xi$ in (3.2), then by virtue of (2.11), we have

$$(\alpha^2 - \beta^2)[h(Y,\xi) - h(\xi,\xi)\eta(Y) = 0. \tag{3.3}$$

With the hypothesis on K_ξ, the above equation yields:

$$h(Y,\xi) = h(\xi,\xi)g(Y,\xi). \tag{3.4}$$

Again by taking $X = Z = \xi$ in (3.1), we obtain

$$(\alpha^2 - \beta^2)[\eta(Y)h(\xi,W) - h(Y,W) + g(Y,W)h(\xi,\xi) - \eta(W)h(\xi,Y)] = 0. \tag{3.5}$$

Since $(\alpha^2 - \beta^2) \neq 0$, we have

$$h(Y,W) = \eta(Y)h(\xi,W) + g(Y,W)h(\xi,\xi) - \eta(W)h(\xi,Y). \tag{3.6}$$

By using (3.4) in (3.6), we get

$$h(X,Y) = h(\xi,\xi)g(X,Y). \tag{3.7}$$

The above equation gives the conclusion :

Theorem 3.1. *Let (M,ϕ,ξ,η,g) be a trans-Sasakian manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in \Gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

We call the notion of η-Ricci solitons from [3].

$$L_V g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.8}$$

where L_V is the Lie derivative operator along the vector field V and λ and μ are real constants.
Because of (2.5), the equation (3.8) becomes:

$$S(X,Y) = -(\lambda+\beta)g(X,Y) + (\beta-\mu)\eta(X)\eta(Y). \tag{3.9}$$

The above equation yields

$$\begin{aligned} S(X,\xi) &= -(\lambda+\mu)\eta(X), & (3.10)\\ QX &= -(\lambda+\beta)X + (\beta-\mu)\eta(X)\xi, & (3.11)\\ Q\xi &= -(\lambda+\mu)\xi, & (3.12)\\ r &= -\lambda n - (n-1)\beta - \mu, & (3.13) \end{aligned}$$

where r is the scalar curvature. Off the two natural situations regarding the vector field V: $V \in Span\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $L_\xi g + 2S + 2\mu\eta \otimes \eta$. A straightforward computation gives

$$L_\xi g(X,Y) = 2\beta[g(X,Y) - \eta(X)\eta(Y)]. \tag{3.14}$$

In a 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.15}$$

$$\begin{aligned} QX =& \left[\frac{r}{2} + (\xi\beta) - (\alpha^2 - \beta^2)\right] X - \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2 - \beta^2)\right] \eta(X)\xi \\ & + \eta(X)[\phi(grad\alpha) - grad\beta] - [((\phi X)\alpha) + (X\beta)]\xi. \end{aligned} \tag{3.16}$$

By using (2.7) and (2.9) in (3.16), we obtain

$$QX = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] X - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\xi, \tag{3.17}$$

$$S(X,Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] g(X,Y) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\eta(Y). \tag{3.18}$$

Next, we consider the equation

$$h(X,Y) = (L_\xi g)(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y). \tag{3.19}$$

By using (3.14) and (3.18) in (3.19), we have

$$\begin{aligned} h(X,Y) =& \left[\frac{r}{2} - 2(\alpha^2 - \beta^2) + 2\beta\right] g(X,Y) \\ & - \left[\frac{r}{2} - 6(\alpha^2 - \beta^2) + 2\beta + 2\mu\right] \eta(X)\eta(Y). \end{aligned} \tag{3.20}$$

Putting $X = Y = \xi$ in (3.20), we get

$$h(\xi,\xi) = 2[2(\alpha^2 - \beta^2) - \mu]. \tag{3.21}$$

So (3.7) becomes

$$h(X,Y) = 2[2(\alpha^2 - \beta^2) - \mu]g(X,Y). \tag{3.22}$$

From (3.19) and (3.22), it follows that g is an η-Ricci soliton.

So we can state:

Theorem 3.2. *Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional trans-Sasakian manifold. Then (g, ξ, μ) yields an η-Ricci soliton on $(M^3, \phi, \xi, \eta, g)$.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0,$$

or

$$\begin{aligned} & bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) \\ & + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned}$$

Using (2.5), we obtain

$$\begin{aligned} & bg(-\alpha\phi X + \beta(X - \eta(X)\xi), Y) + (Xb)\eta(Y) + bg(-\alpha\phi Y + \beta(Y - \eta(Y)\xi), X) \\ & + (Yb)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \end{aligned}$$

which yields

$$\begin{aligned} &2b\beta g(X,Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X,Y) \\ &+ 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{3.23}$$

Replacing Y by ξ in the above equation, we obtain

$$Xb + (\xi b)\eta(X) + 2(2(\alpha^2 - \beta^2) + \lambda + \mu)\eta(X) = 0. \tag{3.24}$$

Again putting $X = \xi$ in (3.24), we obtain $\xi b = -2(\alpha^2 - \beta^2) - \lambda - \mu$.

Plugging this in (3.24), we get

$$Xb + +(2(\alpha^2 - \beta^2) + \lambda + \mu)\eta(X) = 0,$$

or

$$db = -\{\lambda + \mu + 2(\alpha^2 - \beta^2)\}\eta. \tag{3.25}$$

Applying d on (3.25), we get $\{\lambda + \mu + 2(\alpha^2 - \beta^2)\}d\eta = 0$.

Since $d\eta \neq 0$ we have

$$\lambda + \mu + 2(\alpha^2 - \beta^2) = 0. \tag{3.26}$$

Equation (3.26) in (3.25) yields b as a constant. Therefore from (3.23), it follows that

$$S(X,Y) = -(\lambda + b\beta)g(X,Y) + (b\beta - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for a constant β. This leads to the following:

Theorem 3.3. *If in a 3-dimensional trans-Sasakian manifold the metric g is an η-Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided β is a constant.*

Taking $X = Y = \xi$ in (3.7) and (3.18) and comparing, we get

$$\lambda = -2(\alpha^2 - \beta^2) + \mu = -2K_\xi - \mu. \tag{3.27}$$

From (3.13) and (3.27), we obtain

$$r = 6(\alpha^2 - \beta^2) - 2\beta + 2\mu. \tag{3.28}$$

Since λ is a constan,t it follows from (3.27) that K_ξ is a constant.

Theorem 3.4. *Let (g, ξ, μ) be an η-Ricci soliton in $(M^3, \phi, \xi, \eta, g)$. Then the scalar λ and the scalar curvature r satisfy the relations:*
$\lambda + \mu = -2K_\xi, \quad r = 6K_\xi + 2\mu + 2\beta.$

Remark 3.5: For $\mu = 0$, (3.27) reduces to $\lambda = -2K_\xi$, so Ricci soliton in 3-dimensional trans-Sasakian manifold is shrinking.

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